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Existence, uniqueness, and convergence of solutions of strongly damped wave equations with arithmetic-mean terms

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Abstract

In this paper, we study the Robin-Dirichlet problem (P_n) for a strongly damped wave equation with arithmetic-mean terms $S_n u$ and $\hat{S}_n u$, where u is the unknown function, $S_n u = \frac{1}{n}\sum_{i=1}^{n} u(\frac{i-1}{n}, t)$ and $\hat{S}_n u = \frac{1}{n}\sum_{i=1}^{n} u_x^2(\frac{i-1}{n}, t)$. First, under suitable conditions, we prove that, for each $n \in \mathbb{N}$, (P_n) has a unique weak solution u^n. Next, we prove that the sequence of solutions u^n converge strongly in appropriate spaces to the weak solution u of the problem (P), where (P) is defined by (P_n) in which the arithmetic-mean terms $S_n u$

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and $\hat{S}_n u$ are replaced by $\int_0^1 u(y,t)dy$ and $\int_0^1 u_x^2(y,t)dy$, respectively. Finally, some remarks on a couple of open problems are given.

Keywords: Robin-Dirichlet problem, Arithmetic-mean terms, Faedo-Galerkin method, Linear recurrent sequence.
2010 MSC: 35L20, 35L70, 35Q72.

1. Introduction

In this paper, we investigate the Robin-Dirichlet problem for a strongly damped wave equation as follows

$$(P_n) \begin{cases} u_{tt} - \lambda u_{txx} - \Big(1 + (\hat{S}_n u)(t)\Big) u_{xx} \\ \quad = f\left(x, t, u, u_x, u_t, (S_n u)(t)\right), \ 0 < x < 1, \ 0 < t < T, \\ u_x(0,t) - \zeta u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x), \ u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{1.1}$$

where f, $\tilde{u}_0$, $\tilde{u}_1$ are given functions, $\lambda > 0$, $\zeta \geq 0$, are given constants, and $S_n u$, $\hat{S}_n u$ are arithmetic-mean terms defined by

$$\begin{aligned} &(S_n u)(t) = \tfrac{1}{n} \sum\nolimits_{i=1}^{n} u(\tfrac{i-1}{n}, t), \\ &(\hat{S}_n u)(t) = (S_n u_x^2)(t) = \tfrac{1}{n} \sum\nolimits_{i=1}^{n} u_x^2(\tfrac{i-1}{n}, t). \end{aligned} \tag{1.2}$$

The nonlinear wave equations with strong damping have been investigated by many authors for years. These equations arise naturally in various sciences such as classical mechanics, fluid dynamics, quantum field theory, see [1] - [14] and the references given therein.

In [11], Pellicer and Morales considered a model for a damped spring-mass system, precisely a strongly damped wave equation with dynamic boundary conditions as follows

$$\begin{cases} u_{tt} - u_{xx} - \alpha u_{txx} = 0, \ 0 < x < 1, \ t > 0, \\ u(0,t) = 0, \\ u_{tt}(1,t) = -\varepsilon \left[u_x(1,t) + \alpha u_{tx}(1,t) + r u_t(1,t)\right]. \end{cases} \tag{1.3}$$

It is well known that the motion of a mass in a spring-mass-damper system is usually modelled by the following second-order ordinary differential equation (ODE) of damped oscillations

$$m u''(t) = -k u(t) - d u'(t), \tag{1.4}$$

where $k > 0$ is recovery constant of spring and $d \geq 0$ stands for dissipation coefficient. The authors showed that, for some certain values of the parameters in (1.4), the large time behaviour of the solutions is the same as for a classical spring-mass-damper ODE. For more details, they proved that for fixed constants α, $r > 0$ and ε small enough, the partial differential equation model (1.3) admitted two dominant eigenvalues. Therefore, this can be implied the existence of a second-order ODE of type (1.4) which can be considered as the limit of the model (1.3) when $t \to \infty$ and ε is sufficiently small.

In [5], O.M. Jokhadze studied the following Cauchy problem for a wave equation with a nonlinear damping term

$$\begin{cases} u_{tt} - u_{xx} + h(u_t) = f(x,t), \ x \in \mathbb{R}, \ t > 0, \\ u(x,0) = \varphi(x), \ u_t(x,0) = \psi(x), \end{cases} \tag{1.5}$$

where h, f, φ, and ψ are given real functions. The existence, uniqueness, nonuniqueness, and nonexistence of a global classical solution were established.

In [9], Nhan et. al. considered the Robin problem for a nonlinear wave equation with source containing multi-point nonlocal terms as follows

$$\begin{cases} u_{tt} - u_{xx} = f\left(x, t, u(x,t), u_t(x,t), u(\eta_1, t), \cdots, u(\eta_q, t)\right), \\ \qquad\qquad 0 < x < 1, 0 < t < T, \\ u_x(0,t) - h_0 u(0,t) = u_x(1,t) + h_1 u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x), \ u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{1.6}$$

where f, $\tilde{u}_0$, $\tilde{u}_1$ are given functions and h_0, $h_1 \geq 0$, η_1, $\eta_2, \cdots, \eta_q$ are given constants with $h_0 + h_1 > 0$, $0 \leq \eta_1 < \eta_2 < \cdots < \eta_q \leq 1$. The unique existence and the high-order asymptotic expansion in a small parameter of solutions for the problem (1.6) were established. We note that the arithmetic-mean $\frac{1}{n}\sum_{i=1}^{n} u(\frac{i-1}{n})$ in (1.1) can be considered as a special linear combination of $\{u(\eta_i)\}_{1\leq i \leq q}$ in (1.6).

We also note that, if the functions $y \longmapsto u(y,t)$ and $y \longmapsto u_x^2(y,t)$ are continuous on $[0,1]$, with $t \in [0,T]$ fixed, then we have

$$\begin{aligned} &\tfrac{1}{n}\sum\nolimits_{i=1}^{n} u(\tfrac{i-1}{n}, t) \to \int_0^1 u(y,t)dy, \\ &\tfrac{1}{n}\sum\nolimits_{i=1}^{n} u_x^2(\tfrac{i-1}{n}, t) \to \int_0^1 u_x^2(y,t)dy, \ \text{as } n \to \infty, \end{aligned}$$

hence Eq. $(1.1)_1$ may be related to the following equation

$$\begin{aligned} &u_{tt} - \lambda u_{txx} - \left(1 + \int_0^1 u_x^2(y,t)dy\right) u_{xx} \\ &= f\left(x, t, u, u_x, u_t, \int_0^1 u(y,t)dy\right), \ 0 < x < 1, \ 0 < t < T. \end{aligned} \tag{1.7}$$

Therefore, it is possible that the existence of solution for the problem (P_n) (1.1)-(1.2) leads to the existence of solution for the problem (P) $(1.1)_{2,3}$-(1.7).

Motivated by the mentioned works, especially according to the point of view above, we shall consider the problem (P_n) (1.1)-(1.2). Our paper consits of five sections. In Section 2, we present preliminaries and technical lemmas (Lemma 2.1- Lemma 2.4). In Section 3, we prove that (P_n) has a unique weak solution u^n. In Section 4, we show that the solution sequence u^n in appropriate spaces strongly converges to a weak solution u of the problem (P) as $n \to \infty$. In the proofs of results obtained here, the main tools of functional analysis such as the linear approximate method, the Galerkin method, the arguments of continuity with priori estimates, the compact method, the regularized technique are employed. The engery method is also applied to contructing a suitable engery lemma (Lemma 3.3), in which a piecewise linear function on $[0,T]$ and a regularized sequence in $C_c^\infty(\mathbb{R})$ are used to get an engery inequality. Lemma 3.3 is a relative generalization of the lemma given in Lions's book [[7], Lemma 6.1, p. 224], that is the key lemma to establish the convergence of linear approximate sequence associated with the problem (P_n). Finally, in Section 5, we give some remarks on a couple of open problems.

2. Preliminaries

Put $\Omega = (0,1)$. We denote $L^p = L^p(\Omega)$, $H^m = H^m(\Omega)$. Let $\langle \cdot, \cdot \rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $\|\cdot\|$ stands for the norm in L^2 and $\|\cdot\|_X$ for the norm in a Banach space X. We call X' the dual space of X. We consider $L^p(0,T;X)$, $1 \leq p \leq \infty$, that is a Banach space of real functions $u : (0,T) \to X$ measurable, such that $\|u\|_{L^p(0,T;X)} < +\infty$, with

$$\|u\|_{L^p(0,T;X)} = \begin{cases} \left(\int_0^T \|u(t)\|_X^p \, dt\right)^{1/p}, & \text{if} \quad 1 \leq p < \infty, \\ \underset{0<t<T}{ess\sup} \|u(t)\|_X, & \text{if} \quad p = \infty. \end{cases}$$

Let $u(t)$, $u'(t) = u_t(t) = \dot{u}(t)$, $u''(t) = u_{tt}(t) = \ddot{u}(t)$, $u_x(t) = \nabla u(t)$, $u_{xx}(t) = \Delta u(t)$, denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively.

Let $T^* > 0$, with $f \in C^k([0,1] \times [0,T^*] \times \mathbb{R}^4)$, $f = f(x,t,y_1,\cdots,y_4)$, we put $D_1 f = \frac{\partial f}{\partial x}$, $D_2 f = \frac{\partial f}{\partial t}$, $D_{i+2} f = \frac{\partial f}{\partial y_i}$ with $i = 1,\cdots,4$, and $D^\alpha f = D_1^{\alpha_1} \cdots D_6^{\alpha_6} f$, $\alpha = (\alpha_1,\cdots,\alpha_6) \in \mathbb{Z}_+^6$, $|\alpha| = \alpha_1 + \cdots + \alpha_6 = k$, $D^{(0,\cdots,0)} f = f$.

On H^1, we shall use the following norm

$$\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2\right)^{1/2}. \tag{2.1}$$

We put

$$V = \{v \in H^1(\Omega) : v(1) = 0\}, \tag{2.2}$$

$$a(u,v) = \int_0^1 u_x(x) v_x(x) dx + \zeta u(0) v(0), \ u, v \in V. \tag{2.3}$$

V is a closed subspace of H^1 and three norms $v \longmapsto \|v\|_{H^1}$, $v \longmapsto \|v_x\|$ and $v \longmapsto \|v\|_a = \sqrt{a(v,v)}$ on V are equivalent norms.

We have the following lemmas, the proofs of which are straightforward hence we omit the details.

Lemma 2.1. *The imbedding $H^1 \hookrightarrow C^0(\overline{\Omega})$ is compact and*

$$\|v\|_{C^0(\overline{\Omega})} \le \sqrt{2} \|v\|_{H^1} \ \text{for all} \ v \in H^1. \tag{2.4}$$

Lemma 2.2. *Let $\zeta \ge 0$. Then the imbedding $V \hookrightarrow C^0(\overline{\Omega})$ is compact and*

$$\begin{cases} \|v\|_{C^0(\overline{\Omega})} \le \|v_x\| \le \|v\|_a, \\ \frac{1}{\sqrt{2}} \|v\|_{H^1} \le \|v_x\| \le \|v\|_a \le \sqrt{1+\zeta} \|v_x\| \le \sqrt{1+\zeta} \|v\|_{H^1}, \end{cases} \tag{2.5}$$

for all $v \in V$.

