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Study of descriptor multi-fractional integro-differential order nonlinear system by homotopy perturbation method

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Abstract

In this paper we are discussing a nonlinear system that is descriptor, multi derived from the Caputo-derived type as well as Riemann-Liouville fractional integral and has been relying on a system solution using a homotopy perturbation method method. In addition, three examples were given to illustrate the method of solving the system in order to compare the results obtained.

Mathematics Subject Classification (2020): 34A08, 65L05

Key words and phrases: Homotopy perturbation method, descriptor, integro- differential, fractional order.

1. Introduction

We can express descriptor systems, in their other name, which are called descriptor systems, sometimes referred to as algebraic differential systems, which are a generalization of dynamic systems [2]. Problems related to fractional differential equations, including fractional convection and dispersion, are studied. [8], certain types of time-fractional diffusion equations [15], fractional generalized Burgers' fluid [16], fractional KdV-type equations [3], space-time fractional Whitham-Broer-Kaupand

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equations [4], fractional heat- and wave-like equations [12], and space fractional backward Kolmogorov equations [10]. Recently, proposals have been made to solve differential equations using the homotopy perturbation method (HPM) [5,7]. Some mathematicians have used the HPM method to solve discrete domain [28]. Dawood, L., et al., (2020), presents the application of (VIM) and (MHPM) to solve the Volterra integro-differential equations [18]. Eshkuvatov, Z. (2022). developed (NDHAM) is demonstrated for (NIDEs) [19]. Naik, P. A., et al., (2022), proposed and analyze a nonlinear fractional-order SEIR epidemic model to transmit HIV [20]. Olayiwola, M. O., et al. (2023), studied utilized a novel SEITR mathematical model to investigate the impact of treatment on physical limitations in tuberculosis [21]. Polat, S. N. T., et al. (2023). Presented the (HPM) for solving non-linear delay of multi-term fractional order [22]. Kayota, S. et al. (2023). Introduced the application of the (LVIM) and (LHPM) to some (FIDEs) [23]. Yang, H. et al. (2024). Study (CSRP) for fractional-order linear continuous-time systems with order $0 < a < 1$ [27]. Moumen, A., et al. (2024). proposed strategy uses first-order shifted Chebyshev polynomials and a projection method [25]. AlBaidani, M. M. (2025). A comparison of approximate and exact solutions to the TF-GBFE equation was made. [26]. Yisa, B. M., et al. (2025). (HAITM) developed some problems in the results are presented in tabular form, as well as in 2D graphs [27].

In this paper is consists of three sections. section one is introduction, section two deals with some of the basic mathematics concepts and principles of descriptor fractional order system. finally, section three is the descriptor homotopy Perturbation method of solution fractional order system is discussed.

2. Fractional Derivatives and Fractional Integration

Definition (2.1), [11]: *The Riemann-Liouville fractional integral operator of order* $\alpha > 0$ *is defined as:*

$$I_t^{\propto} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x)dx, \alpha > 0, x > 0 \tag{1}$$

Definition (2.2), [11]: *The Riemann-Liouville fractional derivative operator of order* $\propto > 0$ *is defined as:*

$$D_t^{\propto} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-x)^{n-\alpha-1} f(x)dx \tag{2}$$

where n is an integer and $n-1 < \alpha \le n$.

Definition (2.3), [11]: *The Caputo fractional derivative operator of order α is defined as:*

$$^{c}D_t^{\propto} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-x)^{n-\alpha-1} \frac{d^n}{dt^n} f(x)dx \tag{3}$$

where n is an integer and $n-1 < \alpha \le n$.

Now, the following properties of fractional integro-differentail equation can be found in [16], [17], [20].

Properties (2.1):

1) ${}_0I_t^{\propto}\ {}_0I_t^{\beta} f(t) = {}_0I_t^{\propto+\beta}\ f(t)$ for $\propto, \beta \ge 0$.
2) ${}_0I_t^{\propto}\ {}_0I_t^{\beta} f(t) = {}_0I_t^{\beta}\ {}_0I_t^{\propto}\ f(t)$ for $\propto, \beta \ge 0$.
3) ${}_0I_t^{\propto}\ {}_0^cD_t^{\propto} f(t) = f(t) - f(0)$, $0 < \propto < 1$.

3. Homotopy Perturbation (Method of Solution)

Consider the descriptor multi- fractional integro - differential nonlinear system

$$E\left[D^{\delta}u(t) + D^{\gamma}u(t)\right] + FD^{\beta}u(t) = Bg(t) + CI^{\alpha}M(t), t \in [0,G], \tag{4}$$

$$u(0)=u_0=\begin{bmatrix}u_{1,0}\\u_{2,0}\end{bmatrix}, \mathrm{n}<\alpha<\beta<\gamma<\delta\le \mathrm{n}-1,$$

where $D^{\delta}u(t), D^{\gamma}u(t), D^{\beta}u(t)$ is Caputo fractional derivative of u(t) $E, F\in\mathbb{R}^{n\times n}$ be a singular matrix,$B, C\in\mathbb{R}^{n\times n}$, $I^{\alpha}u(t)$ is the fractional order operator in the Riemann-Liouville fractional integral, and M(t) is linear continuous function with the following:

i) $E=\begin{bmatrix}E_{11} & 0\\ 0 & 0\end{bmatrix}, F=\begin{bmatrix}F_{11} & F_{12}\\ F_{21} & F_{22}\end{bmatrix}, E_{11}\in\mathbb{R}^{r_1\times r_1},, F_{22}\in\mathbb{R}^{n-r_1\times n-r_1}$

ii) $I^{\alpha}M(u(t))=\begin{bmatrix}I^{\alpha}M_1(u_1(t))\\ I^{\alpha}M_2(u_2(t))\end{bmatrix},$

