Results in Nonlinear Analysis 9 (2026) No. 2, 59–67
https://doi.org/10.31838/rna/2026.09.02.011
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

Estimate the scale parameter for inverse rayleigh distribution via maximum likelihood with trapezoidal fuzzy data

Hawraa J. Kadhim[1], Amal A. Mohammed[2], Sudad K. Abraheem[3]*

[1]Rusafa 1st Education Directorate, Ministry of Education, Baghdad, Iraq; [2]Department of Ways and Transportation, College of Engineering, Mustansiriyah University, Baghdad, Iraq; [3]Department of Mathematics, College of Science, Mustansiriyah University, Baghdad, Iraq

Abstract

In this work, the scale parameter of the inverse Rayleigh distribution (IRD) is estimated; it is one of the most important distributions for analyzing failure times and has broad applications across different fields. The data of this distribution are presumed to be trapezoidal fuzzy, and they were converted into classical data using two paths based on approaches $C_1(\bar{A})$ and $C_2(\bar{A})$, introduced as novel fuzzy output conversion techniques. Three fuzziness classes were adopted: absolute, asymmetric (severe), and symmetric (moderate) fuzziness, to assess the robustness of the estimators under varying degrees of imprecision. The estimation workflow was performed using maximum likelihood estimation (MLE), with simulations used to generate the data. The comparison between results depends on the mean squared error (MSE). All mathematical procedures were applied in MATLAB R2024b, and the results showed that the $C_2(\bar{A})$ method outperformed $C_1(\bar{A})$ in terms of estimation accuracy and lower error values. Indeterminate environments demand improved estimation accuracy when incorporating reliability models and defuzzification methods. This is evident through the practical importance referenced in these findings.

Mathematics Subject Classification: 62A86, 62F10, 62N05, 03E72

Key words and phrases: Centroid defuzzification methods, Trapezoidal fuzzy data, Inverse Rayleigh distribution, Maximum likelihood estimation, Monte Carlo simulation.

Email addresses: hawraajawad71@yahoo.com (Hawraa J. Kadhim), dr.amal70@uomustansiriyah.edu.iq (Amal A. Mohammed), sudad1968@uomustansiriyah.edu.iq (Sudad K. Abraheem)*

Received May 10, 2026; Accepted June 30, 2026; Online July 23, 2026

1. Introduction

In most previous studies, data are treated as confirmed values, essentially as crisp data in conventional research. Yet, the reality is that, in many cases unclear and fuzzy in the data. This may be a result of measurement error (due to human or machine), conceptual ambiguity, or an absence of exact knowledge of the true value or its expressivity. In such cases, when data is vague, fuzzy set theory is a suitable mathematical basis for modeling the imprecise data [1].

The IRD was first suggested by Trayer in 1964. Since then, IRDs have been frequently used in reliability studies for their ability to fit real data from skewed lifetime distributions. Few-parameter estimates and reliability models dealing with crisp data have been recognized so far in the realm of classical studies. The IRD has been applied to models such as the stress-strength model with MLE, the method of moments, uniformly minimum variance unbiased estimators, and shrinkage estimators. Fuzzy environments should be integrated into reliability models because real data are frequently uncertain [2, 3].

Several generalizations of the IRD have been proposed to improve its flexibility. They include the exponentiated IRD and three-parameter IRD, which is a special case of the standard IRD [4, 5]. Akdoğan, Y. et al. [6] studied the problem of point and interval estimation of the parameter for the IRD based on the type-I hybrid censored samples, and considered the method of maximum likelihood to estimate the parameter of the IRD by using the expectation maximization algorithm, as well as obtained the approximate confidence interval according to the MLE using Fisher's information. Sabry, M. A. H. et al. [7] reparametrized the IRD, and its generalizations have been applied in the fuzzy environment using classical estimation methods, including maximum product of spacing, MLE, and Bayesian estimation methods, by Markov Chain Monte Carlo (MCMC), specifically METROPOLIS–HASTINGS. Tanış, C. [8] introduced a new lifetime distribution called the transmuted lower record type IRD, which extends the IRD, and used to model the recovery times of COVID-19 patients. Also, obtained the new distribution using the first two lower record statistics of the IRD, and examined some statistical inferences and mathematical properties for the suggested distribution. Tanış, C. and Saraçoğlu, B. [9] proposed a new lifetime distribution called the cubic rank transmuted IRD as an alternative to the IRD and examined some distributional properties of the suggested distribution, also considered five methods to estimate the distribution's parameters, and presented a real data application to illustrate the usefulness of the proposed distribution.

