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Characterization of tri quasi purity applicable for $(\flat_1, \flat_2)$-regularities of an ordered semirings

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Abstract
Investigating and categorizing many different kinds of tri-ideals in ordered semirings such as ordered tri quasi ideals, quasi ideals, and (2, 2)-ideals in semirings is the aim of this study. In order to characterize the strongly $(\flat_1, \flat_2)$ regular ordered semirings and fully idempotent ordered semirings. We introduce the notions of left tri purity, right tri purity, tri quasi purity, (2, 2) bi purity, left weak tri purity and right weak tri purity of ordered tri ideals of an ordered semirings. Also, we discussed the 0-minimal tri-ideals, (2, 1)-bi-ideals, (2, 3)-bi-ideals, and (4, 4)-bi-ideals in ordered semigroups. The subsemigroup $\top$ in $\mathcal{Z}$ is a (2, 2)-bi-ideals in $\mathcal{Z}$ if and only if there is a (2, 1)-bi-ideals Ψ and a (1, 2)-bi-ideals $\Bbbk$ in $\mathcal{Z}$ such that $\Psi^2 \circ \Bbbk^2 \subseteq \top \subseteq \Psi^2 \Cap \Bbbk^2$. If and only if $\top$ is a tri-ideal of certain left tri-ideal in $\mathcal{Z}$, then $\top$ is a (4, 4)-bi-ideals in $\mathcal{Z}$. We draw some inferences, even though the examples do not support the opposite

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1. Introduction

An orderable semigroup $\mathcal{Z}$ is a subsemigroup if the set of all idempotents is non-empty. Free semigroups, free commutative semigroups, and free n-step nilpotent semigroups are examples of orderable semigroups. There are several ways to order free semigroups of finite rank $n \succeq 2$. In addition to semigroups from a number of well-known classes (such as semigroups of idempotents and inverse semigroups), certain necessary and sufficient conditions for the order ability of arbitrary semigroups have been identified. Ordered semigroups are common in many fields of mathematics as well as the entire field of our research, including computer science, social science, and economics. However, their simplicity in definition makes them challenging to characterize. The transition from the semigroup to the ordered semigroup case is not simple, against common opinion. For example, it should be noted that

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there is currently no formulation for the ordered factor semigroup. Semirings, ordered semirings, and hemirings appear naturally in several applications to the theory of automata and formal languages. Concepts that extend an associative ring $(R, +, \circ)$, (where + is a usual addition and $\circ$ is a usual multiplication) are called universal algebras. except that it is assumed that $(S, +)$ is a semigroup instead of a commutative group. Ideals of semirings are helpful for a variety of reasons and are essential to structure theory. However, if S is a ring, they typically do not overlap with the standard ring ideals. A semiring can be found in rings but not in semigroups because it is a generalization of a ring. The introduction of ideals for algebraic numbers and their subsequent application to associative rings were initially proposed by Dedekind. Shabir et al. [1] described semirings using a variety of ideals and their characteristics. Adhikari et al. [2] have examined a topological space of several types of ideals of a semiring. The concept of BID for semigroups was first presented by Good et al. [3]. The generalized BID for rings was introduced by Szasz [4]. Dutta et al. [5] investigated semiprime ideals and irreducible ideals of semirings.

As a generalization of rings, Vandiver introduced semirings [6]. Semirings are found naturally in many mathematical domains. The concept of a QID in semigroups and rings was introduced by Steinfeld [7]. The semirings were described by Shabir et al. [8] using QID characteristics. The concept of an ordered semiring in connection to a semiring with a compatible partial order relation by Gan et al. [9]. Palakawong et al. [10] constructed and characterized the BIDs of ordered semirings. Lajos [11] proposed the concept of BIDs in semigroups. The QID, BID and TIDs is a generalization of IDs of a semiring [12]. Grosek et al. [13] developed the notion of ideals in semigroups. Palanikumar et al. created several methods to generate ideals of ordered semirings [14]. The ideas of ordered k-regular semirings via ordered k-ideals were covered by Patchakhieo et al. [15]. The classification of semirings was covered by Sen et al. [16]. Lee et al. introduced complete regularity on ordered semigeoups [17]. If $\top$ is an ID in $\mathcal{Z}$, B is an idempotent of $\top$ of $\mathcal{Z}$, then B is an ID in $\mathcal{Z}$ [18]. Several authors including Alimov [19], Chehata [20] and Vinogradov [21] started studying ordered semigroups. Steinfeld [22] first proposed the concept of QIDs in semigroups. BIDs in ordered semigroups and ordered groups were investigated by Kehayopulu et al. [23]. Kehayopulu et al. [24] demonstrated that any maximal ID in commutative ordered semigroups with identity is a prime ID; the opposite is generally not true. We now understand that ordered QIDs in ordered semigroups are a generalization of the concept of ordered (m_1, m_2) QIDs.

A nonempty subset $\top$ in $\mathcal{Z}$ is the LID (RID) in $\mathcal{Z}$ if $\mathcal{Z}\top\subseteq\top$ (resp. $\top\mathcal{Z}\subseteq\top$). An ID in $\mathcal{Z}$ is a non-empty subset in $\mathcal{Z}$ that is both LID and RID. The intersection of all LIDs (RIDs) in $\mathcal{Z}$ including $\top$ coincides with $L(\top)$ (resp., $R(\top)$), and the intersection of all IDs in $\mathcal{Z}$ containing $\top$ coincides with $I(\top)$. In a semigroup $\mathcal{Z}$, the smallest LID, RID, and ID in $\mathcal{Z}$ containing any set $\top$ are $L(\top)=\top\cup\mathcal{Z}\top$, $R(\top)=\top\cup\top\mathcal{Z}$, and $I(\top)=\top\cup\mathcal{Z}\top\cup\top\mathcal{Z}\cup\mathcal{Z}\top\mathcal{Z}$. These are known as the principal LID, principal RID, and principal ID in $\mathcal{Z}$ generated by $\top$. The principal LID, principal RID, or principal ID in $\mathcal{Z}$, respectively, generated by "δ" ($\delta\in\mathcal{Z}$), are denoted by $L(\delta)$, $R(\delta)$, and $I(\delta)$ instead of $L(\delta)$, $R(\delta)$, and $I(\delta)$. Now, $L(\delta)=\{\gamma\in\mathcal{Z}\mid\gamma=\delta$ or $\gamma=\psi\delta$ for certain $\psi\in\mathcal{Z}\}$, $R(\delta)=\{\gamma\in\mathcal{Z}\mid\gamma=\delta$ or $\gamma=\delta\kappa$ for certain $\kappa\in\mathcal{Z}\}$, $I(\delta)=\{\gamma\in\mathcal{Z}\mid\gamma=\delta$ or $\gamma=\psi\delta$ for certain $\psi\in\mathcal{Z}$ or $\gamma=\delta\kappa$ for certain $\kappa\in\mathcal{Z}$ or $\gamma=\kappa\delta\psi$ for certain $\kappa,\psi\in\mathcal{Z}\}$. The article is presented as follows. Preliminaries come after this section. There are five distinct sections in this study. An introduction can be found in 1. Section 2 provides a brief description of semirings and semigroups and their fundamental characteristics. Characterization of ordered TIDs as in the section 3. Section 4 describes the tri quasi purity of semirings via ordered $(\mathfrak{h}_1,\mathfrak{h}_2)$ regularities. Section 5 presents our conclusions of the paper. The structural organization of the paper's content is shown in Figure 1.

2. Preliminaries

Throughout this paper, $\mathcal{Z}$ represents the ordered semiring.

Definition 2.1. [15] *Suppose that* $\mathfrak{I}$ *is a subset of* $(\mathcal{Z}, +, \leq)$. *Then* $\mathfrak{I}$ *is called an RID (LID) in* $\mathcal{Z}$, *if* $(\mathfrak{I}, +)$ *is a SSG of* $(\mathcal{Z}, +)$

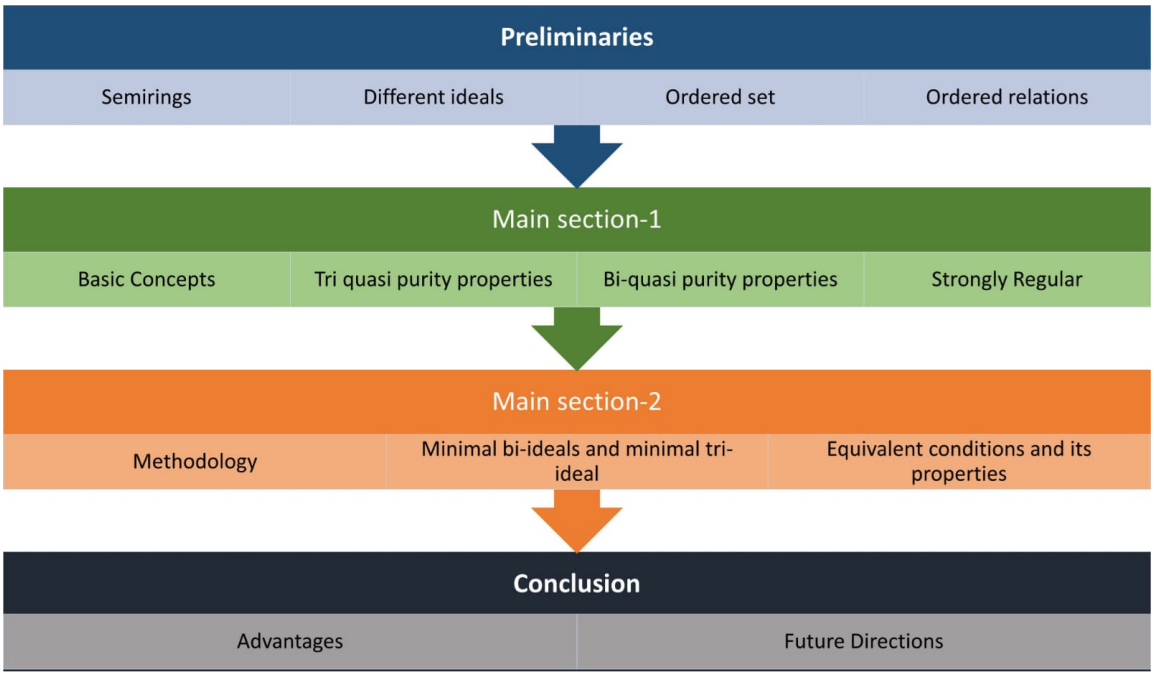

Figure 1: Organization of the paper

(1) $\beth$ *is a RID(LID) in* $\mathcal{Z}$.
(2) *If* $\ni \leq i$ *for some* $i \in \beth$, *then* $\ni \in \beth$ *(i.e.* $\beth = (\underline{\beth}|$). $\beth$ *is called an ID if* $\beth$ *is an RID and LID in* $\mathcal{Z}$.