Lemma 2.3. *Let $\zeta \ge 0$. Then the symmetric bilinear form $a(\cdot,\cdot)$ defined by* (2.3) *is continuous on $V \times V$ and coercive on V.*

Lemma 2.4. *Let $\zeta \ge 0$. Then there exists a Hilbert orthonormal base $\{w_j\}$ of L^2 consisting of eigenfunctions w_j corresponding to eigenvalues λ_j such that*

$$\begin{cases} 0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \le \cdots, \ \lim_{j \to +\infty} \lambda_j = +\infty, \\ a(w_j, v) = \lambda_j \langle w_j, v \rangle \ \text{for all} \ v \in V, \ j = 1, 2, \cdots. \end{cases} \tag{2.6}$$

Furthermore, the sequence $\{w_j/\sqrt{\lambda_j}\}$ is also a Hilbert orthonormal base of V with respect to the scalar product $a(\cdot,\cdot)$ defined by (2.3).

On the other hand, w_j satisfies the following boundary value problem

$$\begin{cases} -\Delta w_j = \lambda_j w_j, \ \text{in} \ (0,1), \\ w_{jx}(0) - \zeta w_j(0) = w_j(1) = 0, \ w_j \in C^\infty(\overline{\Omega}). \end{cases} \tag{2.7}$$

The proof of Lemma 2.4 can be found in ([13], p.87, Theorem 7.7), with $H = L^2$ and V, $a(\cdot,\cdot)$ as defined by (2.2), (2.3).

Definition 2.5. *A weak solution of the initial-boundary value problem (1.1) is a function $u \in \tilde{V}_T = \{v \in L^\infty(0,T; H^2 \cap V) : v' \in L^\infty(0,T; H^2 \cap V),\ v'' \in L^\infty(0,T; L^2) \cap L^2(0,T; V)\}$, such that u satisfies the following variational equation*

$$\langle u''(t), w \rangle + \lambda a(u'(t), w) + \mu[u](t) a(u(t), w) = \langle f[u](t), w \rangle, \tag{2.8}$$

for all $w \in V$, *a.e.,* $t \in (0,T)$, *together with the initial conditions*

$$u(0) = \tilde{u}_0, \ u'(0) = \tilde{u}_1, \tag{2.9}$$

where

$$\begin{aligned} f[u](x,t) &= f\left(x,t,u(x,t),u_x(x,t),u'(x,t),(S_n u)(t)\right), \\ \mu(t) &= \mu[u](t) = 1 + (\hat{S}_n u)(t), \\ (S_n u)(t) &= \tfrac{1}{n}\sum\nolimits_{i=1}^{n} u\left(\tfrac{i-1}{n},t\right), \\ (\hat{S}_n u)(t) &= (S_n u_x^2)(t) = \tfrac{1}{n}\sum\nolimits_{i=1}^{n} u_x^2(\tfrac{i-1}{n},t). \end{aligned} \tag{2.10}$$

3. Existence and uniqueness

In this section, we shall prove the existence and uniqueness of solutions of the problem (P_n) (1.1)-(1.2). It is necessary to make the following assumptions:

(H_1) $\tilde{u}_0, \tilde{u}_1 \in V \cap H^2$, $\tilde{u}_{0x}(0) - \zeta\tilde{u}_0(0) = 0$;

(H_2) $f \in C^1([0,1]\times[0,T^*]\times\mathbb{R}^4)$ such that
$f(1,t,0,y_2,0,y_4) = 0$ for all $t \in [0,T^*]$, $\forall\,(y_2,y_4) \in \mathbb{R}^2$.

For each $M > 0$ given, we set the constant $K_M(f)$ as follows

$$K_M(f) = \|f\|_{C^1(\bar{A}_M)} = \|f\|_{C^0(\bar{A}_M)} + \sum\nolimits_{i=1}^{6} \|D_i f\|_{C^0(\bar{A}_M)},$$

where

$$\begin{cases} \|f\|_{C^0(\bar{A}_M)} = \sup\limits_{(x,t,y_1,\cdots,y_4)\in\bar{A}_M} |f(x,t,y_1,\cdots,y_4)|, \\ \bar{A}_M = [0,1]\times[0,T^*]\times[-M,M]\times[-\sqrt{2}M,\sqrt{2}M]\times[-M,M]^2. \end{cases}$$

For every $T \in (0,T^*]$, we put

$$V_T = \{v \in L^\infty(0,T;H^2\cap V) : v' \in L^\infty(0,T;H^2\cap V),\ v'' \in L^2(0,T;V)\}$$

then V_T is a Banach space with respect to the following norm (see Lions [7])

$$\|v\|_{V_T} = \max\left\{\|v\|_{L^\infty(0,T;H^2\cap V)},\ \|v'\|_{L^\infty(0,T;H^2\cap V)},\ \|v''\|_{L^2(0,T;V)}\right\}.$$

For every $M > 0$, we put

$$\begin{aligned} W(M,T) &= \{v \in V_T : \|v\|_{V_T} \le M\}, \\ W_1(M,T) &= \{v \in W(M,T) : v'' \in L^\infty(0,T;L^2)\}. \end{aligned}$$

Now, we construct a recurrent sequence $\{u_m\}$ which is established by choosing the first term $u_0 \equiv \tilde{u}_0$, and suppose that

$$u_{m-1} \in W_1(M,T). \tag{3.1}$$

Then, we associate (1.1)-(1.2) with the following problem.

Find $u_m \in W(M,T)$ $(m \ge 1)$ satisfying the linear variational problem

$$\begin{cases} \langle u_m''(t),w\rangle + \lambda a(u_m'(t),w) + \mu_m(t)a(u_m(t),w) = \langle F_m(t),w\rangle,\ \forall w \in V, \\ u_m(0) = \tilde{u}_0,\ u_m'(0) = \tilde{u}_1, \end{cases} \tag{3.2}$$

where

$$\begin{aligned} F_m(x,t) &= f[u_{m-1}](x,t) \\ &= f\left(x,t,u_{m-1}(x,t),\nabla u_{m-1}(x,t),u_{m-1}'(x,t),(S_n u_{m-1})(t)\right), \\ (S_n u_{m-1})(t) &= \frac{1}{n}\sum\nolimits_{i=1}^{n} u_{m-1}\left(\tfrac{i-1}{n},t\right), \\ \mu_m(t) &= 1 + (\hat{S}_n u_{m-1})(t) = 1 + \frac{1}{n}\sum\nolimits_{i=1}^{n} \left|\nabla u_{m-1}\left(\tfrac{i-1}{n},t\right)\right|^2. \end{aligned} \tag{3.3}$$

Then, we have the following theorem.

Theorem 3.1. *Let* (H_1), (H_2) *hold. Then there are positive constants* $M, T > 0$ *such that, for* $u_0 \equiv \tilde{u}_0$, *there exists the recurrent sequence* $\{u_m\} \subset W(M,T)$ *defined by* (3.1)-(3.3).

Proof. The proof consists of several steps.

Step 1. The Faedo-Galerkin approximation (introduced by Lions [7]). Consider the basis $\{w_j\}$ for L^2 as in Lemma 2.4. Put

$$u_m^{(k)}(t) = \sum\nolimits_{j=1}^{k} c_{mj}^{(k)}(t) w_j, \tag{3.4}$$

where the coefficients $c_{mj}^{(k)}(t)$, $j = 1, \cdots, k$ satisfy the system of linear differential equations as follows

$$\begin{cases} \langle \ddot{u}_m^{(k)}(t), w_j \rangle + \lambda a(\dot{u}_m^{(k)}(t), w) + \mu_m(t) a(u_m^{(k)}(t), w_j) \\ \qquad\qquad\qquad\qquad = \langle F_m(t), w_j \rangle, \; 1 \le j \le k, \\ u_m^{(k)}(0) = \tilde{u}_{0k}, \; \dot{u}_m^{(k)}(0) = \tilde{u}_{1k}, \end{cases} \tag{3.5}$$

where $F_m(x,t)$ is defined as in (3.3) and

$$\begin{cases} \tilde{u}_{0k} = \sum_{j=1}^{k} \alpha_j^{(k)} w_j \to \tilde{u}_0 \text{ strongly in } H^2 \cap V, \\ \tilde{u}_{1k} = \sum_{j=1}^{k} \beta_j^{(k)} w_j \to \tilde{u}_1 \text{ strongly in } H^2 \cap V. \end{cases} \tag{3.6}$$

After integrating, it can see that, the system (3.5) is equivalent to the system of linear intergal equations

$$c_{mj}^{(k)}(t) = G_{mj}^{(k)}(t) + L_{mj}[c_m^{(k)}](t), \; 1 \le j \le k, \tag{3.7}$$

where

$$\begin{aligned} G_{mj}^{(k)}(t) &= \alpha_j^{(k)} + \frac{\beta_j^{(k)}}{\lambda \lambda_j} \left(1 - e^{-\lambda \lambda_j t}\right) \\ &\quad + \frac{1}{\lambda \lambda_j} \int_0^t \left(1 - e^{-\lambda \lambda_j (t-s)}\right) \langle F_m(s), w_j \rangle \, ds, \\ L_{mj}[c_m^{(k)}](t) &= -\frac{1}{\lambda} \int_0^t \left(1 - e^{-\lambda \lambda_j (t-s)}\right) \mu_m(s) c_{mj}^{(k)}(s) ds, \; 1 \le j \le k, \\ c_m^{(k)} &= \left(c_{m1}^{(k)}, \cdots, c_{mk}^{(k)}\right). \end{aligned} \tag{3.8}$$

Omitting the indexs m, k, the system (3.7)-(3.8) is written as follows

$$c(t) = U[c], \tag{3.9}$$

where

$$\begin{aligned} c(t) &= (c_1(t), \cdots, c_k(t)), \\ U[c](t) &= (U_1[c](t), \cdots, U_k[c](t)), \\ U_j[c](t) &= G_j(t) + L_j[c](t), \; 1 \le j \le k, \\ G_j(t) &= \alpha_j^{(k)} + \frac{\beta_j^{(k)}}{\lambda \lambda_j} \left(1 - e^{-\lambda \lambda_j t}\right) \\ &\quad + \frac{1}{\lambda \lambda_j} \int_0^t \left(1 - e^{-\lambda \lambda_j (t-s)}\right) \langle F_m(s), w_j \rangle \, ds, \\ L_j[c](t) &= -\frac{1}{\lambda} \int_0^t \left(1 - e^{-\lambda \lambda_j (t-s)}\right) \mu_m(s) c_j(s) ds, \; 1 \le j \le k. \end{aligned} \tag{3.10}$$

Applying the contraction principle, we can prove that the equation (3.9) has a unique solution $c(t)$ in $[0, T]$.