In this study, the scale parameter of the IRD is estimated. The data of this distribution are presumed to be trapezoidal fuzzy, and were switched into classical data using two paths based on approaches, $C_1(\bar{A})$ and $C_2(\bar{A})$ introduced as novel fuzzy output conversion techniques. Three fuzziness classes were adopted: absolute fuzziness, asymmetric (severe) fuzziness, and symmetric (moderate) fuzziness, to assess the robustness of the estimators under varying degrees of imprecision.

2. Fundamental Functions of the Inverse Rayleigh Distribution

The probability density function (PDF), cumulative distribution function (CDF), reliability function, and hazard rate function of the IRD are respectively defined as follows [5, 7]:

$$f(x,\alpha)=\frac{2\alpha}{x^3}e^{\frac{-\alpha}{x^2}} \qquad x>0;\alpha>0 \tag{1}$$

$$F(x,\alpha)=e^{\frac{-\alpha}{x^2}} \qquad x>0;\alpha>0 \tag{2}$$

$$R(x,\alpha)=1-F(x,\alpha)$$

$$R(x,\alpha) = 1 - e^{\frac{-\alpha}{x^2}} \qquad x > 0; \alpha > 0 \tag{3}$$

$$h(x,\alpha) = \frac{f(x,\alpha)}{R(x,\alpha)}$$

$$h(x,\alpha) = \frac{\frac{2\alpha}{x^3} e^{\frac{-\alpha}{x^2}}}{1 - e^{\frac{-\alpha}{x^2}}} \qquad x > 0; \alpha > 0 \tag{4}$$

where α is the scale parameter of IRD.

3. The Maximum Likelihood Estimation

The MLE method is widely used and considered important, as it possesses desirable properties such as consistency, efficiency, and the invariance property [10].

Consider a random sample x_1, x_2, ..., x_n of size n drawn from a population whose PDF is given by $f(x_i,\alpha)$, where the underlying distribution is IRD.

The likelihood function for the observed sample, represented by equation (1), is given by:

$$L = 2^n \alpha^n \Pi_{i=1}^n x_i^{-3} e^{-\alpha \Sigma_{i=1}^n x_i^{-2}} \tag{5}$$

Taking the natural logarithm of the likelihood function in equation (5), and simplifying, yields:

$$\ln L = \ln 2^n + n \ln \alpha + \sum_{i=1}^n \ln x_i^{-3} - \alpha \sum_{i=1}^n x_i^{-2} \tag{6}$$

Differentiating equation (6) with respect to α

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^n x_i^{-2} \tag{7}$$

To obtain the maximum likelihood estimator of α, we set equation (7) equal to zero as follows [11, 12]

$$\hat{\alpha}_{ml} = \frac{n}{\sum_{i=1}^n x_i^{-2}} \tag{8}$$

4. Centroid-Based Ranking in Fuzzy Set Theory

The process of ranking fuzzy numbers and converting them into crisp values plays a fundamental role in decision-making under uncertainty. It has been widely applied in various fields such as socio-economic analysis, risk assessment, forecasting, control systems, and medical diagnostics, among other fuzzy system applications, where data are inherently imprecise [13].

Definition: A fuzzy number $\bar{A} = (a_1, a_2, a_3, a_4)$ where a_1,a_2,a_3, and a_4 A real number is said to be a trapezoidal fuzzy number, denoted by (T_r.*F.N*), its membership function [1]:

$$M_{\bar{A}}(x) = \begin{cases} \frac{(x - a_1)}{(a_2 - a_1)}, a_1 \le x \le a_2 \\ 1, a_2 \le x \le a_3 \\ \frac{(a_4 - x)}{(a_4 - a_3)}, a_3 \le x \le a_4 \end{cases} \tag{9}$$

Proposed Method and Justification

To explain the reasons for obtaining a simpler and more computationally efficient closed formula for determining the center of gravity of trapezoidal fuzzy numbers, the following steps apply:

- **Computational efficiency:** allows for direct implementation in MATLAB or similar environments and does away with the necessity for repetitive integration procedures
- **Simpler closed formula:** obtain a simpler closed formula, which offers a single compact formula, in contrast to Wang's method [14], which depends on segment-wise computations.
- **Numerical stability:** minimizes piecewise integration-related rounding errors.
- **The compatibility:** Yields nearly identical results for a broad range of trapezoidal fuzzy numbers, which is totally in line with Wang et al.'s computational approach.
- **The contribution of research:** Introduce a practical device that simplifies the ranking process and defuzzification for fuzzy numbers.