Definition 2.2. *Suppose that* $\beth$ *is a subset of* $(\mathcal{Z}, +, \geq)$. *Then* $\beth$ *is called an QID in* $\mathcal{Z}$, *if* $(\beth, +)$ *is a SSG of* $(\mathcal{Z}, +)$

(1) $(\Sigma\beth\underline{\mathcal{Z}}| \sqcap (\Sigma\mathcal{Z}\underline{\beth}| \subseteq \beth$
(2) *If* $\ni \leq q$ *for some* $q \in \beth$, *then then* $\ni \in \beth$ *(i.e.* $\beth = (\underline{\beth}|$).

Definition 2.3. [15] *Suppose that* $\beth$ *is a subset of* $(\mathcal{Z}, +, \leq)$. *Then* $\beth$ *is called an BID in* $\mathcal{Z}$, *if* $(\beth, +)$ *is a SSG of* $(\mathcal{Z}, +)$

(1) $\beth\mathcal{Z}\beth \subseteq \beth$
(2) *If* $\ni \leq b$ *for some* $b \in \beth$, *then* $\ni \in \beth$ *(i.e.* $\beth = (\underline{\beth}|$).

Lemma 2.4. *For any nonempty subsets* $\beth, \Lambda$ *in* $\mathcal{Z}$.. *Then*

(1) $\beth \subseteq \Sigma\beth$ *and* $\Sigma(\Sigma\beth) = \Sigma\beth$
(2) $\beth \subseteq \Lambda$ *and* $\Sigma\beth \subseteq \Sigma\Lambda$
(3) $\beth(\Sigma\Lambda) \subseteq (\Sigma\beth)(\Sigma\Lambda) \subseteq \Sigma\beth\Lambda$, $(\Sigma\beth)\Lambda \subseteq (\Sigma\beth)(\Sigma\Lambda) \subseteq \Sigma\beth\Lambda$
(4) $\Sigma(\underline{\beth}| \subseteq (\Sigma\underline{\beth}|$
(5) $\beth \subseteq (\underline{\beth}|$ *and* $((\underline{\beth}|| = (\underline{\beth}|$
(6) *If* $\beth \subseteq \Lambda$, *then* $(\underline{\beth}| \subseteq (\underline{\Lambda}|$
(7) $\beth(\underline{\Lambda}| \subseteq (\underline{\beth}|(\underline{\Lambda}| \subseteq (\beth\underline{\Lambda}|$, $(\underline{\beth}|\Lambda \subseteq (\underline{\beth}|(\underline{\Lambda}| \subseteq (\beth\underline{\Lambda}|$

3. Regular approaches applied to tri quasi purity of an ordered semirings

Definition 3.1. *An ID* $\beth$ *in* $\mathcal{Z}$ *is called a LT-purity (RT-purity) if* $\ni \in (\ni \circ \beth \circ \ni \circ \underline{\ni}|(\ni \in (\ni \circ \ni \circ \beth \circ \underline{\ni}|$, $\forall \ni \in \beth$.

Definition 3.2. *An ID* $\beth$ *in* $\mathcal{Z}$ *is called a weakly LT-purity (weakly RT-purity) if* $\beth \cap \beth_1 = (\Sigma\beth_1 \circ \beth \circ \beth_1 \circ \underline{\beth_1}|$ $(\beth_1 \cap \beth = \left(\Sigma\beth_1 \circ \beth_1 \circ \beth \circ \underline{\beth_1}|\right)$, $\forall \beth_1$ *in* $\mathcal{Z}$.

Definition 3.3. *An ID* $\mathfrak{I}$ *in* $\mathcal{Z}$ *is called a TQ-purity if* $\mathfrak{z} \in (\mathfrak{z} \circ \mathfrak{z} \circ \mathfrak{I} \circ \mathfrak{z}] \cap (\mathfrak{z} \circ \mathfrak{I} \circ \mathfrak{z} \circ \mathfrak{z}], \forall\, \mathfrak{z} \in \mathfrak{I}$.

Definition 3.4. *An ID* $\mathfrak{I}$ *in* $\mathcal{Z}$ *is called a (2,2) bi-purity if* $\mathfrak{z} \in (\mathfrak{z} \circ \mathfrak{z} \circ \mathfrak{I} \circ \mathfrak{z} \circ \mathfrak{z}], \forall\, \mathfrak{z} \in \mathfrak{I}$.

Remark 3.5. *Every LT-purity (RT-purity) is a weakly LT-purity (weakly RT-purity).*

Now, we give an example of a weakly LT-purity of which is not LT-purity.

Example 3.6. Let $\mathcal{Z} = \{\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3, \mathfrak{z}_4, \mathfrak{z}_5, \mathfrak{z}_6\}$ be the semiring.

$+$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$	$\mathfrak{z}_4$	$\mathfrak{z}_5$	$\mathfrak{z}_6$
$\mathfrak{z}_1$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$	$\mathfrak{z}_4$	$\mathfrak{z}_5$	$\mathfrak{z}_6$
$\mathfrak{z}_2$	$\mathfrak{z}_2$	$\mathfrak{z}_3$	$\mathfrak{z}_4$	$\mathfrak{z}_5$	$\mathfrak{z}_6$	$\mathfrak{z}_3$
$\mathfrak{z}_3$	$\mathfrak{z}_3$	$\mathfrak{z}_4$	$\mathfrak{z}_5$	$\mathfrak{z}_6$	$\mathfrak{z}_3$	$\mathfrak{z}_4$
$\mathfrak{z}_4$	$\mathfrak{z}_4$	$\mathfrak{z}_5$	$\mathfrak{z}_6$	$\mathfrak{z}_3$	$\mathfrak{z}_4$	$\mathfrak{z}_5$
$\mathfrak{z}_5$	$\mathfrak{z}_5$	$\mathfrak{z}_6$	$\mathfrak{z}_3$	$\mathfrak{z}_4$	$\mathfrak{z}_5$	$\mathfrak{z}_6$
$\mathfrak{z}_6$	$\mathfrak{z}_6$	$\mathfrak{z}_3$	$\mathfrak{z}_4$	$\mathfrak{z}_5$	$\mathfrak{z}_6$	$\mathfrak{z}_3$

$\circ$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$	$\mathfrak{z}_4$	$\mathfrak{z}_5$	$\mathfrak{z}_6$
$\mathfrak{z}_1$	$\mathfrak{z}_1$	$\mathfrak{z}_1$	$\mathfrak{z}_1$	$\mathfrak{z}_1$	$\mathfrak{z}_1$	$\mathfrak{z}_1$
$\mathfrak{z}_2$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{z}_3$	$\mathfrak{z}_4$	$\mathfrak{z}_5$	$\mathfrak{z}_6$
$\mathfrak{z}_3$	$\mathfrak{z}_1$	$\mathfrak{z}_3$	$\mathfrak{z}_5$	$\mathfrak{z}_3$	$\mathfrak{z}_5$	$\mathfrak{z}_3$
$\mathfrak{z}_4$	$\mathfrak{z}_1$	$\mathfrak{z}_4$	$\mathfrak{z}_3$	$\mathfrak{z}_6$	$\mathfrak{z}_5$	$\mathfrak{z}_4$
$\mathfrak{z}_5$	$\mathfrak{z}_1$	$\mathfrak{z}_5$	$\mathfrak{z}_5$	$\mathfrak{z}_5$	$\mathfrak{z}_5$	$\mathfrak{z}_5$
$\mathfrak{z}_6$	$\mathfrak{z}_1$	$\mathfrak{z}_6$	$\mathfrak{z}_3$	$\mathfrak{z}_4$	$\mathfrak{z}_5$	$\mathfrak{z}_6$

Define a binary relation $\preceq$ *in* $\mathcal{Z}$ *by* $\preceq := \{(\mathfrak{z}_1,\mathfrak{z}_1),(\mathfrak{z}_2,\mathfrak{z}_3),(\mathfrak{z}_3,\mathfrak{z}_3),(\mathfrak{z}_4,\mathfrak{z}_4),(\mathfrak{z}_5,\mathfrak{z}_5),(\mathfrak{z}_6,\mathfrak{z}_5),(\mathfrak{z}_1,\mathfrak{z}_5),(\mathfrak{z}_2,\mathfrak{z}_6)\}$. *Then* $(\mathcal{Z}, +, \preceq)$ *is a semiring. Clearly* $\{\mathfrak{z}_1\}, \{\mathfrak{z}_1, \mathfrak{z}_5\}$ *and* $\mathcal{Z}$.

Since,

$$\begin{bmatrix} \{\mathfrak{z}_1\} \cap \{\mathfrak{z}_1\} = \{\mathfrak{z}_1\} = \Sigma(\mathfrak{z}_1 \circ \mathfrak{z}_1 \circ \mathfrak{z}_1 \circ \mathfrak{z}_1), \\ \{\mathfrak{z}_1\} \cap \{\mathfrak{z}_1, \mathfrak{z}_5\} = \{\mathfrak{z}_1\} = \Sigma(\mathfrak{z}_1 \circ \{\mathfrak{z}_1, \mathfrak{z}_5\} \circ \mathfrak{z}_1 \circ \mathfrak{z}_1), \\ \{\mathfrak{z}_1\} \cap \{\mathcal{Z}\} = \{\mathfrak{z}_1\} = \Sigma(\mathfrak{z}_1 \circ \mathcal{Z} \circ \mathfrak{z}_1 \circ \mathfrak{z}_1) \end{bmatrix}$$

and

$$\begin{bmatrix} \{\mathfrak{z}_1, \mathfrak{z}_5\} \cap \{\mathfrak{z}_1\} = \{\mathfrak{z}_1\} = \Sigma(\{\mathfrak{z}_1, \mathfrak{z}_5\} \circ \mathfrak{z}_1 \circ \{\mathfrak{z}_1, \mathfrak{z}_5\} \circ \{\mathfrak{z}_1, \mathfrak{z}_5\}), \\ \{\mathfrak{z}_1, \mathfrak{z}_5\} \cap \{\mathfrak{z}_1, \mathfrak{z}_5\} = \{\mathfrak{z}_1, \mathfrak{z}_5\} = \Sigma(\{\mathfrak{z}_1, \mathfrak{z}_5\} \circ \{\mathfrak{z}_1, \mathfrak{z}_5\} \circ \{\mathfrak{z}_1, \mathfrak{z}_5\} \circ \{\mathfrak{z}_1, \mathfrak{z}_5\}), \\ \{\mathfrak{z}_1, \mathfrak{z}_5\} \cap \{\mathcal{Z}\} = \{\mathfrak{z}_1, \mathfrak{z}_5\} = \Sigma(\{\mathfrak{z}_1, \mathfrak{z}_5\} \circ \mathcal{Z} \circ \{\mathfrak{z}_1, \mathfrak{z}_5\} \circ \{\mathfrak{z}_1, \mathfrak{z}_5\}) \end{bmatrix}$$

and

$$\begin{bmatrix} \{\mathcal{Z}\} \cap \{\mathfrak{z}_1\} = \{\mathfrak{z}_1\} = \Sigma(\mathcal{Z} \circ \mathfrak{z}_1 \circ \mathcal{Z} \circ \mathcal{Z}), \\ \{\mathcal{Z}\} \cap \{\mathfrak{z}_1, \mathfrak{z}_5\} = \{\mathfrak{z}_1, \mathfrak{z}_5\} = \Sigma(\mathcal{Z} \circ \{\mathfrak{z}_1, \mathfrak{z}_5\} \circ \mathcal{Z} \circ \mathcal{Z}), \\ \{\mathcal{Z}\} \cap \{\mathcal{Z}\} = \{\mathcal{Z}\} = \Sigma(\mathcal{Z} \circ \mathcal{Z} \circ \mathcal{Z} \circ \mathcal{Z}) \end{bmatrix}$$

Since, $\{\mathfrak{z}_1, \mathfrak{z}_5\}$ *is a weakly LT-purity. However,* $\{\mathfrak{z}_1, \mathfrak{z}_5\}$ *is not a LT-purity, because* $\mathfrak{z}_3 \notin \{\mathfrak{z}_1, \mathfrak{z}_5\} = (\mathfrak{z}_3 \circ \mathcal{Z} \circ \mathfrak{z}_3 \circ \mathfrak{z}_3]$.