Indeed, let $\gamma > \frac{1}{\lambda}\sup_{0\le s\le T}\mu_m(s)$, it is known that $X = C^0\left([0,T];\mathbb{R}^k\right)$ is a Banach space with respect to the following norm

$$\|c\|_{\gamma,X} = \sup_{0\le t\le T} e^{-\gamma t}\left|c(t)\right|_1, \ \ |c(t)|_1 = \sum\nolimits_{j=1}^{k}|c_j(t)|, \ c\in X.$$

Clearly, $U : X \to X$. We will prove that $U : X \to X$ is contractive as follows.

For all $c = (c_1,\cdots,c_k)$, $d = (d_1,\cdots,d_k)\in X$, $q = c-d$, and (3.10), we have the following estimate

$$\begin{aligned} |(Uc)(t)-(Ud)(t)|_1 &\le \frac{1}{\lambda}\sum\nolimits_{j=1}^{k}\int_0^t\left(1-e^{-\lambda\lambda_j(t-s)}\right)\mu_m(s)\,|q_j(s)|\,ds \\ &\le \frac{1}{\lambda}\int_0^t\mu_m(s)\,|q(s)|_1\,ds \le \frac{1}{\lambda\gamma}\sup_{0\le s\le T}\mu_m(s)e^{\gamma t}\,\|q\|_{\gamma,X} \\ &= \frac{1}{\lambda\gamma}\sup_{0\le s\le T}\mu_m(s)e^{\gamma t}\,\|c-d\|_{\gamma,X}. \end{aligned}$$

It follows that

$$e^{-\gamma t}\,|(Uc)(t)-(Ud)(t)|_1 \le \frac{1}{\lambda\gamma}\sup_{0\le s\le T}\mu_m(s)\,\|c-d\|_{\gamma,X},$$

this leads to

$$\|Uc-Ud\|_{\gamma,X} \le \frac{1}{\lambda\gamma}\sup_{0\le s\le T}\mu_m(s)\,\|c-d\|_{\gamma,X}. \tag{3.11}$$

By $0 < \frac{1}{\lambda\gamma}\sup_{0\le s\le T}\mu_m(s) < 1$ and (3.11), we deduce that $U : X \to X$ is a contractive map. Then, the equation (3.9) has a unique solution $c\in X$. Thus, the system (3.5) has a unique solution $u_m^{(k)}(t)$ in $[0,T]$.

Step 2. Priori estimation. Put

$$\begin{aligned} S_m^{(k)}(t) &= \left\|\dot u_m^{(k)}(t)\right\|^2 + \left\|\dot u_m^{(k)}(t)\right\|_a^2 \\ &+ \mu_m(t)\left(\left\|u_m^{(k)}(t)\right\|_a^2 + \left\|\Delta u_m^{(k)}(t)\right\|^2\right) + \lambda\left\|\Delta\dot u_m^{(k)}(t)\right\|^2 \\ &+ 2\lambda\int_0^t\left(\left\|\dot u_m^{(k)}(s)\right\|_a^2 + \left\|\Delta u_m^{(k)}(s)\right\|^2\right)ds + 2\int_0^t\left\|\ddot u_m^{(k)}(s)\right\|_a^2 ds, \end{aligned} \tag{3.12}$$

then we deduce from (3.5) that

$$\begin{aligned} S_m^{(k)}(t) &= S_m^{(k)}(0) + 2\mu_m(0)\langle\triangle\tilde u_{0k},\triangle\tilde u_{1k}\rangle + 2\int_0^t\mu_m(s)\left\|\Delta\dot u_m^{(k)}(s)\right\|^2 ds \\ &+ \int_0^t\mu_m'(s)\left(\left\|u_m^{(k)}(s)\right\|_a^2 + \left\|\Delta u_m^{(k)}(s)\right\|^2 + 2\langle\Delta u_m^{(k)}(s),\Delta\dot u_m^{(k)}(s)\rangle\right)ds \\ &+ 2\int_0^t\left[\langle F_m(s),\dot u_m^{(k)}(s)\rangle + a\left(F_m(s),\dot u_m^{(k)}(s)\right)\right]ds \\ &+ 2\int_0^t a\left(F_m(s),\ddot u_m^{(k)}(s)\right)ds - 2\mu_m(t)\langle\Delta u_m^{(k)}(t),\Delta\dot u_m^{(k)}(t)\rangle \\ &= S_m^{(k)}(0) + 2\mu_m(0)\langle\triangle\tilde u_{0k},\triangle\tilde u_{1k}\rangle + I_1+\cdots+I_5. \end{aligned} \tag{3.13}$$

We shall estimate the terms $I_1,\cdots,I_4$ on the right-hand side of (3.13) as follows.

Note that

$$\begin{aligned} 1 &\le \mu_m(t) = 1 + \frac{1}{n}\sum\nolimits_{i=1}^{n}\left|\nabla u_{m-1}\left(\tfrac{i-1}{n},t\right)\right|^2 \\ &\le 1 + 2\left\|\nabla u_{m-1}(t)\right\|_{H^1}^2 \le 1 + 2M^2, \end{aligned}$$

$$\mu_m'(t) = \frac{2}{n}\sum_{i=1}^{n} \nabla u_{m-1}\left(\tfrac{i-1}{n}, t\right) \nabla u_{m-1}'\left(\tfrac{i-1}{n}, t\right),$$

we deduce from (3.1) that

$$\begin{aligned} 1 \leq\ & \mu_m(t) \leq 1 + 2\left\|\nabla u_{m-1}(t)\right\|_{H^1}^2 \leq 1 + 2M^2, \\ \left|\mu_m'(t)\right| \leq\ & 4\left\|\nabla u_{m-1}(t)\right\|_{H^1}\left\|\nabla u_{m-1}'(t)\right\|_{H^1} \leq 4M^2. \end{aligned} \tag{3.14}$$

By the estimates (3.14) and the following inequalities

$$\begin{aligned} S_m^{(k)}(t) &\geq \lambda\left\|\Delta\dot{u}_m^{(k)}(t)\right\|^2, \\ S_m^{(k)}(t) &\geq \left\|u_m^{(k)}(t)\right\|_a^2 + \left\|\Delta u_m^{(k)}(t)\right\|^2, \\ S_m^{(k)}(t) &\geq \left\|\Delta u_m^{(k)}(t)\right\|^2 + \lambda\left\|\Delta\dot{u}_m^{(k)}(t)\right\|^2 \geq 2\sqrt{\lambda}\left\|\Delta u_m^{(k)}(t)\right\|\left\|\Delta\dot{u}_m^{(k)}(t)\right\|, \end{aligned}$$

the integrals I_1, I_2 are estimated as follows

$$\begin{aligned} I_1 &= 2\int_0^t \mu_m(s)\left\|\Delta\dot{u}_m^{(k)}(s)\right\|^2 ds \leq \frac{2\left(1+2M^2\right)}{\lambda}\int_0^t S_m^{(k)}(s)ds, \\ I_2 &= \int_0^t \mu_m'(s)\left(\left\|u_m^{(k)}(s)\right\|_a^2 + \left\|\Delta u_m^{(k)}(s)\right\|^2 + 2\langle\Delta u_m^{(k)}(s), \Delta\dot{u}_m^{(k)}(s)\rangle\right) ds \\ &\leq 4M^2\left(1 + \frac{1}{\sqrt{\lambda}}\right)\int_0^t S_m^{(k)}(s)ds. \end{aligned} \tag{3.15}$$

On the other hand, we also have

$$\begin{aligned} \left|(S_n u_{m-1})(t)\right| &\leq \frac{1}{n}\sum_{i=1}^{n}\left|u_{m-1}\left(\tfrac{i-1}{n}, t\right)\right| \leq \frac{1}{n}\sum_{i=1}^{n}\left\|\nabla u_{m-1}(t)\right\| \\ &\leq \left\|u_{m-1}\right\|_{L^\infty(0,T;V)} \leq M, \\ F_{mx}(x,t) &= D_1 f[u_{m-1}](x,t) + D_3 f[u_{m-1}](x,t)\nabla u_{m-1}(x,t) \\ &+ D_4 f[u_{m-1}](x,t)\Delta u_{m-1}(x,t) + D_5 f[u_{m-1}](x,t)\nabla u_{m-1}'(x,t). \end{aligned}$$

Then, we estimate $|F_m(x,t)|$, $\|F_{mx}(t)\|$, $\|F_m(t)\|_a$ and I_3, I_4 on the right hand side of (3.13) as follows

$$\begin{aligned} |F_m(x,t)| \leq\ & K_M(f), \ \ \|F_{mx}(t)\| \leq K_M(f)\left(1+3M\right), \\ \|F_m(t)\|_a \leq\ & \sqrt{1+\zeta}\,\|F_{mx}(t)\| \leq K_M(f)\left(1+3M\right)\sqrt{1+\zeta}, \end{aligned} \tag{3.16}$$

$$\begin{aligned} I_3 &= 2\int_0^t \left[\langle F_m(s), \dot{u}_m^{(k)}(s)\rangle + a\left(F_m(s), \dot{u}_m^{(k)}(s)\right)\right] ds \\ &\leq 2\int_0^t \left(\|F_m(s)\|^2 + \|F_m(s)\|_a^2\right)^{1/2}\left(\left\|\dot{u}_m^{(k)}(s)\right\|^2 + \left\|\dot{u}_m^{(k)}(s)\right\|_a^2\right)^{1/2} ds \\ &\leq TK_M^2(f)\left[1 + (1+3M)^2(1+\zeta)\right] + \int_0^t S_m^{(k)}(s)ds, \\ I_4 &= 2\int_0^t a\left(F_m(s), \ddot{u}_m^{(k)}(s)\right) ds \leq 2\int_0^t \|F_m(s)\|_a^2\, ds + \frac{1}{2}\int_0^t \left\|\ddot{u}_m^{(k)}(s)\right\|_a^2 ds \\ &\leq 2TK_M^2(f)\left(1+3M\right)^2(1+\zeta) + \frac{1}{4}S_m^{(k)}(t). \end{aligned} \tag{3.17}$$

We estimate I_5 as below.

$$I_5 = -2\mu_m(t)\langle\Delta u_m^{(k)}(t), \Delta\dot{u}_m^{(k)}(t)\rangle \leq \frac{1}{4}S_m^{(k)}(t) + \frac{4}{\lambda}\left\|\mu_m(t)\Delta u_m^{(k)}(t)\right\|^2. \tag{3.18}$$