With roots in the Bander-Simonovic-Cheng approach of triangular fuzzy sets, like those in reference [15], this work shifts toward trapezoidal forms. Through such adjustment comes room without losing core traits for grasping fuzzier, richer data patterns. The original spirit stays intact, even as shape evolves.

Considering the centroid model put forward by [14]:

$$C(\bar{A}) = \frac{\int x M_{\bar{A}}(x)dx}{\int M_{\bar{A}}(x)dx} \tag{10}$$

Substituting the trapezoidal membership function, get:

$$C_1(\bar{A}) = \frac{\int_{a_1}^{a_2} x\frac{(x-a_1)}{(a_2-a_1)}dx + \int_{a_2}^{a_3} x dx + \int_{a_3}^{a_4} x\frac{(a_4-x)}{(a_4-a_3)}dx}{\int_{a_1}^{a_2}\frac{(x-a_1)}{(a_2-a_1)}dx + \int_{a_2}^{a_3} dx + \int_{a_3}^{a_4}\frac{(a_4-x)}{(a_4-a_3)}dx} \tag{11}$$

Equation (11) gives:

$$C_1(\bar{A}) = \frac{a_1^2 + a_1a_2 + a_2^2 - a_3^2 - a_3a_4 - a_4^2}{3(a_1 + a_2 - a_3 - a_4)} \tag{12}$$

To obtain the proposed method, we raised the membership function in equation (10) to the eighth power as follows:

$$C_2(\bar{A}) = \frac{\int x\left(M_{\bar{A}}(x)\right)^8 dx}{\int\left(M_{\bar{A}}(x)\right)^8 dx}$$

Substituting equation (9) into the above equation for the trapezoidal membership function, have:

$$C_2(\bar{A}) = \frac{\int_{a_1}^{a_2} x\left(\frac{(x-a_1)}{(a_2-a_1)}\right)^8 dx + \int_{a_2}^{a_3} x dx + \int_{a_3}^{a_4} x\left(\frac{(a_4-x)}{(a_4-a_3)}\right)^8 dx}{\int_{a_1}^{a_2}\left(\frac{(x-a_1)}{(a_2-a_1)}\right)^8 dx + \int_{a_2}^{a_3} dx + \int_{a_3}^{a_4}\left(\frac{(a_4-x)}{(a_4-a_3)}\right)^8 dx} \tag{13}$$

And

$$C_2(\bar{A}) = \frac{a_1^2 + 8a_1a_2 + 36a_2^2 - 36a_3^2 - 8a_3a_4 - a_4^2}{10(a_1 + 8a_2 - 8a_3 - a_4)} \tag{14}$$

The main practical contribution is given by equation (12), while the consistency of the theory of the generalized centroid method is illustrated with that of equation (14).

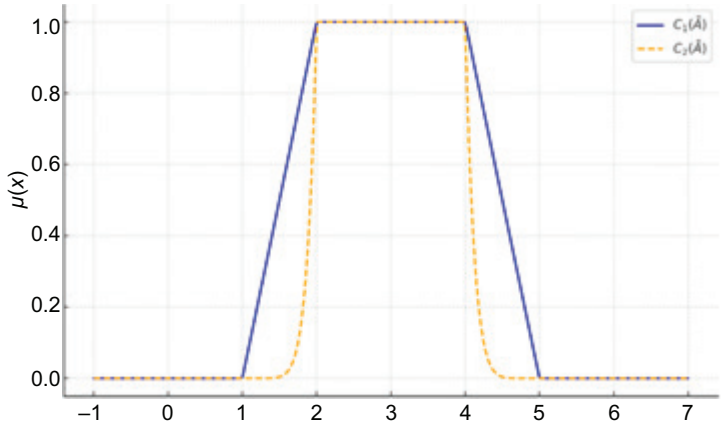

Figure 1: Comparison of trapezoidal fuzzy membership functions for $C_1(\bar{A})$ and $C_2(\bar{A})$.