Theorem 3.7. *Let* $\mathfrak{I}$ *be an TID in* $\mathcal{Z}$. *Then*

(i) $\mathfrak{I}$ *is a LT-purity if and only if* $\mathfrak{I} \cap \Psi = (\mathfrak{I} \circ \Psi]$, *for every LTID in* $\mathcal{Z}$.
(ii) $\mathfrak{I}$ *is a RT-purity if and only if* $\mathbb{k} \cap \mathfrak{I} = (\mathbb{k} \cap \mathfrak{I}]$, *for every RTID in* $\mathcal{Z}$.

Proof. (i) Assume that $\mathfrak{I}$ is a LT-purity. Let Ψ be a LTID in $\mathcal{Z}$. If $\mathfrak{z} \in \mathfrak{I} \cap \Psi$, then $\mathfrak{z} \in (\mathfrak{z} \circ \mathfrak{I} \circ \mathfrak{z} \circ \mathfrak{z}] \subseteq (\mathfrak{I} \circ \mathfrak{I} \circ \Psi \circ \mathfrak{I}]$. So, $\mathfrak{I} \cap \Psi \subseteq (\mathfrak{I} \circ \mathfrak{I} \circ \Psi \circ \mathfrak{I}]$. Clearly, $(\mathfrak{I} \circ \mathfrak{I} \circ \Psi \circ \mathfrak{I}] \subseteq \mathfrak{I} \cap \Psi$. Thus, $\mathfrak{I} \cap \Psi = (\mathfrak{I} \circ \mathfrak{I} \circ \Psi \circ \mathfrak{I}]$.

Conversely, let $\mathfrak{z} \in \mathfrak{I}$ and Lemma 2.4, we get

$$\begin{aligned} \mathfrak{z} \in \mathfrak{I} \cap \Psi(\mathfrak{z}) &= (\mathfrak{I} \circ \mathfrak{I} \circ \Psi(\mathfrak{z}) \circ \mathfrak{I}] \\ &= (\mathfrak{I} \circ \mathfrak{I}(\Sigma\, \mathfrak{z} + \mathfrak{z} \circ \mathcal{Z} \circ \mathfrak{z} \circ \mathfrak{z}] \circ \mathfrak{I}] \\ &\subseteq ((\Sigma \mathfrak{I} \circ \mathfrak{I} \circ \mathfrak{z} + \mathfrak{I} \circ \mathfrak{I} \circ \mathfrak{z} \circ \mathcal{Z} \circ \mathfrak{z} \circ \mathfrak{z}] \circ \mathfrak{I}] \\ &\subseteq (\mathfrak{I} \circ \mathfrak{I} \circ \mathfrak{z} \circ \mathfrak{I}] + (\mathfrak{I} \circ \mathfrak{I} \circ \mathfrak{z} \circ \mathfrak{I}] \\ &\subseteq (\mathfrak{I} \circ \mathfrak{I} \circ \mathfrak{z} \circ \mathfrak{I}] \end{aligned}$$

Thus, $\mathfrak{I}$ is a LT-purity in $\mathcal{Z}$. (ii) It can be proved similarly

Theorem 3.8. *An ID* $\mathfrak{I}$ *of* $\mathcal{Z}$ *is a TQ-purity if and only if* $\mathfrak{I} \cap \Phi = [\mathfrak{I}\circ\Phi\circ\mathfrak{I}\circ\mathfrak{I}] \cap [\mathfrak{I}\circ\mathfrak{I}\circ\Phi\circ\mathfrak{I}]$, *for every QID* Φ *in* $\mathcal{Z}$.

Proof. Let Φ be an QID of $\mathcal{Z}$. Now, $\mathfrak{z} \in (\mathfrak{z}\circ\mathfrak{z}\circ\mathfrak{I}\circ\mathfrak{z}] \cap (\mathfrak{z}\circ\mathfrak{I}\circ\mathfrak{z}\circ\mathfrak{z}] \subseteq (\mathfrak{I}\circ\mathfrak{z}\circ\mathfrak{I}\circ\mathfrak{I}] \cap (\mathfrak{I}\circ\mathfrak{I}\circ\mathfrak{z}\circ\mathfrak{I}] \subseteq (\mathfrak{I}\circ\Phi\circ\mathfrak{I}\circ\mathfrak{I}] \cap (\mathfrak{I}\circ\mathfrak{I}\circ\Phi\circ\mathfrak{I}]$. So, $\mathfrak{I} \cap \Phi \subseteq [\mathfrak{I}\circ\Phi\circ\mathfrak{I}\circ\mathfrak{I}] \cap [\mathfrak{I}\circ\mathfrak{I}\circ\Phi\circ\mathfrak{I}]$. Clearly, $[\mathfrak{I}\circ\Phi\circ\mathfrak{I}\circ\mathfrak{I}] \cap [\mathfrak{I}\circ\mathfrak{I}\circ\Phi\circ\mathfrak{I}] \subseteq \mathfrak{I} \cap \Phi$. Thus, $\mathfrak{I} \cap \Phi = [\mathfrak{I}\circ\Phi\circ\mathfrak{I}\circ\mathfrak{I}] \cap [\mathfrak{I}\circ\mathfrak{I}\circ\Phi\circ\mathfrak{I}]$.

Conversely,

$$\begin{aligned}
\mathfrak{z} \in \mathfrak{I} \cap \Phi(\mathfrak{z}] &= (\Phi(\mathfrak{z})\circ\Phi(\mathfrak{z})\circ\mathfrak{I}\circ\Phi(\mathfrak{z})] \cap (\Phi(\mathfrak{z})\circ\mathfrak{I}\circ\Phi(\mathfrak{z})\circ\Phi(\mathfrak{z})] \\
&= ((\Sigma\,\mathfrak{z} + (\mathfrak{z}\circ\mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}] \cap (\mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}\circ\mathfrak{z}])\circ(\Sigma\,\mathfrak{z} + (\mathfrak{z}\circ\mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}] \\
&\quad \cap (\mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}\circ\mathfrak{z}])\circ\mathfrak{I}\circ(\Sigma\,\mathfrak{z} + (\mathfrak{z}\circ\mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}] \cap (\mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}\circ\mathfrak{z})] \\
&\cap (\Sigma\,\mathfrak{z} + (\mathfrak{z}\circ\mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}] \cap (\mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}\circ\mathfrak{z}])\circ\mathfrak{I}\circ(\Sigma\,\mathfrak{z} + (\mathfrak{z}\circ\mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}] \\
&\quad \cap (\mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}\circ\mathfrak{z}])\circ(\Sigma\,\mathfrak{z} + (\mathfrak{z}\circ\mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}] \cap (\mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}\circ\mathfrak{z}]) \\
&\subseteq (\Sigma\,\mathfrak{z}\circ\mathfrak{z}\circ\mathfrak{I}\circ\mathfrak{z} + \Sigma\,\mathfrak{z}\circ\mathfrak{z}\circ\mathfrak{I}\circ\mathfrak{z}] \cap (\Sigma\,\mathfrak{z}\circ\mathfrak{I}\circ\mathfrak{z}\circ\mathfrak{z} + \Sigma\,\mathfrak{z}\circ\mathfrak{I}\circ\mathfrak{z}\circ\mathfrak{z}] \\
&\subseteq (\mathfrak{z}\circ\mathfrak{z}\circ\mathfrak{I}\circ\mathfrak{z} + \mathfrak{z}\circ\mathfrak{z}\circ\mathfrak{I}\circ\mathfrak{z}] \cap (\mathfrak{z}\circ\mathfrak{I}\circ\mathfrak{z}\circ\mathfrak{z} + \mathfrak{z}\circ\mathfrak{I}\circ\mathfrak{z}\circ\mathfrak{z}] \\
&\subseteq (\mathfrak{z}\circ\mathfrak{z}\circ\mathfrak{I}\circ\mathfrak{z}] \cap (\mathfrak{z}\circ\mathfrak{I}\circ\mathfrak{z}\circ\mathfrak{z}]
\end{aligned}$$

Thus, $\mathfrak{I}$ is a TQ-purity in $\mathcal{Z}$.

Theorem 3.9. *An* $\mathfrak{I}$ *in* $\mathcal{Z}$ *is (2,2) bi-purity if and only if* $\mathfrak{I} \cap \mathfrak{I}_1 = (\mathfrak{I}_1\circ\mathfrak{I}_1\circ\mathfrak{I}\circ\mathfrak{I}_1\circ\mathfrak{I}_1]$, *for every BID* $\mathfrak{I}_1$ *in* $\mathcal{Z}$.

Proof. Suppose that $\mathfrak{I}$ is (2,2) bi-purity. Let $\mathfrak{I}_1$ be a BID of $\mathcal{Z}$. If $\mathfrak{z} \in \mathfrak{I} \cap \mathfrak{I}_1$, then $\mathfrak{z} \in (\mathfrak{z}\circ\mathfrak{z}\circ\mathfrak{I}\circ\mathfrak{z}\circ\mathfrak{z}] \subseteq (\mathfrak{I}_1\circ\mathfrak{I}_1\circ\mathfrak{I}\circ\mathfrak{I}_1\circ\mathfrak{I}_1]$. So, $\mathfrak{I} \cap \mathfrak{I}_1 \subseteq (\mathfrak{I}_1\circ\mathfrak{I}_1\circ\mathfrak{I}\circ\mathfrak{I}_1\circ\mathfrak{I}_1]$. Clearly, $(\mathfrak{I}_1\circ\mathfrak{I}_1\circ\mathfrak{I}\circ\mathfrak{I}_1\circ\mathfrak{I}_1] \subseteq \mathfrak{I} \cap \mathfrak{I}_1$. Thus, $\mathfrak{I} \cap \mathfrak{I}_1 = (\mathfrak{I}_1\circ\mathfrak{I}_1\circ\mathfrak{I}\circ\mathfrak{I}_1\circ\mathfrak{I}_1]$.