Due to

$$\left\|\mu_m(t)\Delta u_m^{(k)}(t)\right\|^2 \leq \left(\|\mu_m(0)\Delta\tilde{u}_0\| + \int_0^t \left\|\frac{\partial}{\partial s}\left[\mu_m(s)\Delta u_m^{(k)}(s)\right]\right\| ds\right)^2,$$

and

$$\begin{aligned}\left\|\frac{\partial}{\partial t}\left[\mu_m(t)\Delta u_m^{(k)}(t)\right]\right\| &= \left\|\mu_m(t)\Delta\dot{u}_m^{(k)}(t) + \mu_m'(t)\Delta u_m^{(k)}(t)\right\| \\ &\leq \left(1+2M^2\right)\sqrt{\frac{S_m^{(k)}(t)}{\lambda}} + 4M^2\sqrt{S_m^{(k)}(t)} \\ &= \left(\frac{1+2M^2}{\sqrt{\lambda}} + 4M^2\right)\sqrt{S_m^{(k)}(t)},\end{aligned}$$

we deduce that

$$\begin{aligned}&\left\|\mu_m(t)\Delta u_m^{(k)}(t)\right\|^2 \\ &\leq \left(\|\mu_m(0)\Delta\tilde{u}_0\| + \int_0^t \left\|\frac{\partial}{\partial s}\left[\mu_m(s)\Delta u_m^{(k)}(s)\right]\right\| ds\right)^2 \\ &\leq 2\left|\mu_m(0)\right|^2\left\|\Delta\tilde{u}_0\right\|^2 + 2T^*\left(\frac{1+2M^2}{\sqrt{\lambda}} + 4M^2\right)^2\int_0^t S_m^{(k)}(s)ds.\end{aligned}$$

Therefore, I_5 is estimated as follows

$$\begin{aligned}I_5 \leq\, &\frac{1}{4}S_m^{(k)}(t) + \frac{8}{\lambda}\left|\mu_m(0)\right|^2\left\|\Delta\tilde{u}_0\right\|^2 \\ &+ \frac{8}{\lambda}T^*\left(\frac{1+2M^2}{\sqrt{\lambda}} + 4M^2\right)^2\int_0^t S_m^{(k)}(s)ds.\end{aligned} \tag{3.19}$$

Combining (3.15), (3.17) and (3.19), it derives from (3.13) that

$$S_m^{(k)}(t) \leq S_{0m}^{(k)} + TD_1(M) + D_2(M)\int_0^t S_m^{(k)}(s)ds, \tag{3.20}$$

where

$$\begin{aligned}S_{0m}^{(k)} &= 2S_m^{(k)}(0) + 4\mu_m(0)\langle\triangle\tilde{u}_{0k}, \triangle\tilde{u}_{1k}\rangle + \frac{16}{\lambda}\left|\mu_m(0)\right|^2\left\|\Delta\tilde{u}_0\right\|^2, \\ D_1(M) &= 2K_M^2(f)\left[1 + 3\left(1+3M\right)^2\left(1+\zeta\right)\right], \\ D_2(M) &= 2 + \frac{4\left(1+2M^2\right)}{\lambda} + 8M^2\left(1 + \frac{1}{\sqrt{\lambda}}\right) \\ &\quad + \frac{16}{\lambda}T^*\left(\frac{1+2M^2}{\sqrt{\lambda}} + 4M^2\right)^2.\end{aligned} \tag{3.21}$$

Estimate $S_{0m}^{(k)}$. We have

$$\begin{aligned}S_{0m}^{(k)} &= 2\left\|\tilde{u}_{1k}\right\|^2 + 2\left\|\tilde{u}_{1k}\right\|_a^2 + 2\mu_m(0)\left(\left\|\tilde{u}_{0k}\right\|_a^2 + \left\|\Delta\tilde{u}_{0k}\right\|^2\right) \\ &\quad + 2\lambda\left\|\Delta\tilde{u}_{1k}\right\|^2 + 4\mu_m(0)\langle\triangle\tilde{u}_{0k}, \triangle\tilde{u}_{1k}\rangle + \frac{16}{\lambda}\mu_m^2(0)\left\|\Delta\tilde{u}_0\right\|^2, \\ \mu_m(0) &= 1 + \frac{1}{n}\sum\nolimits_{i=1}^n \tilde{u}_{0x}^2\left(\tfrac{i-1}{n}\right) \leq 1 + 2\left\|\tilde{u}_{0x}\right\|_{H^1}^2.\end{aligned} \tag{3.22}$$

By (3.6), it follows from (3.22) that

$$S_{0m}^{(k)} \leq \frac{1}{2}M^2, \text{ for all } m,\ k, \tag{3.23}$$

where M is a constant depending only on λ, ζ, $\tilde{u}_0$, $\tilde{u}_1$.

We choose $T \in (0, T^*]$, such that

$$\left(\frac{1}{2}M^2 + TD_1(M)\right) e^{D_2(M)T} \leq M^2, \tag{3.24}$$

and

$$k_T = 2\left(2\sqrt{2} + \frac{1}{\lambda}\right)\sqrt{9K_M^2(f) + 16M^4}\sqrt{T}e^{4TM^2} < 1. \tag{3.25}$$

Finally, by using Gronwall's lemma, we obtain from (3.20), (3.23) and (3.24) that

$$S_m^{(k)}(t) \leq M^2 e^{-D_2(M)T} e^{D_2(M)t} \leq M^2, \tag{3.26}$$

for all $t \in [0, T]$, for all m and k.

Therefore, we have

$$u_m^{(k)} \in W(M, T), \text{ for all } m \text{ and } k \in \mathbb{N}. \tag{3.27}$$

Step 3. Limiting process. From (3.27), we deduce the existence of a subsequence of $\{u_m^{(k)}\}$ still so denoted by the same symbol such that

$$\begin{cases} u_m^{(k)} \to u_m & \text{in} \quad L^\infty(0, T; H^2 \cap V) \text{ weak*}, \\ \dot{u}_m^{(k)} \to u_m' & \text{in} \quad L^\infty(0, T; H^2 \cap V) \text{ weak*}, \\ \ddot{u}_m^{(k)} \to u_m'' & \text{in} \quad L^2(0, T; V) \text{ weak}, \\ u_m \in W(M, T). \end{cases} \tag{3.28}$$

Passing to limit in (3.5), we have u_m satisfying (3.2), (3.3) in $L^2(0, T)$ weak.

Furthermore, $(3.2)_1$ and $(3.28)_4$ imply that

$$u_m'' = \lambda \Delta u_m' + \mu_m(t)\Delta u_m + F_m \in L^\infty(0, T; L^2),$$

so we obtain $u_m \in W_1(M, T)$, Theorem 3.1 is proved. □

In next part, we introduce the space

$$H_T = \{v \in L^\infty(0, T; H^2 \cap V) : v' \in L^2(0, T; H^2 \cap V) \cap L^\infty(0, T; V)\}. \tag{3.29}$$

Note that H_T is a Banach space with respect to the norm (see Lions [7]).

$$\|v\|_{H_T} = \|v\|_{L^\infty(0,T;H^2\cap V)} + \|v'\|_{L^2(0,T;H^2\cap V)} + \|v'\|_{L^\infty(0,T;V)}. \tag{3.30}$$

We use the result given in Theorem 3.1 and the compact imbedding theorems to prove the existence and uniqueness of weak solution of (1.1)-(1.2). Hence, we get the main result in this section as follows.

Theorem 3.2. *Let* (H_1), (H_2) *hold. Then*

(i) Prob. (1.1)-(1.2) *has a unique weak solution* $u \in W_1(M, T)$, *where the constants* $M > 0$ *and* $T > 0$ *are chosen as in Theorem 3.1.*

(ii) The recurrent sequence $\{u_m\}$ *defined by* (3.1)-(3.3) *converges to the solution* u *of* (1.1)-(1.2) *strongly in* H_T.

Furthermore, we also have the estimation

$$\|u_m - u\|_{H_T} \leq C_T k_T^m, \text{ for all } m \in \mathbb{N}, \tag{3.31}$$

where the constant $k_T \in [0, 1)$ *is defined as in* (3.25) *and* C_T *is a constant depending only on* T, f, $\tilde{u}_0$, $\tilde{u}_1$ *and* k_T.

Proof of Theorem 3.2. (a) *Existence of solutions.*

We shall prove that $\{u_m\}$ is a Cauchy sequence in H_T. Let $w_m = u_{m+1} - u_m$. Then w_m satisfies the variational problem

$$\begin{cases} \langle w_m''(t), w\rangle + \lambda a(w_m'(t), w) + \mu_{m+1}(t) a(w_m(t), w) \\ = \langle F_{m+1}(t) - F_m(t) + (\mu_{m+1}(t) - \mu_m(t))\, \Delta u_m(t), w\rangle\,,\ \forall w \in V, \\ w_m(0) = w_m'(0) = 0. \end{cases} \tag{3.32}$$

Taking $w = w_m'$ in $(3.32)_1$ and integrating in t, we get

$$\begin{aligned} X_m(t) = {} & 2\int_0^t \left\langle F_{m+1}(s) - F_m(s), w_m'(s)\right\rangle ds + \int_0^t \mu_{m+1}'(s)\,\|w_m(s)\|_a^2\, ds \\ & + 2\int_0^t (\mu_{m+1}(s) - \mu_m(s))\,\langle \Delta u_m(s), w_m'(s) ds, \end{aligned} \tag{3.33}$$

where

$$X_m(t) = \left\|w_m'(t)\right\|^2 + \mu_{m+1}(t)\,\|w_m(t)\|_a^2 + 2\lambda\int_0^t \left\|w_m'(s)\right\|_a^2 ds. \tag{3.34}$$

Now, we require the following lemma.

Lemma 3.3. *Let $u \in \tilde{V}_T$ (as in Definition 2.5) be a weak solution of the following problem*

$$\begin{cases} u'' - \lambda u_{xx}' - \mu(t) u_{xx} = F(x,t),\ 0 < x < 1,\ 0 < t < T, \\ u_x(0,t) - \zeta u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \\ \tilde{u}_0,\ \tilde{u}_1 \in V \cap H^2,\ \tilde{u}_{0x}(0) - \zeta \tilde{u}_0(0) = 0, \\ F \in L^2(0,T;V),\ \mu \in H^1(0,T),\ \mu(t) \geq \mu_* > 0. \end{cases} \tag{3.35}$$

Then, we have

$$\begin{aligned} \frac{1}{2}\,\|u'(t)\|_a^2 + \frac{1}{2}\mu(t)\,\|\Delta u(t)\|^2 + \lambda \textstyle\int_0^t \|\Delta u'(s)\|^2 ds \\ \geq \frac{1}{2}\,\|\tilde{u}_1\|_a^2 + \frac{1}{2}\mu(0)\,\|\Delta \tilde{u}_0\|^2 + \frac{1}{2}\textstyle\int_0^t \mu'(s)\,\|\Delta u(s)\|^2 ds \\ + \textstyle\int_0^t \langle F(s), -\Delta u'(s)\rangle\, ds,\ \text{a.e. } t \in [0,T]. \end{aligned} \tag{3.36}$$

Furthermore, if $\tilde{u}_0 = \tilde{u}_1 = 0$, *then there is an equality in* (3.36).