5. Levels of Fuzziness

Established that the shape and slope of trapezoidal functions affect the result of the model [16]. Showed the behavior of a fuzzy rule-based classifier as the conversion of the fuzziness slope, mainly in such a low confidence area. In this study, three specific fuzziness levels are: severe asymmetric S_1, moderate symmetric S_2, light symmetric S_3 to determine the robustness of the proposed estimator for various degrees of certainty. As a result, trapezoidal fuzzy numbers $\bar{A} = (a_1, a_2, a_3, a_4)$ with varying spreads were used to establish three diverse fuzziness levels (S_1, S_2 and S_3), detailed in Table 1 below:

Table 1: Definition of fuzziness levels (S_1, S_2, and S_3)

Level	Definition of $\tilde{A} = (a_1, a_2, a_3, a_4)$	Fuzziness type	Description
S_1	a_1 = x–0.04x, a_2 = x–0.02x, a_3 = x+0.02x, a_4 = x+0.20x	Heavy Asymmetric Fuzziness	The right-side spread is significantly larger than the left-side spread
S_2	a_1 = x–0.08x, a_2 = x–0.04x, a_3 = x+0.04x, a_4 = x+0.08x	Moderate Symmetric Fuzziness	Left and right spreads are approximately equal
S_3	a_1 = x–0.10x, a_2 = x–0.05x, a_3 = x+0.05x, a_4 = x+0.10x	Light Symmetric Fuzziness	Spreads are narrower compared to S_2

The choice of these three fuzziness levels is aimed at testing whether, for various uncertainty situations, the proposed fuzzy estimators outperform the crisp estimator. S_1 indicates an asymmetric case of uncertainty; S_2 indicates an equal amount of fuzziness is symmetrically distributed (the moderate); and S_3 indicates more light symmetric fuzziness (moderate). The proposed test allows us to evaluate the accuracy of the estimators not only for extreme uncertainty but also at realistic moderate levels.

6. The simulation

The random samples directly from the IRD's cumulative distribution are generated by using the inverse transform method, by setting:

$$u_i = e^{\frac{-\alpha}{x_i^2}}$$

Rearranging the above equation to find x_i as:

$$x_i = \pm \sqrt{\frac{\alpha}{\ln \frac{1}{u_i}}}.$$

But the IRD only deals with positive values x>0; the negative roots are ignored, so the sampling formula is:

$$x_i = \sqrt{\frac{\alpha}{\ln(\frac{1}{u_i})}} \tag{15}$$

The simulation procedure adopted in this study is as follows:

1) **Generation of the Data:** Random samples are generated from the IRD for different sample sizes (n=10, 40,100) with scale parameters (α=0.8, 1.6, 2.4).
2) **Fuzzification:** Each sample is converted into a trapezoidal fuzzy number under three levels of fuzziness: S_1 (heavy asymmetric), S_2 (moderate symmetric), and S_3 (light symmetric).
3) **Defuzzification:** Two centroid-based methods, $C_1(\bar{A})$ and $C_2(\bar{A})$, are applied to transform fuzzy numbers into crisp values.
4) **Parameter estimation:** The scale parameter α is estimated using the MLE method for both the original and the defuzzified data.
5) **Performance evaluation:** A Monte Carlo simulation with 4000 replications is conducted, and the results are compared using the MSE as the accuracy measure [17].

7. Results of Simulation

The simulation results will be presented in the table of the mean squared errors for each of the sample sizes, the parameter value, and the level of fuzziness, $C_1(\bar{A})$ and $C_2(\bar{A})$ methods. In addition, the same results were plotted in the graph and bar chart, so that the differences between $C_1(\bar{A})$ and $C_2(\bar{A})$ are more obvious.

Table 2 provides the MSE values of the MLE for the scale parameter of the IRD under the two defuzzification methods (centroid-based: $C_1(\bar{A})$ and $C_2(\bar{A})$) for various sample sizes, parameter values, and fuzziness levels. The following results were obtained:

- The accuracy improves as the sample size increases, as exhibited by the MSE values. That is consistent with the property of MLE: larger samples provide more accurate parameter estimates.
- The proposed $C_2(\bar{A})$ method yielded lower MSE values than $C_1(\bar{A})$, showing an outstanding ability to preserve information in trapezoidal fuzzy observations of the defuzzification procedures.

Table 2: MSE values of MLE estimators under $C_1(\bar{A})$ and $C_2(\bar{A})$.