Conversely, let $\mathfrak{z} \in \mathfrak{I}$ and Lemma 2.4, we get

$$\begin{aligned}
\mathfrak{z} \in \mathfrak{I} \cap \mathfrak{I}_1(\mathfrak{z}) &= (\mathfrak{I}_1(\mathfrak{z})\circ\mathfrak{I}_1(\mathfrak{z})\circ\mathfrak{I}\circ\mathfrak{I}_1(\mathfrak{z})\circ\mathfrak{I}_1(\mathfrak{z})] \\
&= ((\Sigma\,\mathfrak{z} + \Sigma\,\mathfrak{z}^2 + \mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}]\circ(\Sigma\,\mathfrak{z} + \Sigma\,\mathfrak{z}^2 + \mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}]\circ\mathfrak{I}\circ \\
&\quad (\Sigma\,\mathfrak{z} + \Sigma\,\mathfrak{z}^2 + \mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}]\circ(\Sigma\,\mathfrak{z} + \Sigma\,\mathfrak{z}^2 + \mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}]) \\
&\subseteq (\Sigma\,\mathfrak{z}\circ\mathfrak{z}\circ\mathfrak{I}\circ\mathfrak{z}\circ\mathfrak{z}] \\
&= (\mathfrak{z}\circ\mathfrak{z}\circ\mathfrak{I}\circ\mathfrak{z}\circ\mathfrak{z}]
\end{aligned}$$

Thus, $\mathfrak{I}$ is a bi-purity in $\mathcal{Z}$.

Theorem 3.10. *(i) if* $\mathfrak{z} \in (\Sigma\,\mathfrak{z}^2 + \mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}^2 + \Sigma\mathcal{Z}\circ\mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}^2]$ *for any* $\mathfrak{z} \in \mathcal{Z}$, *then* $\mathcal{Z}$ *is a strongly left regular. (ii) if* $\mathfrak{z} \in (\Sigma\,\mathfrak{z}^2 + \mathfrak{z}^2\circ\mathcal{Z}\circ\mathfrak{z} + \Sigma\,\mathfrak{z}^2\circ\mathcal{Z}\circ\mathfrak{z}\circ\mathcal{Z}]$ *for any* $\mathfrak{z} \in \mathcal{Z}$, *then* $\mathcal{Z}$ *is a strongly right regular.*

Proof. (i) Let $\mathfrak{z} \in \mathcal{Z}$. Suppose that $\mathfrak{z} \in (\Sigma\,\mathfrak{z}^2 + \mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}^2 + \Sigma\mathcal{Z}\circ\mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}^2] \subseteq (\mathcal{Z}\circ\mathfrak{z} + \mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z} + \Sigma\mathcal{Z}\circ\mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}^2]$.

We obtain that $\mathfrak{z}^2 \in ((\Sigma\,\mathfrak{z}^2 + \mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z} + \Sigma\mathcal{Z}\circ\mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}^2]\circ(\Sigma\,\mathfrak{z}^2 + \mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z} + \Sigma\mathcal{Z}\circ\mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}^2] \subseteq (\Sigma\mathcal{Z}\circ\mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}^2]$.

and $\mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}^2 \in (\mathcal{Z}\circ\mathfrak{z} + \mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z} + \Sigma\mathcal{Z}\circ\mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}^2]\circ\mathcal{Z}\circ(\Sigma\mathcal{Z}\circ\mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}^2] \subseteq (\Sigma\mathcal{Z}\circ\mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}^2]$.

Thus,

$$\begin{aligned}
&\mathfrak{z} \in (\Sigma\,\mathfrak{z}^2 + \mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}^2 + \Sigma\mathcal{Z}\circ\mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}^2] \\
&\subseteq (\Sigma(\Sigma\mathcal{Z}\circ\mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}^2] + (\Sigma\mathcal{Z}\circ\mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}^2] + (\Sigma(\Sigma\mathcal{Z}\circ\mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}^2] \\
&\subseteq ((\Sigma\mathcal{Z}\circ\mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}^2] + (\Sigma\mathcal{Z}\circ\mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}^2] + (\Sigma\mathcal{Z}\circ\mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}^2] \\
&\subseteq (\Sigma\mathcal{Z}\circ\mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}^2 + \Sigma\mathcal{Z}\circ\mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}^2 + \Sigma\mathcal{Z}\circ\mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}^2] \\
&= \Sigma\mathcal{Z}\circ\mathfrak{z}\circ\mathcal{Z}\circ\mathfrak{z}^2]
\end{aligned}$$

Hence, $\mathcal{Z}$ is a strongly left regular.

Theorem 3.11. *Let $\mathcal{Z}$ be an semiring and m, n are positive integers. Then*
(i) $\mathcal{Z}$ is strongly $(\natural_1,0)$ regular if and only if $\Bbbk = (\Bbbk^{\natural_1} \circ \Bbbk^{\natural_1} \circ \mathcal{Z} \circ \Bbbk^{\natural_1}\rfloor$, $\forall (\natural_1,0)$ TID $\Bbbk$ in $\mathcal{Z}$.
(ii) $\mathcal{Z}$ is strongly $(0,\natural_2)$ regular if and only if $\Psi = (\Psi^{\natural_2} \circ \mathcal{Z} \circ \Psi^{\natural_2} \circ \Psi^{\natural_2}\rfloor$, $\forall (0,\natural_2)$ TID Ψ in $\mathcal{Z}$.

Proof. Assume strongly $(\natural_1,0)$ regular. Let $\Bbbk$ be an $(\natural_1,0)$ TID. Let $\mathfrak{z} \in \Bbbk$, then $\mathfrak{z} \in (\mathfrak{z}^{\natural_1} \circ \mathfrak{z}^{\natural_1} \circ \mathcal{Z} \circ \mathfrak{z}^{\natural_1}\rfloor \subseteq (\Bbbk^{\natural_1} \circ \Bbbk^{\natural_1} \circ \mathcal{Z} \circ \Bbbk^{\natural_1}\rfloor$ implies $\Bbbk \subseteq (\Bbbk^{\natural_1} \circ \Bbbk^{\natural_1} \circ \mathcal{Z} \circ \Bbbk^{\natural_1}\rfloor$. Clearly, $(\Bbbk^{\natural_1} \circ \Bbbk^{\natural_1} \circ \mathcal{Z} \circ \Bbbk^{\natural_1}\rfloor \subseteq \Bbbk$. Thus $(\Bbbk^{\natural_1} \circ \Bbbk^{\natural_1} \circ \mathcal{Z} \circ \Bbbk^{\natural_1}\rfloor = \Bbbk$.

$$
\begin{aligned}
\mathfrak{z} &\in [\mathfrak{z}\rfloor_{(\natural_1,0)} \\
&= \left(([\mathfrak{z}\rfloor_{(\natural_1,0)})^{\natural_1} \circ ([\mathfrak{z}\rfloor_{(\natural_1,0)})^{\natural_1} \circ \mathcal{Z} \circ ([\mathfrak{z}\rfloor_{(\natural_1,0)})^{\natural_1}\right\rfloor \\
&= \Big(\big(\Sigma\, \mathfrak{z} + \Sigma\, \mathfrak{z}^2 + \ldots + \Sigma\, \mathfrak{z}^{2\natural_1} + \mathfrak{z}^{\natural_1} \circ \mathfrak{z}^{\natural_1} \circ \mathcal{Z} \circ \mathfrak{z}^{\natural_1}\big)^{\natural_1} \circ \\
&\quad \big(\Sigma\, \mathfrak{z} + \Sigma\, \mathfrak{z}^2 + \ldots + \Sigma\, \mathfrak{z}^{2\natural_1} + \mathfrak{z}^{\natural_1} \circ \mathfrak{z}^{\natural_1} \circ \mathcal{Z} \circ \mathfrak{z}^{\natural_1}\big)^{\natural_1} \\
&\quad \circ \mathcal{Z} \circ \big(\Sigma\, \mathfrak{z} + \Sigma\, \mathfrak{z}^2 + \ldots + \Sigma\, \mathfrak{z}^{2\natural_1} + \mathfrak{z}^{\natural_1} \circ \mathfrak{z}^{\natural_1} \circ \mathcal{Z} \circ \mathfrak{z}^{\natural_1}\big)^{\natural_1}\Big\rfloor \\
&\subseteq \Big(\big(\Sigma\, \mathfrak{z}^{2\natural_1} + \Sigma\, \mathfrak{z}^{\natural_1} \circ \mathfrak{z}^{\natural_1} \circ \mathcal{Z} \circ \mathfrak{z}^{\natural_1}\big) \circ \big(\Sigma\, \mathfrak{z}^{2\natural_1} + \Sigma\, \mathfrak{z}^{\natural_1} \circ \mathfrak{z}^{\natural_1} \circ \mathcal{Z} \circ \mathfrak{z}^{\natural_1}\big) \circ \mathcal{Z} \circ \\
&\quad (\Sigma\, \mathfrak{z}^{2\natural_1} + \Sigma\, \mathfrak{z}^{\natural_1} \circ \mathfrak{z}^{\natural_1} \circ \mathcal{Z} \circ \mathfrak{z}^{\natural_1})\Big\rfloor \\
&\subseteq \Big(\Sigma\, \mathfrak{z}^{\natural_1} \circ \mathfrak{z}^{\natural_1} \circ \mathcal{Z} \circ \mathfrak{z}^{\natural_1} + \Sigma\, \mathfrak{z}^{\natural_1} \circ \mathfrak{z}^{\natural_1} \circ \mathcal{Z} \circ \mathfrak{z}^{\natural_1} + \Sigma\, \mathfrak{z}^{\natural_1} \circ \mathfrak{z}^{\natural_1} \circ \mathcal{Z} \circ \mathfrak{z}^{\natural_1} \\
&\quad + \Sigma\, \mathfrak{z}^{\natural_1} \circ \mathfrak{z}^{\natural_1} \circ \mathcal{Z} \circ \mathfrak{z}^{\natural_1} + \Sigma\, \mathfrak{z}^{\natural_1} \circ \mathfrak{z}^{\natural_1} \circ \mathcal{Z} \circ \mathfrak{z}^{\natural_1} + \Sigma\, \mathfrak{z}^{\natural_1} \circ \mathfrak{z}^{\natural_1} \circ \mathcal{Z} \circ \mathfrak{z}^{\natural_1}\Big\rfloor \\
&\subseteq \Big(\Sigma\, \mathfrak{z}^{\natural_1} \circ \mathfrak{z}^{\natural_1} \circ \mathcal{Z} \circ \mathfrak{z}^{\natural_1} \\
&= \Big(\mathfrak{z}^{\natural_1} \circ \mathfrak{z}^{\natural_1} \circ \mathcal{Z} \circ \mathfrak{z}^{\natural_1}
\end{aligned}
$$