Proof of Lemma 3.3. The idea of the proof is the same as in [[7], Lemma 2.1, p. 79]. Fix t_1, t_2, $0 < t_1 < t_2 < T$ and let $w_{km}(x,t)$ be a function defined as follows

$$w_{km}(x,t) = \left[\left(\theta_m(t)\Delta u'(x,t)\right) * \rho_k(t) * \rho_k(t)\right]\theta_m(t), \tag{3.37}$$

where

(i) θ_m is a continuous, piecewise linear function on $[0,T]$ defined by

$$\theta_m(t) = \begin{cases} 0, & t \notin [t_1 + 1/m,\ t_2 - 1/m], \\ 1, & t \in [t_1 + 2/m,\ t_2 - 2/m], \\ m(t - t_1 - 1/m), & t \in [t_1 + 1/m,\ t_1 + 2/m], \\ -m(t - t_2 + 1/m), & t \in [t_2 - 2/m,\ t_2 - 1/m]; \end{cases} \tag{3.38}$$

(ii) $\{\rho_k\}$ is a regularized sequence in $C_c^\infty(\mathbb{R})$, i.e.,

$$\rho_k \in C_c^\infty(\mathbb{R}),\ \operatorname{supp}\rho_k \subset [-1/k, 1/k],\ \rho_k(-t) = \rho_k(t),\ \int_{-\infty}^{\infty}\rho_k(t)dt = 1; \tag{3.39}$$

(iii) $(*)$ is a convolution product in time variable, i.e.,

$$(u * \rho_k)(x,t) = \int_{-\infty}^{\infty} u(x, t-s)\rho_k(s) ds. \tag{3.40}$$

Taking the scalar product of the function $w_{km}(x,t)$ in $(3.35)_1$, and then integrating with respect to t from 0 to T, we have

$$A_{km} + B_{km} + C_{km} = D_{km}, \tag{3.41}$$

where

$$\begin{aligned} A_{km} &= \int_0^T \langle u''(t), w_{km}(t)\rangle dt, \\ B_{km} &= \lambda \int_0^T a(u'(t), w_{km}(t))dt, \\ C_{km} &= \int_0^T \mu(t) a(u(t), w_{km}(t))dt, \\ D_{km} &= \int_0^T \langle F(t), w_{km}(t)\rangle dt. \end{aligned} \tag{3.42}$$

By using the properties of the functions $\theta_m(t)$ and $\rho_k(t)$, and making some lengthy calculations, we have that

$$\begin{aligned} \lim_{k\to\infty} A_{km} &= \int_0^T \theta_m(t)\theta_m'(t) \left\|u'(t)\right\|_a^2 dt, \\ \lim_{k\to\infty} B_{km} &= -\lambda \int_0^T \theta_m^2(t) \left\|\Delta u'(t)\right\|^2 dt, \\ \lim_{k\to\infty} C_{km} &= \int_0^T \theta_m(t)\theta_m'(t)\mu(t) \left\|\Delta u(t)\right\|^2 dt + \frac{1}{2}\int_0^T \theta_m^2(t)\mu'(t) \left\|\Delta u(t)\right\|^2 dt, \\ \lim_{k\to\infty} D_{km} &= \int_0^T \theta_m^2(t)\langle F(t), \Delta u'(t)\rangle dt. \end{aligned} \tag{3.43}$$

Letting $m \to \infty$, we obtain from (3.41)-(3.43) that

$$\begin{aligned} &\frac{1}{2}\left\|u'(t_1)\right\|_a^2 - \frac{1}{2}\left\|u'(t_2)\right\|_a^2 - \lambda\int_{t_1}^{t_2} \left\|\Delta u'(t)\right\|^2 dt \\ &+\frac{1}{2}\mu(t_1)\left\|\Delta u(t_1)\right\|^2 - \frac{1}{2}\mu(t_2)\left\|\Delta u(t_2)\right\|^2 + \frac{1}{2}\int_{t_1}^{t_2} \mu'(t)\left\|\Delta u(t)\right\|^2 dt \\ = &\int_{t_1}^{t_2} \langle F(t), \Delta u'(t)\rangle dt, \text{ a.e., } t_1, t_2 \in (0,T),\ t_1 < t_2 < T, \end{aligned}$$

or

$$\begin{aligned} &\frac{1}{2}\left\|u'(t_2)\right\|_a^2 + \frac{1}{2}\mu(t_2)\left\|\Delta u(t_2)\right\|^2 + \lambda\int_0^{t_2} \left\|\Delta u'(s)\right\|^2 ds \\ &\quad -\frac{1}{2}\int_0^{t_2} \mu'(s)\left\|\Delta u(s)\right\|^2 ds + \int_0^{t_2}\langle F(s), \Delta u'(s)\rangle ds \\ = &\frac{1}{2}\left\|u'(t_1)\right\|_a^2 + \frac{1}{2}\mu(t_1)\left\|\Delta u(t_1)\right\|^2 + \lambda\int_0^{t_1} \left\|\Delta u'(s)\right\|^2 ds \\ &\quad -\frac{1}{2}\int_0^{t_1} \mu'(s)\left\|\Delta u(s)\right\|^2 ds + \int_0^{t_1}\langle F(s), \Delta u'(s)\rangle ds, \end{aligned} \tag{3.44}$$

a.e., $t_1, t_2 \in (0,T)$, $t_1 < t_2 < T$.

From (3.44), by taking $t_2 = t$ and passing to the limit as $t_1 \to 0_+$ and using the property of weak lower semicontinuity of the functional $v \longmapsto \|v\|^2$, we obtain (3.36).

To get the equality in (3.36), we extend u, F by 0 and μ by $\mu(0)$, respectively as $t < 0$. Moreover, we note that the equality (3.44) is true for almost $t_1 < t_2 < T$. Hence, by taking $t_1 < 0$, the integrals on the right-hand side of (3.44) is 0. Then, by letting $t_1 \to 0_-$ and using $\tilde{u}_0 = \tilde{u}_1 = 0$, we have the equality in (3.36).

The proof of Lemma 3.3 is completed. □

Remark 3.1. Lemma 3.3 is a relative generalization of a lemma given in Lions's book [[7], Lemma 6.1, p. 224].

Note that $w_m = u_{m+1} - u_m \in \tilde{V}_T$ be the weak solution of the problem (3.35) coresponding to $\tilde{u}_0 = \tilde{u}_1 = 0$, $\mu(t) = \mu_{m+1}(t)$,

$$F(t) = [\mu_{m+1}(t) - \mu_m(t)]\,\Delta u_m + F_{m+1}(t) - F_m(t).$$

By using Lemma 3.3 with $\tilde{u}_0 = \tilde{u}_1 = 0$, we have

$$\begin{aligned}
&\frac{1}{2}\left\|w_m'(t)\right\|_a^2 + \frac{1}{2}\mu_{m+1}(t)\left\|\Delta w_m(t)\right\|^2 + \lambda\int_0^t \left\|\Delta w_m'(s)\right\|^2 ds \\
&= \frac{1}{2}\int_0^t \mu_{m+1}'(s)\left\|\Delta w_m(s)\right\|^2 ds \\
&+ \int_0^t \left\langle [\mu_{m+1}(s) - \mu_m(s)]\,\Delta u_m(s) + F_{m+1}(s) - F_m(s), -\Delta w_m'(s)\right\rangle ds.
\end{aligned} \tag{3.45}$$

Put

$$Y_m(t) = \left\|w_m'(t)\right\|_a^2 + \mu_{m+1}(t)\left\|\Delta w_m(t)\right\|^2 + 2\lambda\int_0^t \left\|\Delta w_m'(s)\right\|^2 ds, \tag{3.46}$$

we have

$$\begin{aligned}
Y_m(t) &= \int_0^t \mu_{m+1}'(s)\left\|\Delta w_m(s)\right\|^2 ds \\
&+ 2\int_0^t [\mu_{m+1}(s) - \mu_m(s)]\left\langle \Delta u_m(s), -\Delta w_m'(s)\right\rangle ds \\
&+ 2\int_0^t \left\langle F_{m+1}(s) - F_m(s), -\Delta w_m'(s)\right\rangle ds.
\end{aligned} \tag{3.47}$$

It follows from (3.33), (3.34), (3.46) and (3.47) that

$$\begin{aligned}
S_m(t) &= \int_0^t \mu_{m+1}'(s)\left(\left\|w_m(s)\right\|_a^2 + \left\|\Delta w_m(s)\right\|^2\right) ds \\
&+ 2\int_0^t \left\langle F_{m+1}(s) - F_m(s), w_m'(s) - \Delta w_m'(s)\right\rangle ds \\
&+ 2\int_0^t (\mu_{m+1}(s) - \mu_m(s))\left\langle \Delta u_m(s), w_m'(s) - \Delta w_m'(s)\right\rangle ds \\
&= J_1 + J_2 + J_3,
\end{aligned} \tag{3.48}$$

where

$$\begin{aligned}
S_m(t) &= X_m(t) + Y_m(t) \\
&= \left\|w_m'(t)\right\|_a^2 + \left\|w_m'(t)\right\|^2 + \mu_{m+1}(t)\left(\left\|w_m(t)\right\|_a^2 + \left\|\Delta w_m(t)\right\|^2\right) \\
&+ 2\lambda\int_0^t \left(\left\|w_m'(s)\right\|_a^2 + \left\|\Delta w_m'(s)\right\|^2\right) ds.
\end{aligned} \tag{3.49}$$

We shall estimate the terms J_1, J_2, J_3 on the right-hand side of (3.48) as follows.