		$C_1(\bar{A})$			$C_2(\bar{A})$		
N	α	S_1	S_2	S_3	S_1	S_2	S_3
10	0.8	0.1549	0.1325	0.1560	0.1379	0.1221	0.1049
40		0.0357	0.0270	0.0667	0.0290	0.0235	0.0220
100		0.0201	0.0131	0.0551	0.0146	0.0104	0.0112
10	1.6	0.6482	0.5563	0.4288	0.0290	0.5137	0.4199
40		0.1473	0.1116	0.0911	0.1197	0.0971	0.0734
100		0.0811	0.0532	0.0471	0.0594	0.0425	0.0287
10	2.4	1.4132	1.2130	0.9876	1.2615	1.1203	0.9147
40		0.3331	0.2519	0.1633	0.2705	0.2188	0.1618
100		0.1849	0.1210	0.0674	0.1353	0.0965	0.0617

- The performance of $C_2(\bar{A})$ becomes more distinct at moderate and large sample sizes; the proposed centroid method benefits from increased sample size to more stabilize parameter estimation.
- Overall, Table 2 demonstrates that the proposed $C_2(\bar{A})$ defuzzification method with MLE provides a more accurate and more powerful estimate of the scale parameter for the IRD under fuzzy settings.

Figures 2 and 3 below provide graphical representations of the simulation results presented in Table 2, making it easier to observe the differences in performance between $C_1(\bar{A})$ and $C_2(\bar{A})$.

The plots show the differences in the MSE between estimators of $C_1(\bar{A})$ and $C_2(\bar{A})$ for various sample sizes (n), fuzziness levels (S_1, S_2, S_3), and various parameter values (α), giving a perspective on how the values of both approaches compare.

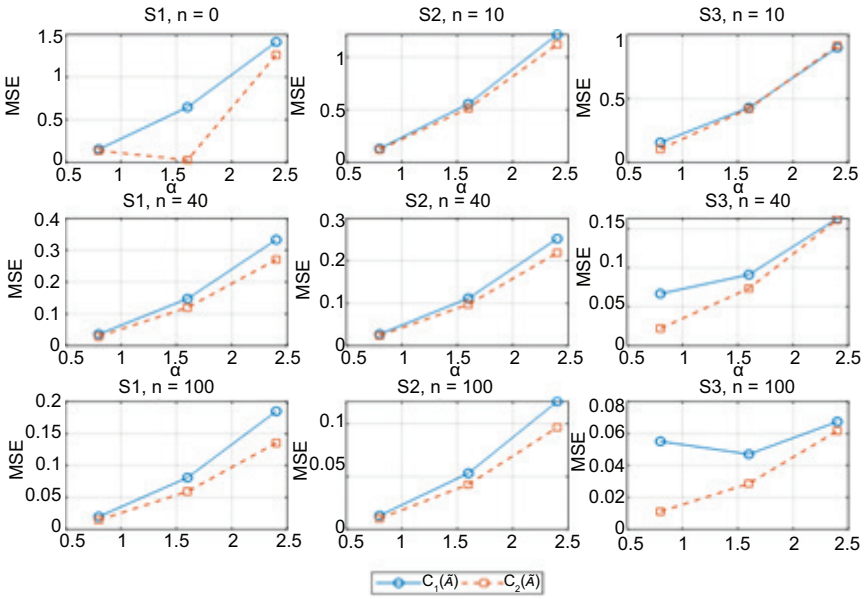

Figure 2: Comparison of $C_1(\bar{A})$ and $C_2(\bar{A})$ across α, n, and fuzziness levels.