Hence $\mathcal{Z}$ is a strongly $(\natural_1,0)$ regular. Similarly, $\mathcal{Z}$ is a strongly $(0,\natural_2)$ regular

Theorem 3.12. *Let m, n are positive integers. Then $\mathcal{Z}$ is a strongly $(\natural_1,\natural_2)$ regular if and only if $\Bbbk \cap \Psi = (\Bbbk^{\natural_1} \circ \Psi^{\natural_2}\rfloor, \forall (\natural_1,0)$ TID $\Bbbk$ and $(0,\natural_2)$ TID ψ in $\mathcal{Z}$.*

Proof. If $(\natural_1 \neq 0, \natural_2 = 0)$ and $(\natural_1 = 0, \natural_2 \neq 0)$ is directly from Theorem 3.11. We assume that $\natural_1 \neq 0$ and $\natural_2 \neq 0$. Assume strongly $(\natural_1,\natural_2)$ regular. Let $\Bbbk$ and ψ be an $(\natural_1,0)$ TID, $(0,\natural_2)$ TID in $\mathcal{Z}$, respectively. Let $\mathfrak{z} \in \Bbbk \cap \Psi$, then $\mathfrak{z} \in (\mathfrak{z}^{2\natural_1} \circ \mathcal{Z} \circ \mathfrak{z}^{\underline{2\natural_1}}\rfloor = (\mathfrak{z}^{2\natural_1} \circ \mathcal{Z} \circ \mathfrak{z}^{\underline{\natural_1}}\rfloor \circ (\mathfrak{z}^{\natural_2} \circ \mathcal{Z} \circ \mathfrak{z}^{\underline{2\natural_2}}\rfloor \subseteq (\Bbbk^{\natural_1} \circ \Psi^{\natural_2}\rfloor$, implies $\Bbbk \cap \Psi \subseteq (\Bbbk^{\natural_1} \circ \Psi^{\natural_2}\rfloor$. Clearly, $(\Bbbk^{\natural_1} \circ \Psi^{\underline{\natural_2}}\rfloor \subseteq \Bbbk \cap \Psi$. Thus, $\Bbbk \cap \Psi = (\Bbbk^{\natural_1} \circ \Psi^{\underline{\natural_2}}\rfloor$.

Conversely, $\mathfrak{z} \in \mathcal{Z}$,

$$
\begin{aligned}
\mathfrak{z} &\in [\mathfrak{z}\rfloor_{(\natural_1,0)} \cap [\mathfrak{z}\rfloor_{(0,\natural_2)} \\
&= \Big(\big([\mathfrak{z}\rfloor_{(\natural_1,0)}\big)^{\natural_1} \circ \big([\mathfrak{z}\rfloor_{(0,\natural_2)}\big)^{\underline{\natural_2}}\Big\rfloor \\
&= ((\Sigma\, \mathfrak{z} + \Sigma\, \mathfrak{z}^2 + \ldots + \Sigma\, \mathfrak{z}^{2\natural_1} + \mathfrak{z}^{2\natural_1} \circ \mathcal{Z} \circ \mathfrak{z}^{\natural_1})^{\natural_1} \circ \\
&\quad ((\Sigma\, \mathfrak{z} + \Sigma\, \mathfrak{z}^2 + \ldots + \Sigma\, \mathfrak{z}^{2\natural_2} + \mathfrak{z}^{\natural_2} \circ \mathcal{Z} \circ \mathfrak{z}^{2\natural_2})^{\underline{\natural_2}}\Big\rfloor \\
&\subseteq ((\Sigma\, \mathfrak{z} + \Sigma\, \mathfrak{z}^2 + \ldots + \Sigma\, \mathfrak{z}^{2\natural_1} + \mathfrak{z}^{2\natural_1} \circ \mathcal{Z} \circ \mathfrak{z}^{\natural_1})^{\natural_1} \circ \\
&\quad \big(\Sigma\, \mathfrak{z} + \Sigma\, \mathfrak{z}^2 + \ldots + \Sigma\, \mathfrak{z}^{2\natural_2} + \mathfrak{z}^{\natural_2} \circ \mathcal{Z} \circ \mathfrak{z}^{2\natural_2}\big)^{\underline{\natural_2}}\Big\rfloor \\
&\subseteq \Big(\big(\Sigma\, \mathfrak{z}^{2\natural_1} + \mathfrak{z}^{2\natural_1} \circ \mathcal{Z} \circ \mathfrak{z}^{\natural_1}\big) \circ \big(\Sigma\, \mathfrak{z}^{2\natural_2} + \mathfrak{z}^{\natural_2} \circ \mathcal{Z} \circ \mathfrak{z}^{2\natural_2}\big)\Big) \\
&\subseteq \Big(\Sigma\, \mathfrak{z}^{2\natural_1+2\natural_2} + \mathfrak{z}^{2\natural_1} \circ \mathcal{Z} \circ \mathfrak{z}^{\natural_1} \circ \mathfrak{z}^{\natural_2} \circ \mathcal{Z} \circ \mathfrak{z}^{\underline{2\natural_2}}\Big\rfloor \\
&\subseteq \Big(\mathfrak{z}^{2\natural_1} \circ \mathcal{Z} \circ \mathfrak{z}^{\underline{2\natural_2}}\Big\rfloor
\end{aligned}
$$

Hence $\mathcal{Z}$ is a strongly $(\flat_1,\flat_2)$ regular.

Theorem 3.13. *$\mathcal{Z}$ is strongly $(\flat_1,\flat_2)$ regular if and only if every TID of $\mathcal{Z}$ is a LT-purity (RT-purity).*

Proof. Assume that $\mathcal{Z}$ is a strongly $(\flat_1,\flat_2)$ regular, $\flat_1 \geq 2$ and $\flat_2 \geq 4$. Let Υ be an ID in $\mathcal{Z}$ and let $\mathfrak{z} \in \Upsilon$. By assumption, $\mathfrak{z} \in (\Sigma\, \mathfrak{z} \circ \mathfrak{z} \circ \mathcal{Z} \circ \mathfrak{z} \circ \mathfrak{z} \circ \mathfrak{z} \circ \underline{\mathfrak{z}}\rfloor \subseteq (\Sigma\, \mathfrak{z} \circ \Upsilon \circ \mathcal{Z} \circ \Upsilon \circ \Upsilon \circ \mathfrak{z} \circ \underline{\mathfrak{z}}\rfloor \subseteq (\Sigma\, \mathfrak{z} \circ \Upsilon \circ \mathfrak{z} \circ \underline{\mathfrak{z}}\rfloor = (\mathfrak{z} \circ \Upsilon \circ \mathfrak{z} \circ \underline{\mathfrak{z}}\rfloor$. Thus, Υ is a LT-purity.

Conversely, $\mathfrak{z} \in \mathcal{Z}$, by assumption, we obtain that $\mathfrak{I}(\mathfrak{z})$ is a LT-purity. Using Lemma 2.4, we have

$$
\begin{aligned}
&\mathfrak{z} \in \mathfrak{I}(\mathfrak{z}) \cap L(\mathfrak{z}) \\
&= \mathfrak{I}(\mathfrak{z}) \circ \mathfrak{I}(\mathfrak{z}) \circ L(\mathfrak{z}) \circ \mathfrak{I}(\mathfrak{z}) \\
&= (\Sigma\, \mathfrak{z} + \Sigma\, \mathfrak{z} \circ \mathcal{Z} \circ \mathfrak{z} \circ \mathfrak{z} + \Sigma\, \mathfrak{z} \circ \mathfrak{z} \circ \mathcal{Z} \circ \underline{\mathfrak{z}}\rfloor \circ (\Sigma\, \mathfrak{z} + \Sigma\, \mathfrak{z} \circ \mathcal{Z} \circ \mathfrak{z} \circ \mathfrak{z} + \Sigma\, \mathfrak{z} \circ \mathfrak{z} \circ \mathcal{Z} \circ \underline{\mathfrak{z}}\rfloor \\
&\quad (\Sigma\, \mathfrak{z} + \Sigma\, \mathfrak{z} \circ \mathcal{Z} \circ \mathfrak{z} \circ \underline{\mathfrak{z}}\rfloor \circ (\Sigma\, \mathfrak{z} + \Sigma\, \mathfrak{z} \circ \mathcal{Z} \circ \mathfrak{z} \circ \mathfrak{z} + \Sigma\, \mathfrak{z} \circ \mathfrak{z} \circ \mathcal{Z} \circ \underline{\mathfrak{z}}\rfloor \\
&= ((\Sigma\, \mathfrak{z} + \Sigma\, \mathfrak{z} \circ \mathcal{Z} \circ \mathfrak{z} \circ \mathfrak{z} + \Sigma\, \mathfrak{z} \circ \mathfrak{z} \circ \mathcal{Z} \circ \mathfrak{z}) \circ (\Sigma\, \mathfrak{z} + \Sigma\, \mathfrak{z} \circ \mathcal{Z} \circ \mathfrak{z} \circ \mathfrak{z} + \Sigma\, \mathfrak{z} \circ \mathfrak{z} \circ \mathcal{Z} \circ \mathfrak{z}) \\
&\quad \left(\Sigma\, \mathfrak{z} + \Sigma\, \mathfrak{z} \circ \mathcal{Z} \circ \mathfrak{z} \circ \mathfrak{z}\right) \circ \left(\Sigma\, \mathfrak{z} + \Sigma\, \mathfrak{z} \circ \mathcal{Z} \circ \mathfrak{z} \circ \mathfrak{z} + \Sigma\, \mathfrak{z} \circ \mathfrak{z} \circ \mathcal{Z} \circ \mathfrak{z}\right) \\
&\subseteq (\Sigma\, \mathfrak{z}^{\flat_1} \circ \mathcal{Z} \circ \underline{\mathfrak{z}^{\flat_2}}\rfloor \\
&\subseteq (\mathfrak{z}^{\flat_1} \circ \mathcal{Z} \circ \underline{\mathfrak{z}^{\flat_2}}\rfloor
\end{aligned}
$$

Hence $\mathcal{Z}$ is a strongly $(\flat_1,\flat_2)$ regular.