Estimate of J_1. Note that

$$\left|\mu_{m+1}'(t)\right| \le 4\left\|\nabla u_m(t)\right\|_{H^1}\left\|\nabla u_m'(t)\right\|_{H^1} \le 4M^2, \tag{3.50}$$

we deduce from (3.49) that

$$J_1 = \int_0^t \mu_{m+1}'(s)\left(\left\|w_m(s)\right\|_a^2 + \left\|\Delta w_m(s)\right\|^2\right) ds \le 4M^2\int_0^t Z_m(s)ds. \tag{3.51}$$

Estimate of J_2. By (H_2), it is clear that

$$\|F_{m+1}(t)-F_m(t)\| \le 3K_M(f)\left[\|\nabla w_{m-1}(t)\| + \left\|w'_{m-1}(t)\right\|\right] \le 3K_M(f)\,\|w_{m-1}\|_{H_T},$$

hence

$$\begin{aligned} J_2 &= 2\int_0^t \left\langle F_{m+1}(s)-F_m(s), w'_m(s)-\Delta w'_m(s)\right\rangle ds \\ &\le 6\sqrt{2}K_M(f)\,\|w_{m-1}\|_{H_T} \int_0^t \left(\left\|w'_m(s)\right\|_a^2 + \left\|\Delta w'_m(s)\right\|^2\right)^{1/2} ds \\ &\le \frac{\lambda}{2}\int_0^t \left(\left\|w'_m(s)\right\|_a^2 + \left\|\Delta w'_m(s)\right\|^2\right)^{1/2} ds + \frac{36}{\lambda}TK_M^2(f)\,\|w_{m-1}\|_{H_T}^2 \\ &\le \frac{1}{4}S_m(t) + \frac{36}{\lambda}TK_M^2(f)\,\|w_{m-1}\|_{H_T}^2. \end{aligned} \tag{3.52}$$

Estimate of J_3. We have

$$\begin{aligned} |\mu_{m+1}(t)-\mu_m(t)| &\le \frac{1}{n}\sum\nolimits_{i=1}^n \left|\left|\nabla u_m\left(\tfrac{i-1}{n},t\right)\right|^2 - \left|\nabla u_{m-1}\left(\tfrac{i-1}{n},t\right)\right|^2\right| \\ &\le \left(\|\nabla u_m(t)\|_{C^0([0,1])} + \|\nabla u_{m-1}(t)\|_{C^0([0,1])}\right)\|\nabla w_{m-1}(t)\|_{C^0([0,1])} \\ &\le 4M\left(\|\nabla w_{m-1}(t)\|^2 + \|\Delta w_{m-1}(t)\|^2\right)^{1/2} \\ &\le 4M\,\|w_{m-1}\|_{H_T}. \end{aligned} \tag{3.53}$$

Hence, J_3 is estimated as follows

$$\begin{aligned} J_3 &= 2\int_0^t (\mu_{m+1}(s)-\mu_m(s))\,\langle\Delta u_m(s), w'_m(s)-\Delta w'_m(s)\rangle ds \\ &\le 8\sqrt{2}M^2\,\|w_{m-1}\|_{H_T}\int_0^t \left(\left\|w'_m(s)\right\|_a^2 + \left\|\Delta w'_m(s)\right\|^2\right)^{1/2} ds \\ &\le \frac{\lambda}{2}\int_0^t \left(\left\|w'_m(s)\right\|_a^2 + \left\|\Delta w'_m(s)\right\|^2\right)^{1/2} ds + \frac{64}{\lambda}TM^4\,\|w_{m-1}\|_{H_T}^2 \\ &\le \frac{1}{4}S_m(t) + \frac{64}{\lambda}TM^4\,\|w_{m-1}\|_{H_T}^2. \end{aligned} \tag{3.54}$$

It derives from (3.48), (3.51), (3.52) and (3.54) that

$$S_m(t) \le \frac{8}{\lambda}T\left(9K_M^2(f)+16M^4\right)\|w_{m-1}\|_{H_T}^2 + 8M^2\int_0^t Z_m(s)ds. \tag{3.55}$$

Using Gronwall's lemma, we deduce from (3.55) that

$$\|w_m\|_{H_T} \le k_T\,\|w_{m-1}\|_{H_T} \quad \forall m\in\mathbb{N}, \tag{3.56}$$

where $k_T\in(0,1)$ is defined as in (3.25), which implies that

$$\|u_m - u_{m+p}\|_{H_T} \le \|u_0-u_1\|_{H_T}(1-k_T)^{-1}k_T^m \;\forall m,\ p\in\mathbb{N}. \tag{3.57}$$

It follows that $\{u_m\}$ is a Cauchy sequence in H_T. Then there exists $u\in H_T$ such that

$$u_m \to u \text{ strongly in } H_T. \tag{3.58}$$

Note that $u_m\in W(M,T)$, then there exists a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$\left\{\begin{array}{lll} u_{m_j}\to u & \text{in} & L^\infty(0,T;H^2\cap V) \text{ weak*}, \\ u'_{m_j}\to u' & \text{in} & L^\infty(0,T;H^2\cap V) \text{ weak*}, \\ u''_{m_j}\to u'' & \text{in} & L^2(0,T;V) \text{ weak}, \\ u\in W(M,T). & & \end{array}\right. \tag{3.59}$$

We also note that

$$\begin{aligned} \|F_m - f[u]\|_{L^\infty(0,T;L^2)} &\le 3K_M(f)\,\|u_{m-1} - u\|_{H_T}\,, \\ \|\mu_m - \mu[u]\|_{L^\infty(0,T)} &\le 4M\,\|u_{m-1} - u\|_{H_T}\,. \end{aligned} \tag{3.60}$$

Hence, from (3.58) and (3.60), we obtain

$$\begin{aligned} F_m &\to f[u] \text{ strongly in } L^\infty(0,T;L^2), \\ \mu_m &\to \mu[u] \text{ strongly in } L^\infty(0,T). \end{aligned} \tag{3.61}$$

Finally, passing to the limit in (3.2)-(3.3) as $m = m_j \to \infty$, it implies from (3.58), $(3.59)_3$ and (3.61) that there exists $u \in W(M,T)$ satisfying (2.8)-(2.10).

Furthermore, $(1.1)_1$ and $(3.59)_4$ imply that

$$u'' = \lambda\Delta u' + \mu[u](t)\Delta u + f[u] \in L^\infty(0,T;L^2),$$

so we obtain $u \in W_1(M,T)$. The existence proof is completed.

(b) *Uniqueness of solutions.*

Let u_1, $u_2 \in W_1(M,T)$ be two various weak solutions of Prob. (1.1)-(1.2). Then $u = u_1 - u_2 \in \tilde{V}_T$ be the weak solution of the problem (3.35) coresponding to $\tilde{u}_0 = \tilde{u}_1 = 0$, $\mu(t) = \bar{\mu}_1(t)$, $F(t) = [\bar{\mu}_1(t) - \bar{\mu}_2(t)]\,\Delta u_2 + \bar{F}_1(t) - \bar{F}_2(t)$, where

$$\begin{aligned} \bar{F}_i(x,t) &= f[u_i](x,t) = f\left(x,t,u_i(x,t),\nabla u_i(x,t),u_i'(x,t),(S_n u_i)(t)\right), \\ \bar{\mu}_i(t) &= \mu[u_i](t) = 1 + (\hat{S}_n u_i)(t),\ i = 1,2. \end{aligned} \tag{3.62}$$

Similarly, by using Lemma 3.3 with $\tilde{u}_0 = \tilde{u}_1 = 0$, we have

$$\begin{aligned} Z(t) &= \int_0^t \bar{\mu}_1'(s)\left(\|u(s)\|_a^2 + \|\Delta u(s)\|^2\right)ds \\ &\quad + 2\int_0^t \left\langle \bar{F}_1(s) - \bar{F}_2(s), u'(s) - \Delta u'(s)\right\rangle ds \\ &\quad + 2\int_0^t [\bar{\mu}_1(s) - \bar{\mu}_2(s)]\,\langle \Delta u_2(s), u'(s) - \Delta u'(s)\rangle ds, \end{aligned} \tag{3.63}$$

where

$$\begin{aligned} Z(t) &= \left\|u'(t)\right\|^2 + \left\|u'(t)\right\|_a^2 + \bar{\mu}_1(t)\left(\|u(t)\|_a^2 + \|\Delta u(t)\|^2\right) \\ &\quad + 2\lambda\int_0^t \left(\left\|u'(s)\right\|_a^2 + \left\|\Delta u'(s)\right\|^2\right)ds. \end{aligned} \tag{3.64}$$

Moreover, we also obtain the following estimate

$$Z(t) \le 8\left(M^2 + \frac{2}{\lambda}(9K_M^2(f) + 2M^4)\right)\int_0^t Z(s)ds. \tag{3.65}$$

Using Gronwall's lemma, it follows from (3.65) that $Z(t) \equiv 0$, ie., $u_1 \equiv u_2$.

Theorem 3.2 is proved completely. □

4. The convergence of solutions of (1.1)-(1.2) as $n \to \infty$

In this section, we shall consider the convergence of solutions of (P_n) to the solution of (P) $(1.1)_{2,3}$-(1.7) as $n \to \infty$ as follows.

For each n, (P_n) has a unique weak solution u^n, i.e. u^n satisfies the following problem

$$\begin{aligned} &\langle u_{tt}^n(t), w\rangle + \lambda a(u_t^n(t), w) + \left(1 + (\hat{S}_n u)(t)\right)a(u^n(t), w) \\ &= \left\langle f\left(\cdot, t, u^n(t), u_x^n(t), u_t^n(t), (S_n u^n)(t)\right), w\right\rangle, \end{aligned} \tag{4.1}$$

for all $w \in V$, a.e., $t \in (0,T)$, together with the initial conditions

$$u^n(0) = \tilde{u}_0, \ u_t^n(0) = \tilde{u}_1. \tag{4.2}$$

By Theorem 3.2, there exist positive constants M, T independing on n such that (P_n) has a unique weak solution u^n which satisfies

$$u^n \in W_1(M,T), \text{ for all } n \in \mathbb{N}. \tag{4.3}$$

From (4.3), we deduce that there exists a subsequence of $\{u^n\}$, used the same notation, such that

$$\begin{cases} u^n \to u & \text{in} \quad L^\infty(0,T;H^2 \cap V) \text{ weak}^*, \\ u_t^n \to u' & \text{in} \quad L^\infty(0,T;H^2 \cap V) \text{ weak}^*, \\ u_{tt}^n \to u'' & \text{in} \quad L^2(0,T;V) \text{ weak}^*. \end{cases} \tag{4.4}$$

Applying the lemma compact embeding of Lions [7], there exists a subsequence $\{u^n\}$, used the same symbol, such that

$$\begin{cases} u^n \to u & \text{in} \quad L^2(0,T;V) \text{ strongly}, \\ u_t^n \to u' & \text{in} \quad L^2(0,T;V) \text{ strongly}. \end{cases} \tag{4.5}$$

Because u^n is the unique weak solution of (P_n), so

$$\begin{aligned} &\int_0^T \langle u_{tt}^n(t), w\rangle \varphi(t)dt + \lambda \int_0^T a(u_t^n(t), w)\varphi(t)dt \\ &+ \int_0^T a(u^n(t), w)\varphi(t)dt + \int_0^T (\hat{S}_n u^n)(t) a(u^n(t), w)\varphi(t)dt \\ &= \int_0^T \langle f\left(t, u^n(t), u_x^n(t), u_t^n(t), (S_n u^n)(t)\right), w\rangle \varphi(t)dt, \end{aligned} \tag{4.6}$$

$\forall w \in V$, $\forall \varphi \in C_c^\infty(0,T)$.

By $(4.4)_3$ and $(4.5)_1$ we get

$$\begin{aligned} \int_0^T \langle u_{tt}^n(t), w\rangle \varphi(t)dt &\to \int_0^T \langle u''(t), w\rangle \varphi(t)dt, \\ \int_0^T a(u^n(t), w)\varphi(t)dt &\to \int_0^T a(u(t), w)\varphi(t)dt, \\ \lambda \int_0^T a(u_t^n(t), w)\varphi(t)dt &\to \lambda \int_0^T a(u'(t), w)\varphi(t)dt. \end{aligned} \tag{4.7}$$

We have to check the convergences

$$\begin{aligned} &\text{(i)} \ \ \int_0^T (\hat{S}_n u^n)(t) a(u^n(t), w)\varphi(t)dt \to \int_0^T \|u_x(t)\|^2 a(u(t), w)\varphi(t)dt, \\ &\text{(ii)} \ \ \int_0^T \langle f\left(t, u^n(t), u_x^n(t), u_t^n(t), (S_n u^n)(t)\right), w\rangle \varphi(t)dt \\ &\qquad \to \int_0^T \left\langle f\left(t, u(t), u_x(t), u'(t), \int_0^1 u(y,t)dy\right), w\right\rangle \varphi(t)dt. \end{aligned} \tag{4.8}$$

Then, we need the following lemmas.