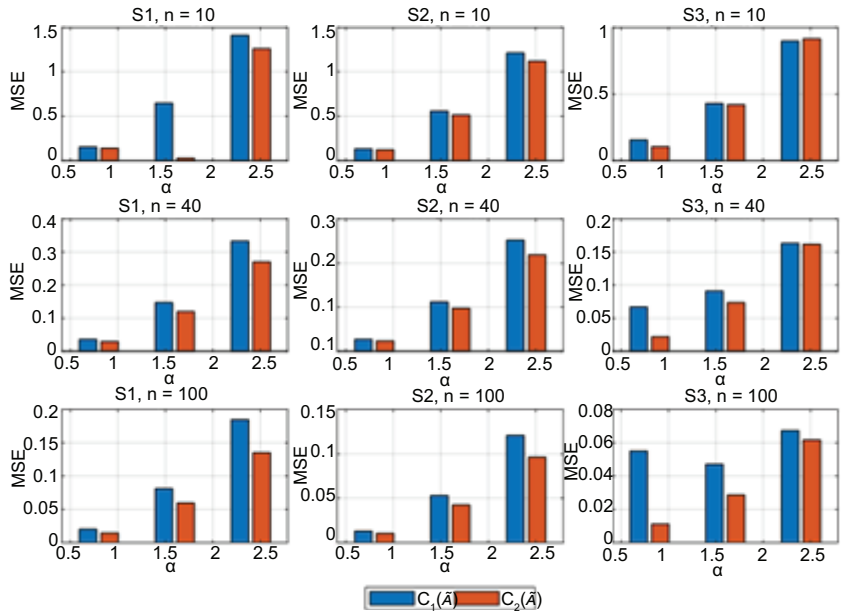

Figure 3: Bar chart of MSE values under $C_1(\bar{A})$ and $C_2(\bar{A})$ across α, n, and fuzziness levels.

Figure 2 shows that MSE values decrease as the sample size increases, verifying the convergence properties of the MLE. For all sample sizes, scale parameter values, and fuzziness levels, the curves for $C_2(\bar{A})$ are lower than $C_1(\bar{A})$, confirming the superior estimation accuracy of the proposed method. In addition, the plots show that the performance disparity between the two methods becomes more apparent with higher uncertainty, especially at the asymmetric fuzziness level (S_1), where the proposed method exhibits increased robustness.

Figure 3 complements these results with bar charts; the lower bar heights in $C_2(\bar{A})$ represent smaller MSE values than in $C_1(\bar{A})$. The bar chart further highlights the stability of $C_2(\bar{A})$ in various parameter values and sample sizes; that is, the performance is not limited to a particular simulation.

Overall, Figures 2 and 3 clearly demonstrate the excellence of the $C_2(\bar{A})$ method in achieving lower MSE values and offer a more reliable and accurate approach for parameter estimation in the presence of trapezoidal fuzzy data.

8. Statistical and Practical Significance of the Simulation Findings

Statistical Significance: The simulation findings show a steady decrease in MSE when the proposed defuzzification method $C_2(\bar{A})$ is used instead of $C_1(\bar{A})$ for all sample sizes, scale parameter values, and fuzziness levels. Since MSE measures both the estimator's bias and variance, its systematic decrease shows that the proposed method $C_2(\bar{A})$ provides more accurate, efficient, and stable estimates of the scale parameter. In addition, the continuous decrease in MSE as the sample size increases confirms the consistency of the MLE, and the proposed defuzzification method keeps this statistical property in fuzzy environments.

Practical Significance: The performance of $C_2(\bar{A})$ refers to more trustworthy parameter estimation when samples are influenced by ambiguity and inaccuracy. Consistently lower estimation errors across different fuzziness levels. The proposed method provides a powerful and effective tool for analyzing trapezoidal fuzzy data, making it relevant in practical use when findings are difficult or impossible to obtain.

9. Conclusions

This paper focused on the MLE of the scale parameter for the IRD amid trapezoidal fuzzy data. Three fuzziness levels (S_1, S_2, S_3) were applied using two centroid-based defuzzification methods $C_1(\bar{A})$ and the suggested $C_2(\bar{A})$. The simulation results clearly demonstrate the superiority of the suggested method $C_2(\bar{A})$. When α=2.4 and n=100, the MSE value for $C_2(\bar{A})$ was 0.0617, compared to 0.0674 for $C_1(\bar{A})$, and when α=0.8 and n=40, the MSE decreased to 0.0220 with the proposed $C_2(\bar{A})$, while it remained higher at 0.0667 for $C_1(\bar{A})$. These results confirm that the proposed $C_2(\bar{A})$ provides lower and more stable MSE values for all conditions, validating its robustness as a defuzzification strategy for parameter estimation amid uncertainty.

Acknowledgments

The creator expresses gratitude to the Rusafa 1st education directorate (Ministry of Education) and Mustansiriyah University (https://uomustansiriyah.edu.iq) in Baghdad, Iraq, for their help with this research.

References

[1] Z. Roohanizadeh, E.B. Jamkhaneh, and E. Deiri, "Parameters and reliability estimation for the Weibull distribution based on intuitionistic fuzzy lifetime data," Complex Intell. Syst., 8, 4881–4896, 2022. https://doi.org/10.1007/s40747-022-00720-x.