Corollary 3.14. *Let $\mathcal{Z}$ be the ordered semiring. Then*

(i) $[\underline{\mathfrak{I}}\rfloor_{(\flat_1,\flat_2)} = (\Sigma\mathfrak{I} + \Sigma\mathfrak{I}^2 + \ldots + \Sigma\mathfrak{I}^{2\flat_1} + \Sigma\mathfrak{I}^{2\flat_1} \circ \mathcal{Z} \circ \mathfrak{I}^{\flat_1}\underline{\rfloor}, \forall$ *non-negative integers* $\flat_1$. *To prove that* $[\underline{\mathfrak{I}}\rfloor_{(\flat_1,\flat_2)}$ *to the smallest* $(\flat_1,0)$ *TID in* $\mathcal{Z}$ *containing* $\mathfrak{I}$.

(ii) $[\underline{\mathfrak{I}}\rfloor_{(\flat_1,\flat_2)} = (\Sigma\mathfrak{I} + \Sigma\mathfrak{I}^2 + \ldots + \Sigma\mathfrak{I}^{2\flat_2} + \Sigma\mathfrak{I}^{\flat_2} \circ \mathcal{Z} \circ \mathfrak{I}^{2\flat_2}\underline{\rfloor}, \forall$ *non-negative integers* $\flat_2$. *To prove that* $[\underline{\mathfrak{I}}\rfloor_{(\flat_1,\flat_2)}$ *to the smallest* $(0,\flat_2)$ *TID in* $\mathcal{Z}$ *containing* $\mathfrak{I}$.

Theorem 3.15. *Every TID is a tri bi-purity of a strongly* $\ddot{\mathfrak{v}}_{\flat_1\ \flat_2}$ *regular.*

Proof. Assume that $\mathcal{Z}$ is a strongly $(\flat_1,\flat_2)$ regular. Let Υ be an TID in $\mathcal{Z}$ and $\zeta_2 \in \Upsilon$. We have

$$
\begin{aligned}
&\zeta_2 \in (\zeta_2^{2\flat_1} \circ \mathcal{Z} \circ \underline{\zeta_2^{2\flat_2}}\rfloor \\
&\subseteq (\zeta_2^{2\flat_1} \circ \mathcal{Z} \circ \underline{\zeta_2^{2\flat_2}}\rfloor \\
&\subseteq (\zeta_2^{2\flat_1} \circ \mathcal{Z} \circ (\zeta_2^{2\flat_1} \circ \mathcal{Z} \circ \zeta_2^{2\flat_2})^{2\flat_2}\rfloor \\
&\subseteq (\zeta_2^{2\flat_1} \circ \mathcal{Z} \circ (\zeta_2^{2\flat_1})^{2\flat_2} \circ \mathcal{Z}^{2\flat_2} \circ (\zeta_2^{2\flat_2})^{2\flat_2}\rfloor \\
&\subseteq (\zeta_2^{2\flat_1} \circ \mathcal{Z} \circ \zeta_2^{2(\flat_1+\flat_2)} \circ \mathcal{Z} \circ (\zeta_2^{2(n+\flat_2)}\rfloor \\
&\subseteq (\zeta_2^{2\flat_1} \circ \mathcal{Z} \circ \zeta_2^{2\flat_1} \circ \mathcal{Z} \circ \zeta_2^{2\flat_2}\rfloor \\
&\subseteq (\zeta_2 \circ \zeta_2 \circ \zeta_2 \circ \zeta_2 \circ \mathcal{Z} \circ \zeta_2 \circ \mathcal{Z} \circ \zeta_2 \circ \zeta_2 \circ \zeta_2 \circ \underline{\zeta_2}\rfloor \\
&\subseteq (\zeta_2 \circ \zeta_2 \circ \zeta_2 \circ \mathcal{Z} \circ \Upsilon \circ \mathcal{Z} \circ \zeta_2 \circ \zeta_2 \circ \underline{\zeta_2}\rfloor \\
&\subseteq (\zeta_2 \circ \zeta_2 \circ \Upsilon \circ \mathcal{Z} \circ \Upsilon \circ \mathcal{Z} \circ \Upsilon \circ \zeta_2 \circ \underline{\zeta_2}\rfloor, \textit{Since BID is a TID} \\
&\subseteq (\zeta_2 \circ \zeta_2 \circ \Upsilon \circ \zeta_2 \circ \underline{\zeta_2}\rfloor
\end{aligned}
$$

Thus, Υ is a tri bi-purity.

4. Characterization of ordered TIDs

Definition 4.1. *Let m_1, m_2 be non-negative integers. The set* $\top^{lcm(m_1,m_2)} \Cup \top^{m_1} \circ \mathcal{Z} \circ \top^{m_2}$ *is a* (m_1, m_2)*-BID in* $\mathcal{Z}$ *if* $\top \subseteq \mathcal{Z}$.

Definition 4.2. *If* $\top \neq \{0\}$ *and* $\{0\}$ *are the only* (2, 2)*-BID in* $\mathcal{Z}$ *that is correctly included in* $\top$*, then a* (2, 2)*-BID* $\top$ *in* $\mathcal{Z}$ *with a zero element* 0 *will be considered* 0*-minimal.*

Theorem 4.3. *Let* $\top \subseteq \mathcal{Z}$*,* $\top$ *is a* (2,4)*-BID iff* $\top$ *is a LTID of certain LTID in* $\mathcal{Z}$*.*

Proof. Suppose that $\top$ is a (2, 4)-BID in $\mathcal{Z}$. Now, $(\top \circ(\top \cup \top \circ \mathcal{Z}^{\circ} \top \circ \top]^{\circ} \top \circ \top] \subseteq (\top^{4} \cup \circ \top^{2} \circ \mathcal{Z}^{\circ} \top^{4}] \subseteq (\top] = \top$ and $((\top]] = (\top]$. Therefore, $\top$ is a LTID of the LTID $(\top \cup \top \circ \mathcal{Z}^{\circ} \top \circ \top]$ in $\mathcal{Z}$.

Conversely, Suppose that $\top$ is a LTID of a LTID $\Bbbk$ in $\mathcal{Z}$. Now,

$$\begin{aligned}(\top^{2} \circ \mathcal{Z}^{\circ} \top^{4}] &\subseteq (\top \circ \top \circ \mathcal{Z}^{\circ} \top \circ \top \circ \top \circ \top] \\ &\subseteq (\top \circ (Q^{\circ} \mathcal{Z}^{\circ} Q^{\circ} Q)^{\circ} \top \circ \top] \\ &\subseteq (\top \circ Q^{\circ} \top \circ \top] \\ &\subseteq (\top].\end{aligned}$$

Let $\kappa \in \top$, $\psi \in \mathcal{Z}$ such that $\psi \preceq \kappa$. Since, $\kappa \in \Bbbk$, thus $\psi \in \Bbbk$.
The assumption applies $\psi \in \top$. Therefore, $\top$ is a (2,4)-BID in $\mathcal{Z}$.

Theorem 4.4. *Let* $\top \subseteq \mathcal{Z}$*,* $\top$ *is a* (4,2)*-BID iff* $\top$ *is a RTID of certain RTID in* $\mathcal{Z}$*.*

Proof. Suppose that, $\top$ is a (4,2)-BID in $\mathcal{Z}$. Now, $(\top \circ \top \circ(\top \cup \top \circ \top \circ \mathcal{Z}^{\circ} \top]^{\circ} \top] \subseteq (\top^{4} \cup \circ \top^{4} \circ \mathcal{Z}^{\circ} \top^{2}] \subseteq (\top] = \top$ and $((\top]] = (\top]$. Therefore, $\top$ is a RTID of the RTID $(\top \cup \top \circ \top \circ \mathcal{Z}^{\circ} \top]$ in $\mathcal{Z}$.

Conversely, Suppose that $\top$ is a RTID of a RTID ψ in $\mathcal{Z}$. Now,

$$\begin{aligned}(\top^{4} \circ \mathcal{Z}^{\circ} \top^{2}] &\subseteq (\top \circ \top \circ \top \circ \top \circ \mathcal{Z}^{\circ} \top \circ \top] \\ &\subseteq (\top \circ \top \circ (Q^{\circ} Q^{\circ} \mathcal{Z}^{\circ} Q)^{\circ} \top] \\ &\subseteq (\top \circ \top \circ Q^{\circ} \top] \\ &\subseteq (\top].\end{aligned}$$

Let $\kappa \in \top$, $\psi \in \mathcal{Z}$ such that $\psi \preceq \kappa$. Since, $\kappa \in \Psi$, thus $\psi \in \Bbbk$ implies $\psi \in \top$. Therefore, $\top$ is a (4,2)-BID in $\mathcal{Z}$.

Theorem 4.5. *Let* $\top \subseteq \mathcal{Z}$*. Then the statements are equivalent.*

(a) $\top$ *is a* (4,4)*-BID in* $\mathcal{Z}$
(b) $\top$ *is a LTID of certain BID in* $\mathcal{Z}$
(c) $\top$ *is a BID of certain LTID in* $\mathcal{Z}$
(d) $\top$ *is a* (2,3)*-BID of certain RTID in* $\mathcal{Z}$
(e) $\top$ *is a RTID of certain* (2,3)*-BID of* $\mathcal{Z}$*.*

Proof. (a) $\Rightarrow$ (b). If $\top$ is a (4,4)-BID in $\mathcal{Z}$, then

$$\begin{aligned}(\top \circ(\top^{4} \cup \top^{4} \circ \mathcal{Z}^{\circ} \top^{2}]^{\circ} \top \circ \top] &\subseteq (\top^{7} \cup \top^{5} \circ \mathcal{Z}^{\circ} \top^{4}] \\ &\subseteq (\top^{4} \cup \top^{4} \circ \mathcal{Z}^{\circ} \top^{4}] \\ &\subseteq (\top] \\ &\subseteq \top\end{aligned}$$

Let $\kappa \in \top$, $\psi \in (\top^{4} \cup \top^{4} \circ \mathcal{Z}^{\circ} \top^{2}]$ such that $\psi \preceq \kappa$ then $\psi \in \top$.
Therefore, $\top$ is a LTID of the BID $(\top^{4} \cup \top^{4} \circ \mathcal{Z}^{\circ} \top^{2}]$ in $\mathcal{Z}$.

(b) $\Rightarrow$ (c). If $\top$ LTID of certain BID in $\mathcal{Z}$. Since, $\top \cup \top \circ \mathcal{Z}^{\circ} \top \circ \top$ is a LTID in $\mathcal{Z}$. Now,

$$\begin{aligned}(\top^{\circ}(\top\uplus\top^{\circ}\mathcal{Z}^{\circ}\top^{\circ}\top]^{\circ}\top] &\subseteq (\top^{\ddot{u}}\uplus\top^{\circ}\mathcal{Z}^{\circ}\top] \\ &\subseteq (\top\uplus\top^{\circ}(Q^{\circ}\mathcal{Z}^{\circ}Q)^{\circ}\top^{\circ}\top] \\ &\subseteq (\top\uplus\top^{\circ}Q^{\circ}\top^{\circ}\top] \\ &\subseteq (\top] \\ &\subseteq \top\end{aligned}$$

Let $\kappa\in\top$, $\psi\in(\top\uplus\top^{\circ}\mathcal{Z}^{\circ}\top^{\circ}\top]$ such that $\psi\preceq\kappa$. Since $\kappa\in\top$ $\kappa\in$. Thus $\psi\in Q$, so $\psi\in\top$. Therefore, $\top$ is a BID of the LTID $(\top\uplus\top^{\circ}\mathcal{Z}^{\circ}\top^{\circ}\top]$ in $\mathcal{Z}$.