Lemma 4.1. *The following convergences are confirmed*

$$\begin{aligned} &\text{(i)} \ \ \left\| S_n u - \int_0^1 u(y,\cdot)dy \right\|_{L^2(0,T)}^2 \\ &\qquad = \int_0^T \left| (S_n u)(t) - \int_0^1 u(y,t)dy \right|^2 dt \to 0, \ \text{as } n \to \infty, \\ &\text{(ii)} \ \ \left\| \hat{S}_n u - \int_0^1 u_x^2(y,\cdot)dy \right\|_{L^2(0,T)}^2 \\ &\qquad = \int_0^T \left| \hat{S}_n u(t) - \|u_x(t)\|^2 \right|^2 dt \to 0, \ \text{as } n \to \infty. \end{aligned} \tag{4.9}$$

Proof of Lemma 4.1.
Proof (i). We note that

$$\frac{1}{n}\sum_{i=1}^{n} g\left(\tfrac{i-1}{n}\right) \to \int_0^1 g(y)dy,\ \forall g \in C^0([0,1]). \tag{4.10}$$

Since $u \in L^\infty(0,T;V) \hookrightarrow L^\infty(0,T;C^0(\bar{\Omega}))$, so the function $y \longmapsto u(y,t)$, a.e. $t \in [0,T]$ belongs to $C^0(\bar{\Omega})$, then,

$$(S_n u)(t) = \frac{1}{n}\sum_{i=1}^{n} u\left(\tfrac{i-1}{n}, t\right) \to \int_0^1 u(y,t)dy,\ \text{as } n \to \infty. \tag{4.11}$$

Note that

$$\begin{aligned} |(S_n u)(t)| &\le \frac{1}{n}\sum_{i=1}^{n} \left|u\left(\tfrac{i-1}{n}, t\right)\right| \le \frac{1}{n}\sum_{i=1}^{n} \|u_x(t)\| \le M, \\ \left|\int_0^1 u(y,t)dy\right| &\le \|u_x(t)\| \le M, \end{aligned} \tag{4.12}$$

so

$$\left|(S_n u)(t) - \int_0^1 u(y,t)dy\right| \le 2M, \tag{4.13}$$

for all $n \in \mathbb{N}$ and a.e. $t \in [0,T]$. Applying the dominated convergence theorem, we deduce that (i) is valid.
Proof (ii). By $u \in L^\infty(0,T;H^2 \cap V)$, we have $u_x \in L^\infty(0,T;V) \hookrightarrow L^\infty(0,T;C^0(\bar{\Omega}))$. With the same argument as in proof of (i), we have

$$\left\|\hat{S}_n u - \int_0^1 u_x^2(y,\cdot)dy\right\|_{L^2(0,T)}^2 = \int_0^T \left|\hat{S}_n u(t) - \|u_x(t)\|^2\right|^2 dt \to 0,\ \textit{as } n \to \infty. \tag{4.14}$$

Lemma 4.1 is proved. □

Lemma 4.2: *The following convergences are confirmed*

$$\begin{aligned} &\text{(i)}\ \ \|S_n u^n - S_n u\|_{L^2(0,T)}^2 \to 0,\ \textit{as } n \to \infty, \\ &\text{(ii)}\ \ \left\|S_n u^n - \textstyle\int_0^1 u(y,\cdot)dy\right\|_{L^2(0,T)}^2 \\ &\qquad = \textstyle\int_0^T \left|S_n u^n(t) - \int_0^1 u(y,t)dy\right|^2 dt \to 0,\ \textit{as } n \to \infty. \end{aligned} \tag{4.15}$$

Proof of Lemma 4.2.
Proof (i). We note that

$$\begin{aligned} |S_n u^n(t) - S_n u(t)| &\le \frac{1}{n}\sum_{i=1}^{n} \left|u^n\left(\tfrac{i-1}{n}, t\right) - u\left(\tfrac{i-1}{n}, t\right)\right| \\ &\le \|u^n(t) - u(t)\|_{C^0([0,1])} \le \|u^n(t) - u(t)\|_V. \end{aligned} \tag{4.16}$$

By $(4.5)_1$, we deduce from (4.16) that

$$\|S_n u^n - S_n u\|_{L^2(0,T)} \le \|u^n - u\|_{L^2(0,T;V)} \to 0. \tag{4.17}$$

Proof (ii). It follows from Lemma 4.1 (i) and (4.17) that

$$\begin{aligned} &\left\|S_n u^n - \int_0^1 u(y,\cdot)dy\right\|_{L^2(0,T)} \\ \le\ &\|S_n u^n - S_n u\|_{L^2(0,T)} + \left\|S_n u - \int_0^1 u(y,\cdot)dy\right\|_{L^2(0,T)} \\ \to\ &0,\ \text{as } n \to \infty. \end{aligned}$$

Lemma 4.2 is proved. □

Lemma 4.3: *There exists a subsequence of $\{u^n\}$, still denoted by $\{u^n\}$, such that*

$$\begin{aligned} &\text{(i)} \ \left\|\hat{S}_n u^n - \hat{S}_n u\right\|_{C^0([0,T])} \to 0, \ \textit{as} \ n \to \infty, \\ &\text{(ii)} \ \left\|\hat{S}_n u^n - \textstyle\int_0^1 u_x^2(y,\cdot)dy\right\|_{L^2(0,T)}^2 \\ &\qquad = \textstyle\int_0^T \left|\hat{S}_n u^n(t) - \|u_x(t)\|^2\right|^2 dt \to 0, \ \textit{as} \ n \to \infty. \end{aligned} \tag{4.18}$$

Proof of Lemma 4.3. By $(4.9)_{(ii)}$, we obtain

$$\begin{aligned} &\left\|\hat{S}_n u^n - \int_0^1 u_x^2(y,\cdot)dy\right\|_{L^2(0,T)} \\ &\leq \left\|\hat{S}_n u^n - \hat{S}_n u\right\|_{L^2(0,T)} + \left\|\hat{S}_n u - \int_0^1 u_x^2(y,\cdot)dy\right\|_{L^2(0,T)} \\ &\leq \sqrt{T}\left\|\hat{S}_n u^n - \hat{S}_n u\right\|_{C^0([0,T])} + \left\|\hat{S}_n u - \int_0^1 u_x^2(y,\cdot)dy\right\|_{L^2(0,T)} \\ &\to 0, \ \text{as} \ n \to \infty. \end{aligned} \tag{4.19}$$

This implies $(4.18)_{(ii)}$ holds. We prove $(4.18)_{(i)}$ only.
By $u^n \in W(M,T)$, we get that

$$\begin{aligned} &u^n \in C^0\left([0,T]; H^2 \cap V\right) \cap C^1\left([0,T]; V\right) \cap L^\infty(0,T; H^2 \cap V), \\ &u_t^n \in C^0\left([0,T]; V\right) \cap L^\infty(0,T; H^2 \cap V), \\ &\|u^n\|_{L^\infty(0,T;H^2\cap V)} \leq M, \ \|u_t^n\|_{L^\infty(0,T;H^2\cap V)} \leq M. \end{aligned} \tag{4.20}$$

Consider the sequence $\{h_n\}$ defined by $h_n = u_x^n$.
Then, by $H^1 \hookrightarrow C^0([0,1]) \equiv E$, we have $\{h_n\} \subset C^0\left([0,T]; H^1\right) \subset C^0\left([0,T]; E\right)$.
We shall show that there exists a subsequence of $\{h_n\}$, still denoted by $\{h_n\}$, such that

$$h_n \to u_x \text{ strongly in } C^0\left([0,T]; E\right). \tag{4.21}$$

Using Ascoli-Arzela theorem in $C^0([0,T]; E)$, we shall prove that

$$\begin{aligned} &\text{(j)} \ \{h_n\} \text{ is equicontinuous in } C^0\left([0,T]; E\right), \\ &\text{(jj)} \ \text{For every } t \in [0,T], \ \{h_n(t) : n \in \mathbb{N}\} \text{ is relatively compact in } E. \end{aligned} \tag{4.22}$$

Proof $(4.22)_{(j)}$. For all $t_1, t_2 \in [0,T]$, $t_1 \leq t_2$, $\forall n \in \mathbb{N}$, by $(4.20)_{(ii)}$, we have

$$\begin{aligned} &\|h_n(t_2) - h_n(t_1)\|_E \\ &= \left\|\int_{t_1}^{t_2} h_n'(t)dt\right\|_E \leq \int_{t_1}^{t_2} \left\|h_n'(t)\right\|_E dt \\ &= \int_{t_1}^{t_2} \|u_{xt}^n(t)\|_E dt \leq \sqrt{2}\int_{t_1}^{t_2} \|u_{xt}^n(t)\|_{H^1} dt \\ &\leq \sqrt{2}\,|t_2 - t_1|\,\|u_t^n\|_{L^\infty(0,T;H^2\cap V)} \leq \sqrt{2}M\,|t_2 - t_1|. \end{aligned} \tag{4.23}$$

This implies $(4.22)_{(j)}$ holds.
Proof $(4.22)_{(jj)}$. By $(4.20)_{(i)}$, we have

$$\|h_n(t)\|_{H^1} = \|u_x^n(t)\|_{H^1} \leq \|u^n(t)\|_{H^2\cap V} \leq \|u^n\|_{L^\infty(0,T;H^2\cap V)} \leq M. \tag{4.24}$$

Because the imbedding $H^1 \hookrightarrow C^0([0,1]) = E$ is compact, then there exists a convergent subsequence of $\{h_n\}$ (in E). This implies $(4.22)_{(jj)}$ holds.

From (4.22), we deduce that there exists a subsequence of $\{h_n\}$, still denoted by $\{h_n\}$, such that

$$h_n \to h \text{ strongly in } C^0([0,T];E). \tag{4.25}$$

Due to $C^0([0,T];E) \hookrightarrow L^2(Q_T)$, we have that

$$h_n \to h \text{ strongly in } L^2(Q_T). \tag{4.26}$$

On the other hand, from $(4.5)_{(i)}$, we obtain

$$h_n = u_x^n \to u_x \text{ strongly in } L^2(Q_T). \tag{4.27}$$

It follows from (4.26) and (4.27) that $h = u_x$, thus (4.21) is proved.