[2] S.H. Raheem, H.K. Mansor, B.A. Kalaf, and A.N. Salman, "A Comparison for some of the Estimation Methods of the Parallel Stress-Strength Model in the Case of Inverse Rayleigh Distribution," Proc. 2019 First Int. Conf. Computer and Applied Sciences (CAS), Mustansiriyah Univ., Baghdad, Iraq, pp. 22–27, 2019. https://doi.org/10.1109/CAS47993.2019.907568.

[3] S.H. Raheem, B.A. Kalaf, and A.N. Salman, "Estimation of the reliability system in model of stress–strength according to distribution of inverse Rayleigh," Period. Eng. Nat. Sci., 9(2), 524–533, 2021.

[4] G.S. Raw and S. Mbwambo, "Exponentiated Inverted Rayleigh Distribution and an Application to Coating Weights of Iron Sheets Data," J. Probab. Stat., (Hindawi), Article ID 7519429, 2019. https://doi.org/10.1155/2019/7519429.

[5] M. Shala and F. Merovci, "A New Three-Parameter Inverse Rayleigh Distribution: Simulation and Application to Real Data," Symmetry, 16, 634, 2024. https://doi.org/10.3390/sym16050634.

[6] Akdoğan, Y., Özkan, E., Karakaya, K., & Tanış, C., "Estimation Of Parameter For Inverse Rayleigh Distribution Under Type-I Hybrid Censored Samples," *Sigma Journal of Engineering and Natural Sciences*, 38(4), 1705–1711, 2020.

[7] M.A.H. Sabry, E.M. Almetwally, O.A. Alamri, M. Yusuf, H.M. Almongy, and A.S. Eldeeb, "Inference of fuzzy reliability model for Inverse Rayleigh Distribution," AIMS Math., 6(9), 9770–9785, 2021.

[8] Tanış, C., "Transmuted lower record type inverse Rayleigh distribution: estimation, characterizations and applications," *Ricerche di Matematica*, 71(2), 777–802, 2022.

[9] Tanış, C., & Saraçoğlu, B. (2022). Cubic rank transmuted inverse Rayleigh distribution: Properties and applications. *Sigma Journal of Engineering and Natural Sciences*, 40(2), 421–432, 2022.

[10] S.A.K. Al-Sultany, "Bayesian Estimations with Fuzzy Data to Estimation Inverse Rayleigh Scale Parameter," Open J. Appl. Sci., 9, 673–681, 2019.

[11] S.K. Abraheem, A.A. Mohammed, and H.J. Kadhim, "A Numerical Comparison between the Approximation and the Estimation of the Reversed Hazard Rate Function for the Inverted Kumaraswamy Distribution," AIP Conf. Proc., vol.3219, p.040002, 2024. https://doi.org/10.1063/5.0237170.

[12] A.H.A. Noor, A.A. Mohammed, and S.K. Abraheem, "Utilizing various statistical techniques to estimate the scale parameter for the Rayleigh distribution," Results Nonlinear Anal., vol. 8, no. 2, pp. 182–206, 2025, https://doi.org/10.31838/rna/2025.08.02.015.

[13] D. Marimuthu and G.S. Mahapatra, "Multi-criteria decision-making using a complete ranking of generalized trapezoidal fuzzy numbers," Soft Comput., 25(15), 9859–9871, 2021.

[14] Y.-M. Wang, J.-B. Yang, D.-L. Xu, and K.-S. Chin, "Centroid for ranking generalized fuzzy numbers," Fuzzy Sets Syst., 157(7), 899–911, 2006. https://doi.org/10.1016/j.fss.2005.11.012.

[15] H.J. Kadhim, "Estimating the parameters of Generalized Inverted Exponential Distribution with Fuzziness," M.Sc. thesis, Dept. of Mathematics, College of Science for Women, Univ. of Baghdad, Baghdad, Iraq 2016.

[16] F. Casalino, F. Marcelloni, and R. Minervini, "Impact of fuzziness on fuzzy rule-based classifiers forming strong fuzzy partitions," Soft Comput., 26, 13207–13224, 2022.

[17] S.K. Abraheem, Y.H. Abd, and A.A. Mohammed, "Estimate the shape parameter for the Kumaraswamy distribution via some estimation methods," Results Nonlinear Anal., vol. 7, no. 1, pp. 142–155, 2024.