(c) ⇒ (d). If $\top$ is a BID of certain LTID in $\mathcal{Z}$. Since, $\top\uplus\top^{\circ}\top^{\circ}\mathcal{Z}^{\circ}\top$ is a RTID in $\mathcal{Z}$. Now,

$$\begin{aligned}(\top^{2}\circ(\top\uplus\top^{\circ}\top\circ\mathcal{Z}^{\circ}\top]^{\circ}\top^{3}] &\subseteq (\top^{6}\uplus\top^{4}\circ\mathcal{Z}^{\circ}\top^{4}] \\ &\subseteq (\top\uplus\top^{\circ}\Bbbk^{3}\circ\mathcal{Z}^{\circ}\Bbbk^{3}\circ\top] \\ &\subseteq (\top\uplus\top^{\circ}(\Bbbk^{\circ}\mathcal{Z}^{\circ}\Bbbk^{\circ}\Bbbk)^{\circ}\top] \\ &\subseteq (\top\uplus\top^{\circ}\Bbbk^{\circ}\top] \\ &\subseteq (\top] \\ &\subseteq \top\end{aligned}$$

Let $\kappa\in\top$, $\psi\in(\top\uplus\top^{\circ}\top^{\circ}\mathcal{Z}^{\circ}\top]$ such that $\psi\preceq\kappa$, then $\kappa\in\Bbbk$. Thus $\psi\in\Bbbk$, so $\psi\in\top$.
Therefore, $\top$ is a (2,3)-BID of the RTID $(\top\uplus\top^{\circ}\top^{\circ}\mathcal{Z}^{\circ}\top]$ in $\mathcal{Z}$.

(d) ⇒ (e). If $\top$ is a (2,3)-BID of certain RTID in $\mathcal{Z}$. Since, $\top\uplus\top^{\circ}\top^{\circ}\mathcal{Z}^{\circ}\top$ is a RTID in $\mathcal{Z}$. Now,

$$\begin{aligned}(\top^{\circ}\top^{\circ}(\top\uplus\top^{2}\circ\mathcal{Z}^{\circ}\top^{3}]^{\circ}\top] &\subseteq (\top^{4}\uplus\top^{4}\circ\mathcal{Z}^{\circ}\top^{4}] \\ &\subseteq (\top\uplus\top^{\circ}\top^{\circ}(\top^{2}\circ\mathcal{Z}^{\circ}\top^{3})^{\circ}\top] \\ &\subseteq (\top\uplus\top^{2}\circ(\Psi^{\circ}\Psi^{\circ}\mathcal{Z}^{\circ}\Psi)^{\circ}\top^{3}] \\ &\subseteq (\top\uplus(\top^{2}\circ\Psi^{\circ}\top^{3}] \\ &\subseteq (\top] \\ &\subseteq \top\end{aligned}$$

Let $\kappa\in\top$,$\psi\in(\top\uplus\top^{2}\circ\mathcal{Z}^{\circ}\top^{3}]$ such that $\psi\preceq\kappa$, then $\kappa\in\Psi$. Thus $\psi\in\Psi$, so $\psi\in\top$.

Therefore, $\top$ is a RTID of the (2,3)-BID in $\mathcal{Z}$.

(e) ⇒ (a). If $\top$ is a RTID of certain (2,3)-BID in $\mathcal{Z}$. Now,

$$\begin{aligned}(\top^{4}\circ\mathcal{Z}^{\circ}\top^{4}] &\subseteq (\top^{\circ}\top^{\circ}(\Psi^{2}\circ\mathcal{Z}^{\circ}\Psi^{3})^{\circ}\top] \\ &\subseteq (\top^{\circ}\top^{\circ}\Psi^{\circ}\top] \\ &\subseteq (\top] \\ &\subseteq \top\end{aligned}$$

Assume that $\kappa\in\top$, $\psi\in\mathcal{Z}$ such that $\psi\preceq\kappa$. Since, $\kappa\in\Psi$, so $\psi\in\Psi$, thus $\psi\in\top$.
Therefore, $\top$ is a (4,4)-BID in $\mathcal{Z}$.

Lemma 4.6. *Let $\top$ be a SSG in $\mathcal{Z}$ such that $\top=(\top]$. Then $\top$ is a (2,2)-BID in $\mathcal{Z}$ iff there exist a* (2,1)*-BID ψ in $\mathcal{Z}$ and* (1,2)*-BID in $\mathcal{Z}$ such that $\Psi^{2}\circ\Bbbk^{2}\subseteq\top\subseteq\Psi^{2}\Cap\Bbbk^{2}$.*

Proof. Suppose that, $\top$ is a (2,2)-BID in $\mathcal{Z}$. Now, $\top^{2}\uplus\top^{2}\circ\mathcal{Z}^{\circ}\top$ and $\top^{2}\uplus\top\circ\mathcal{Z}^{\circ}\top^{2}$ are RTID and LTID in $\mathcal{Z}$, respectively. Now $\Psi=\top^{2}\uplus\top^{2}\circ\mathcal{Z}^{\circ}\top$ and $\Bbbk=\top^{2}\uplus\top\circ\mathcal{Z}^{\circ}\top^{2}$. Now,

$$\begin{aligned}\boldsymbol{\Psi}^2\circ\Bbbk^2 &\subseteq (\top^2\Cup\top^2\circ\mathcal{Z}^\circ\top)^2\circ(\top^2\Cup\top\circ\mathcal{Z}^\circ\top^2)^2]\\ &\subseteq (\top^2\Cup\top^2\circ\mathcal{Z}^\circ\top^2]\\ &\subseteq (\top]\\ &\subseteq \top\end{aligned}$$

Conversely, let ψ be a RTID and $\Bbbk$ be a LTID in $\mathcal{Z}$ such that $\boldsymbol{\Psi}^2\circ\Bbbk^2\subseteq\top\subseteq\boldsymbol{\Psi}^2\Cap\Bbbk^2$. Then,

$$\begin{aligned}\top^2\circ\mathcal{Z}^\circ\top^2 &\subseteq (\boldsymbol{\Psi}^2\Cap\Bbbk^2)^2\circ\mathcal{Z}^\circ(\boldsymbol{\Psi}^2\Cap\Bbbk^2)^2\\ &= (\boldsymbol{\Psi}^2\Cap\Bbbk^2)^\circ(\boldsymbol{\Psi}^2\Cap\Bbbk^2)^\circ\mathcal{Z}^\circ(\boldsymbol{\Psi}^2\Cap\Bbbk^2)^\circ(\boldsymbol{\Psi}^2\Cap\Bbbk^2)\\ &\subseteq \boldsymbol{\Psi}^2\circ\mathcal{Z}^\circ\mathcal{Z}^\circ\boldsymbol{\Psi}^2\circ\Bbbk^2\\ &\subseteq \boldsymbol{\Psi}^\circ\boldsymbol{\Psi}^\circ\mathcal{Z}^\circ\mathcal{Z}^\circ\boldsymbol{\Psi}^\circ\boldsymbol{\Psi}^\circ\Bbbk^2\\ &\subseteq (\boldsymbol{\Psi}^\circ\boldsymbol{\Psi}^\circ\mathcal{Z}^\circ\boldsymbol{\Psi})^\circ\boldsymbol{\Psi}^\circ\Bbbk^2\\ &\subseteq \boldsymbol{\Psi}^2\circ\Bbbk^2\\ &\subseteq \top\end{aligned}$$

Therefore, $\top$ is (2,2)-BID in $\mathcal{Z}$.

Theorem 4.7. *Suppose that* $\Bbbk$ *be a 0-minimal LTID in* $\mathcal{Z}$ *and* $\top$ *is a SSG of* $\Bbbk$ *such that* $(\top]=\top$. *Then* $\top$ *is a (2,2)-BID in* $\mathcal{Z}$ *contained in* $\Bbbk$ *iff* $(\top^2]=\{0\}$ *or* $\top=\Bbbk$.

Proof. Suppose that $\top$ is a (2,2)-BID in $\mathcal{Z}$ contained in $\Bbbk$. Then, $(\top^2\circ\mathcal{Z}^\circ\top^2]\subseteq\Bbbk$. Since $(\top^2\circ\mathcal{Z}^\circ\top^2]$ is a LTID in $\mathcal{Z}$. Now, $(\top^2\circ\mathcal{Z}^\circ\top^2]^\circ\mathcal{Z}^\circ(\top^2\circ\mathcal{Z}^\circ\top^2\circ(\top^2\circ\mathcal{Z}^\circ\top^2]\subseteq(\top^2\circ\mathcal{Z}^\circ\top^2\circ\mathcal{Z}^\circ\top^2\circ\mathcal{Z}^\circ\top^2\circ\top^2\circ\mathcal{Z}^\circ\top^2]\subseteq(\top^2\circ\mathcal{Z}^\circ\top^2]$. Now, $(\top^2\circ\mathcal{Z}^\circ\top^2]=\{0\}$ or $(\top^2\circ\mathcal{Z}^\circ\top^2]=\Bbbk$.

If $(\top^2\circ\mathcal{Z}^\circ\top^2]=\Bbbk$, then $\Bbbk=(\top^2\circ\mathcal{Z}^\circ\top^2]\subseteq(\top]$. Therefore, $\top=\Bbbk$. Let $(\top^2\circ\mathcal{Z}^\circ\top^2]=\{0\}$. Since

$$\begin{aligned}(\top^2]^\circ\mathcal{Z}^\circ(\top^2]^\circ(\top^2] &\subseteq (\top^2\circ\mathcal{Z}^\circ\top^2\circ\top^2]\\ &\subseteq (\top^2\circ\mathcal{Z}^\circ\top^2]\\ &= \{0\}\\ &\subseteq (\top^2].\end{aligned}$$

It follows that $(\top^2]$ is a LTID in $\mathcal{Z}$ contained in $\Bbbk$.

By the minimality of $\Bbbk$, $(\top^2]=\{0\}$ or $(\top^2]=\Bbbk$.

If $(\top^2]=\Bbbk$, then $\top=\Bbbk$. Conversely, other part is a direct proof.

Theorem 4.8. *Let* $\Bbbk$ *be a 0-minimal (2,2)-BID* $\mathcal{Z}$. *Then* $(\Bbbk^2]=\{0\}$ *or* $\Bbbk$ *is a minimal LTID in* $\mathcal{Z}$.