On the other hand, from (4.3), we obtain the following estimation

$$\begin{aligned} \left|\hat{S}_n u^n(t) - \hat{S}_n u(t)\right| &\le \frac{1}{n}\sum\nolimits_{i=1}^{n} \left| \left|u_x^n\left(\tfrac{i-1}{n},t\right)\right|^2 - \left|u_x\left(\tfrac{i-1}{n},t\right)\right|^2 \right| \\ &\le \left(\|u_x^n(t)\|_E + \|u_x(t)\|_E\right)\|u_x^n(t) - u_x(t)\|_E \\ &\le \sqrt{2}\left(\|u_x^n(t)\|_{H^1} + \|u_x(t)\|_{H^1}\right)\|u_x^n(t) - u_x(t)\|_E \\ &\le 2\sqrt{2}M\,\|u_x^n - u_x\|_{C^0([0,T];E)}. \end{aligned} \tag{4.28}$$

Hence

$$\left\|\hat{S}_n u^n - \hat{S}_n u\right\|_{C^0([0,T])} \le 2\sqrt{2}M\,\|u_x^n - u_x\|_{C^0([0,T];E)}. \tag{4.29}$$

From (4.21) and (4.29), we obtain $(4.18)_{(i)}$ holds.

Lemma 4.3 is proved. □

Now, we continue the proof of (4.8).

Proof $(4.8)_{(i)}$. Note that $\left|(\hat{S}_n u^n)(t)\right| \le 2M^2$, we obtain

$$\begin{aligned} &\left|\int_0^T (\hat{S}_n u^n)(t) a(u^n(t),w)\varphi(t)dt - \int_0^T \|u_x(t)\|^2 a(u(t),w)\varphi(t)dt\right| \\ &\le \int_0^T (\hat{S}_n u^n)(t)\,|a(u^n(t)-u(t),w)\varphi(t)|\,dt \\ &+ \int_0^T \left|(\hat{S}_n u^n)(t) - \|u_x(t)\|^2\right| |a(u(t),w)\varphi(t)|\,dt \\ &\le 2M^2\|\varphi\|_{L^2(0,T)}\|w\|_V\|u^n - u\|_{L^2(0,T;V)} \\ &+ \|u\|_{L^\infty(0,T;V)}\|w\|_V\|\varphi\|_{L^2(0,T)}\left\|\hat{S}_n u^n - \int_0^1 u_x^2(y,\cdot)dy\right\|_{L^2(0,T)} \\ &\le M\|w\|_V\|\varphi\|_{L^2(0,T)}\left[2M\|u^n - u\|_{L^2(0,T;V)} + \left\|\hat{S}_n u^n - \int_0^1 u_x^2(y,\cdot)dy\right\|_{L^2(0,T)}\right]. \end{aligned} \tag{4.30}$$

It follows from $(4.5)_1$, $(4.18)_{(ii)}$ and (4.30) that $(4.8)_{(i)}$ holds.

Proof $(4.8)_{(ii)}$. We have

$$\begin{aligned}
&\int_0^T \langle f\left(t, u^n(t), u_x^n(t), u_t^n(t), (S_n u^n)(t)\right), w\rangle \varphi(t)dt \\
&- \int_0^T \left\langle f\left(t, u(t), u_x(t), u'(t), \int_0^1 u(y,t)dy\right), w\right\rangle \varphi(t)dt \\
&= \int_0^T \langle f\left(t, u^n(t), u_x^n(t), u_t^n(t), (S_n u^n)(t)\right) \\
&\qquad - f\left(t, u^n(t), u_x^n(t), u_t^n(t), \int_0^1 u(y,t)dy\right), w\rangle \varphi(t)dt \\
&+ \int_0^T \left\langle f\left(t, u^n(t), u_x^n(t), u_t^n(t), \int_0^1 u(y,t)dy\right)\right. \\
&\qquad \left. - f\left(t, u(t), u_x(t), u'(t), \int_0^1 u(y,t)dy\right), w\right\rangle \varphi(t)dt \\
&= \tilde{J}_1 + \tilde{J}_2.
\end{aligned} \tag{4.31}$$

Proof $\tilde{J}_1 \to 0$. We note that

$$\begin{aligned}
&\left| f\left(t, u^n(t), u_x^n(t), u_t^n(t), (S_n u^n)(t)\right) - f\left(t, u^n(t), u_x^n(t), u_t^n(t), \int_0^1 u(y,t)dy\right)\right| \\
&\leq K_M(f)\left|(S_n u^n)(t) - \int_0^1 u(y,t)dy\right|.
\end{aligned} \tag{4.32}$$

Therefore, we deduce from $(4.15)_{(ii)}$ and (4.32), that

$$\begin{aligned}
\tilde{J}_1 &\leq K_M(f)\, \|w\|\, \|\varphi\|_{L^2(0,T)} \left\| S_n u^n - \int_0^1 u(y,\cdot)dy\right\|_{L^2(0,T)} \\
&\to 0, \text{ as } n \to \infty.
\end{aligned} \tag{4.33}$$

Proof $\tilde{J}_2 \to 0$. We have

$$\begin{aligned}
&\left\| f\left(t, u^n(t), u_x^n(t), u_t^n(t), \int_0^1 u(y,t)dy\right) - f\left(t, u(t), u_x(t), u'(t), \int_0^1 u(y,t)dy\right)\right\| \\
&\leq 2K_M(f)\left(\|u_x^n(t) - u_x(t)\| + \|u_t^n(t) - u'(t)\|\right).
\end{aligned} \tag{4.34}$$

Therefore, we deduce from (4.5) and (4.34), that

$$\begin{aligned}
\tilde{J}_2 &\leq 2K_M(f)\, \|w\|\, \|\varphi\|_{L^2(0,T)} \left[\|u^n - u\|_{L^2(0,T;V)} + \|u_t^n - u'\|_{L^2(0,T;V)}\right] \\
&\to 0, \text{ as } n \to \infty.
\end{aligned} \tag{4.35}$$

Thus, it follows from (4.31), (4.33), (4.35), that $(4.8)_{(ii)}$ holds. □
Finally, letting $n \to \infty$ in (4.6), we deduce from (4.7), (4.8), that $u \in W(M,T)$ and satisfies

$$\begin{aligned}
&\int_0^T \langle u''(t), w\rangle \varphi(t)dt + \lambda \int_0^T a(u'(t), w)\varphi(t)dt \\
&+ \int_0^T \left(1 + \|u_x(t)\|^2\right) a(u(t), w)\varphi(t)dt \\
&= \int_0^T \left\langle f\left(t, u(t), u_x(t), u'(t), \int_0^1 u(y,t)dy\right), w\right\rangle \varphi(t)dt,
\end{aligned} \tag{4.36}$$

for all $w \in V$, $\varphi \in C_c^\infty(0,T)$, together with the initial conditions

$$u(0) = \tilde{u}_0, \;\; u'(0) = \tilde{u}_1. \tag{4.37}$$

Consequently,

$$\begin{cases} \langle u''(t), w\rangle + \lambda a(u'(t), w) + \left(1 + \|u_x(t)\|^2\right) a(u(t), w) \\ \qquad = \left\langle f\left(t, u(t), u_x(t), u'(t), \int_0^1 u(y,t)dy\right), w\right\rangle, \;\; \forall w \in V, \\ u(0) = \tilde{u}_0, \;\; u'(0) = \tilde{u}_1, \\ u \in W(M,T). \end{cases} \tag{4.38}$$

Furthermore, (1.7) and $(3.59)_4$ imply that

$$u'' = \lambda \Delta u' + \left(1 + \|u_x(t)\|^2\right) \Delta u + f[u] \in L^\infty(0,T; L^2),$$

so we obtain $u \in W_1(M,T)$. The proof of the existence is completed.

Next, we are easy to prove the uniqueness of solutions of (P).

Finally, we have the following theorem.

Theorem 4.4. *Let (H_1)– (H_2) hold. Then there exist positive constants M, $T > 0$ such that*

(i) (P) has a unique weak solution $u \in W_1(M,T)$.

(ii) The solution sequence $\{u^n\}$ of (P_n) converges to the weak solution u of (P) in sense

$$\begin{cases} u^n \to u & in \quad L^\infty(0,T; H^2 \cap V) \text{ weak}^*, \\ u_t^n \to u' & in \quad L^\infty(0,T; H^2 \cap V) \text{ weak}^*, \\ u_{tt}^n \to u'' & in \quad L^2(0,T; V) \text{ weak}, \\ u^n \to u & in \quad L^2(0,T; V) \text{ strongly}, \\ u_t^n \to u' & in \quad L^2(0,T; V) \text{ strongly}. \end{cases} \tag{4.39}$$

Remark 4.5. The above method still holds for the problem (1.1)-(1.2) in which $(S_n u)(t)$ and $(\hat{S}_n u)(t)$ are replaced by the following arithmetic-mean terms

$$(S_n u)(t) = \frac{1}{n} \sum_{i=0}^{n-1} u\left(\tfrac{i+\theta_i}{n}, t\right), \;\; (\hat{S}_n u)(t) = \frac{1}{n} \sum_{i=0}^{n-1} u_x^2\left(\tfrac{i+\theta_i}{n}, t\right), \tag{4.40}$$

respectively, where $\theta_i \in [0,1)$, $i = \overline{0, n-1}$, are given constants.

5. Remark

We remark that the methods used in the above sections can be applied to the following problem again, and we also obtain the same results as above.

$$(\bar{P}_n) \begin{cases} u_{tt} - \lambda u_{txx} - B\left((\bar{S}_n u)(t), (\hat{S}_n u)(t)\right) u_{xx} \\ \qquad = f\left(x, t, u, u_x, u_t, (\bar{S}_n u)(t), (\hat{S}_n u)(t)\right), \;\; 0 < x < 1, \;\; 0 < t < T, \\ u_x(0,t) - \zeta u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x), \;\; u_t(x,0) = \tilde{u}_1(x), \end{cases}$$

where $\lambda > 0$, $\zeta \geq 0$ are given constants, B, f, $\tilde{u}_0$, $\tilde{u}_1$ are given functions and $(\bar{S}_n u)(t) = \frac{1}{n}\sum_{i=0}^{n-1} u^2\left(\frac{i+\theta_i}{n}, t\right)$, $(\hat{S}_n u)(t) = \frac{1}{n}\sum_{i=0}^{n-1} u_x^2\left(\frac{i+\theta_i}{n}, t\right)$, $\theta_i \in [0,1)$, $i = 0, \cdots, n-1$ are given constants.

Moreover, we can prove that the weak solution of $(\bar{P}_n)$ converges strongly in appropriate spaces to the weak solution of the following problem

$$(\bar{P}) \begin{cases} u_{tt} - \lambda u_{txx} - B\left(\|u(t)\|^2, \|u_x(t)\|^2\right) u_{xx} \\ \quad = f\left(x, t, u, u_x, u_t, \|u(t)\|^2, \|u_x(t)\|^2\right), \ 0 < x < 1, \ 0 < t < T, \\ u_x(0,t) - \zeta u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x), \ u_t(x,0) = \tilde{u}_1(x), \end{cases}$$

where $\|u(t)\|^2 = \int_0^1 u^2(y,t)dy$, $\|u_x(t)\|^2 = \int_0^1 u_x^2(y,t)dy$.

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