Proof. Now,

$$\begin{aligned}(\Bbbk^2]^2\circ\mathcal{Z}^\circ(\Bbbk^2]^2 &= (\Bbbk^2]^\circ(\Bbbk^2]^\circ\mathcal{Z}^\circ(\Bbbk^2]^\circ(\Bbbk^2]\\ &\subseteq (\Bbbk^2]^\circ(\Bbbk^2\circ\mathcal{Z}^\circ\Bbbk^2]^\circ(\Bbbk^2]\\ &\subseteq (\Bbbk^4]^\circ\mathcal{Z}^\circ\Bbbk^2]^\circ(\Bbbk^2]\\ &\subseteq (\Bbbk^2]^\circ\mathcal{Z}^\circ\Bbbk^2]^\circ(\Bbbk^2]\\ &\subseteq (\Bbbk]^\circ(\Bbbk^2]\\ &\subseteq (\Bbbk^2]\end{aligned}$$

Then, $(\Bbbk^2]$ is a (2,2)-BID in $\mathcal{Z}$ contained in $\Bbbk$.
Therefore, $(\Bbbk^2]=\{0\}$ or $(\Bbbk^2]=\Bbbk$.
Suppose that $(\Bbbk^2]=\Bbbk$.

Since,

$$\begin{aligned}\mathbb{k}\circ\mathcal{Z}\circ\mathbb{k}\circ\mathbb{k} &= (\mathbb{k}^2]\circ\mathcal{Z}\circ(\mathbb{k}^2]\circ(\mathbb{k}^2]\\ &\subseteq (\mathbb{k}^2\circ\mathcal{Z}\circ\mathbb{k}^2\circ\mathbb{k}^2]\\ &\subseteq (\mathbb{k}^2\circ\mathcal{Z}\circ\mathbb{k}^4]\\ &\subseteq (\mathbb{k}^2\circ\mathcal{Z}\circ\mathbb{k}^2]\\ &\subseteq (\mathbb{k}]\\ &\subseteq \mathbb{k}\end{aligned}$$

Therefore, $\mathbb{k}$ is a LTID in $\mathcal{Z}$.
Let Q be a LTID in $\mathcal{Z}$ contained in .
It follows that $Q^2\circ\mathcal{Z}\circ Q^2 \subseteq Q\circ\mathcal{Z}\circ Q\circ Q \subseteq Q \subseteq \mathbb{k}$.
This shows that Q is a (2,2)-BID in $\mathcal{Z}$ contained in $\mathbb{k}$.
Thus, $Q = \{0\}$ or $Q = \mathbb{k}$.

Theorem 4.9. *$\top$ is a minimal* (2,2)*-BID in $\mathcal{Z}$ iff $\top$ is a minimal TID.*

Proof. Now, $\top$ is a minimal (2,2)-BID in $\mathcal{Z}$. Then $(\top^2\circ\mathcal{Z}\circ\top^2]$ is a (2,2)-BID in $\mathcal{Z}$ contained in $\top$. And hence, $(\top^2\circ\mathcal{Z}\circ\top^2] = \top$. Now,

$$\begin{aligned}\top^2\circ\mathcal{Z}\circ\top &= (\top^2\circ\mathcal{Z}\circ\top^2]\circ(\top^2\circ\mathcal{Z}\circ\top^2]\circ\mathcal{Z}\circ(\top^2\circ\mathcal{Z}\circ\top^2]\\ &\subseteq (\top^2\circ\mathcal{Z}\circ\top^2\circ\top^2\circ\mathcal{Z}\circ\top^2\circ\mathcal{Z}\circ\top^2\circ\mathcal{Z}\circ\top^2]\\ &\subseteq (\top^2\circ\mathcal{Z}\circ\top^2]\\ &\subseteq (\top]\\ &\subseteq \top\end{aligned}$$

and

$$\begin{aligned}\top\circ\mathcal{Z}\circ\top^2 &= (\top^2\circ\mathcal{Z}\circ\top^2]\circ\mathcal{Z}\circ(\top^2\circ\mathcal{Z}\circ\top^2]\circ(\top^2\circ\mathcal{Z}\circ\top^2]\\ &\subseteq (\top^2\circ\mathcal{Z}\circ\top^2\circ\mathcal{Z}\circ\top^2\circ\mathcal{Z}\circ\top^2\circ\top^2\circ\mathcal{Z}\circ\top^2]\\ &\subseteq (\top^2\circ\mathcal{Z}\circ\top^2]\\ &\subseteq (\top]\\ &\subseteq \top\end{aligned}$$

It follows that $\top$ is a TID in $\mathcal{Z}$. Suppose that there exist a TID $\mathcal{W}$ in $\mathcal{Z}$ contained in $\top$. Then $\mathcal{W}^2\circ\mathcal{Z}\circ\mathcal{W}^2 \subseteq (\mathcal{W}^2\circ\mathcal{Z}\circ\mathcal{W})\circ\mathcal{W} \subseteq \mathcal{W}^2 \subseteq \mathcal{W} \subseteq \top$. Thus $\mathcal{W}$ is a (2,2)-BID in $\mathcal{Z}$ contained in $\top$.

Conversely, assume that $\top$ is a minimal RTID in $\mathcal{Z}$. Then $\top$ is a (2,2)-BID in $\mathcal{Z}$. Let $\mathcal{Y}$ be a (2,2)-BID in $\mathcal{Z}$ contained in $\top$. Since, $(\mathcal{Y}^2\circ\mathcal{Z}\circ\mathcal{Y}^2]\circ(\mathcal{Y}^2\circ\mathcal{Z}\circ\mathcal{Y}^2]\circ\mathcal{Z}\circ(\mathcal{Y}^2\circ\mathcal{Z}\circ\mathcal{Y}^2] \subseteq (\mathcal{Y}^2\circ\mathcal{Z}\circ\mathcal{Y}^2\circ\mathcal{Y}^2\circ\mathcal{Z}\circ\mathcal{Y}^2\circ\mathcal{Z}\circ\mathcal{Y}^2\circ\mathcal{Z}\circ\mathcal{Y}^2] \subseteq (\mathcal{Y}^2\circ\mathcal{Z}\circ\mathcal{Y}^2]$. Now, $(\mathcal{Y}^2\circ\mathcal{Z}\circ\mathcal{Y}^2]$ is a RTID in $\mathcal{Z}$. Similarly, we find that $(\times^{2\circ}\ \circ\ ^2]$ is a LTID in $\mathcal{Z}$. This implies that $(\mathcal{Y}^2\circ\mathcal{Z}\circ\mathcal{Y}^2] = \top$. Since, $\top = (\mathcal{Y}^2\circ\mathcal{Z}\circ\mathcal{Y}^2] \subseteq (\mathcal{Y}] = \mathcal{Y}$. Therefore, $\top = \mathcal{Y}$. Therefore, $\top$ is a minimal (2,2)-BID in $\mathcal{Z}$.

Theorem 4.10. *Let $\top$ is a* (4,4)*-BID in $\mathcal{Z}$ iff $\top$ is a TID of certain LTID in $\mathcal{Z}$.*

Proof. Assume that $\top$ is a (4,4)-BID in $\mathcal{Z}$. Then

$$(\top^4\cup\top^4\circ\mathcal{Z}\circ\top^4]\circ\mathcal{Z}\circ(\top^4\cup\top^4\circ\mathcal{Z}\circ\top^4]\circ(\top^4\cup\top^4\circ\mathcal{Z}\circ\top^4] \subseteq (\top^4\circ\mathcal{Z}\circ\top^4] \subseteq \top^4\cup(\top^4\circ\mathcal{Z}\circ\top^4].$$

Therefore, $\top^4\cup(\top^4\circ\mathcal{Z}\circ\top^4]$ is a LTID in $\mathcal{Z}$.
Since,

$$\begin{aligned}\top\circ\top\circ(\top^4\cup\top^4\circ\mathcal{Z}\circ\top^4]\circ\top &\subseteq (\top^7\cup\top^6\circ\mathcal{Z}\circ\top^5]\\ &\subseteq (\top^4\cup\top^4\circ\mathcal{Z}\circ\top^4]\\ &\subseteq (\top]\\ &\subseteq \top\end{aligned}$$

and

$$\begin{aligned}\top \circ (\top^4 \Cup \top^4 \circ \mathcal{Z} \circ \top^4] \circ \top \circ \top &\subseteq (\top^7 \Cup \top^5 \circ \mathcal{Z} \circ \top^6] \\ &\subseteq (\top^4 \Cup \top^4 \circ \mathcal{Z} \circ \top^4] \\ &\subseteq (\top] \\ &\subseteq \top\end{aligned}$$

Therefore, $\top$ is a TID of $(\top^4 \Cup \top^4 \circ \mathcal{Z} \circ \top^4]$.

Conversely, $\top$ is a TID of LTID $\Bbbk$ in $\mathcal{Z}$, then

$$\begin{aligned}\top^4 \circ \mathcal{Z} \circ \top^4 &\subseteq \top \circ \top \circ \top \circ \top \circ \mathcal{Z} \circ \top \circ \top \circ \top \circ \top \\ &\subseteq \top \circ \top \circ \mathcal{Z} \circ \mathcal{Z} \circ \mathcal{Z} \circ \top \circ \top \circ \top \circ \top \\ &\subseteq \top \circ \Bbbk \circ \mathcal{Z} \circ \Bbbk \circ \Bbbk \circ \top \circ \top \\ &\subseteq \top \circ \Bbbk \circ \top \circ \top \\ &\subseteq \top\end{aligned}$$

and

$$\begin{aligned}\top^4 \circ \mathcal{Z} \circ \top^4 &\subseteq \top \circ \top \circ \top \circ \top \circ \mathcal{Z} \circ \top \circ \top \circ \top \circ \top \\ &\subseteq \top \circ \top \circ \top \circ \top \circ \mathcal{Z} \circ \mathcal{Z} \circ \mathcal{Z} \circ \top \circ \top \\ &\subseteq \top \circ \top \circ \boldsymbol{\Psi} \circ \boldsymbol{\Psi} \circ \mathcal{Z} \circ \boldsymbol{\Psi} \circ \top \\ &\subseteq \top \circ \top \circ \boldsymbol{\Psi} \circ \top \\ &\subseteq \top\end{aligned}$$

Therefore, $\top$ is a (4,4)-BID in $\mathcal{Z}$.

5. Conclusions and upcoming studies

Our study introduced novel ideas about the tri purity of semirings based on TIDs. This section provides an overview of this emerging topic for the benefit of future researchers. Our research can be expanded to include the generalization of BIDs in terms of ternary BIDs and minimal BIDs of ternary semirings. We will also try to analyze b-semirings and bi-semirings in the future.

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