iii) $B=\begin{bmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{bmatrix}, C=\begin{bmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{bmatrix}$, by conditions above, we can get

$$E\left[D^{\delta}u(t)+D^{\gamma}u(t)\right]+FD^{\beta}u(t)=Bg(t)+CI^{\alpha}M(t)$$

$$\begin{bmatrix}E_{11} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}D^{\delta}u_1(t) & D^{\gamma}u_1(t)\\ D^{\delta}u_2(t) & D^{\gamma}u_2(t)\end{bmatrix}+\begin{bmatrix}F_{11} & F_{12}\\ F_{21} & F_{22}\end{bmatrix}\begin{bmatrix}D^{\beta}u_1(t)\\ D^{\beta}u_2(t)\end{bmatrix}=\begin{bmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{bmatrix}\begin{bmatrix}g_1(t)\\ g_2(t)\end{bmatrix}+\begin{bmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{bmatrix}\begin{bmatrix}I^{\alpha}M_1(u_1(t))\\ I^{\alpha}M_2(u_2(t))\end{bmatrix}$$

Hence

$$\begin{bmatrix}E_{11}D^{\delta}u_1(t)+E_{11}D^{\gamma}u_1(t)\\ 0\end{bmatrix}+\begin{bmatrix}F_{11}D^{\beta}u_1(t)+F_{12}D^{\beta}u_2(t)\\ F_{21}D^{\beta}u_1(t)+F_{22}D^{\beta}u_2(t)\end{bmatrix}=\begin{bmatrix}B_{11}g_1(t)+B_{12}g_2(t)\\ B_{21}g_1(t)+B_{22}g_2(t)\end{bmatrix}$$
$$+\begin{bmatrix}C_{11}I^{\alpha}M_1(u_1(t))+C_{12}I^{\alpha}M_2(u_1(t))\\ C_{21}I^{\alpha}M_1(u_1(t))+C_{22}I^{\alpha}M_2(u_2(t))\end{bmatrix}$$

We can get,

$$\begin{bmatrix}D^{\delta}u_1(t)+D^{\gamma}u_1(t)\\ D^{\beta}u_2(t)+F_{22}^{-1}F_{21}D^{\beta}u_1(t)\end{bmatrix}$$
$$=\begin{bmatrix}E_{11}^{-1}B_{11}g_1(t)+E_{11}^{-1}B_{12}g_2(t)+E_{11}^{-1}C_{11}I^{\alpha}M_1(u_1(t))+E_{11}^{-1}C_{12}I^{\alpha}M_2(u_2(t))\\ -E_{11}^{-1}F_{11}D^{\beta}u_1(t)-E_{11}^{-1}F_{12}D^{\beta}u_2(t)\\ F_{22}^{-1}B_{21}g_1(t)+F_{22}^{-1}B_{22}g_2(t)+F_{22}^{-1}C_{21}I^{\alpha}M_1(u_1(t))+F_{22}^{-1}C_{22}I^{\alpha}M_2(u_2(t))\end{bmatrix} \quad (5)$$

We can rewrite equation (5) as: $\mu(u)-g(x)=0$, here $\mu(u)=\begin{bmatrix}\mu_1(u)\\ \mu_2(u)\end{bmatrix}$. The operator μ can be divided into two parts φ_μ and ω_μ where φ_μ is a linear operator and ω_μ is a nonlinear operator, we can get $\varphi_\mu(u)+\omega_\mu(u)-g(x)=0$, Where $\varphi_\mu(u)=\begin{bmatrix}\varphi_{1,\mu}(u)\\ \varphi_{2,\mu}(u)\end{bmatrix}=\begin{bmatrix}D^{\delta}u_1(t)+D^{\gamma}u_1(t)\\ D^{\beta}u_2(t)\end{bmatrix}$ and

$$\omega_\mu(u)=\begin{bmatrix}\omega_{1,\mu}(u)\\ \omega_{2,\mu}(u)\end{bmatrix}=\begin{bmatrix}E_{11}^{-1}B_{11}g_1(t)+E_{11}^{-1}B_{12}g_2(t)+E_{11}^{-1}C_{11}I^{\alpha}M_1(u_1(t))+E_{11}^{-1}C_{12}I^{\alpha}M_2M_2(u_2(t))\\ -E_{11}^{-1}F_{11}D^{\beta}u_1(t)-E_{11}^{-1}F_{12}D^{\beta}u_2(t)\\ F_{22}^{-1}B_{21}g_1(t)+F_{22}^{-1}B_{22}g_2(t)+F_{22}^{-1}C_{21}I^{\alpha}M_1(u_1(t))+F_{22}^{-1}C_{22}I^{\alpha}M_2(u_2(t))\\ -F_{22}^{-1}F_{21}D^{\beta}u_1(t)\end{bmatrix}$$

According to homotopy perturbation method, we can construct a homotopy $v_h(x,p)=\begin{bmatrix} v_1(x_1,p_1) \\ v_2(x_2,p_2) \end{bmatrix}$ which satisfy:

$$v_1(x_1,p_1):\mathbb{R}^{r_1}\times\mathbb{R}^{r_1}\to\mathbb{R},\ v_2(x_2,p_2):\mathbb{R}^{n-r_1}\times\mathbb{R}^{n-r_1}\to\mathbb{R},$$

$$p=\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}, p\in\mathbb{R}^n,\ p_1\in\mathbb{R}^{r_1},\ p_2\in\mathbb{R}^{n-r_1},\text{ such that } p_1=\begin{bmatrix} p_{1,1} \\ p_{2,1} \\ \vdots \\ p_{m,1} \end{bmatrix}, p_2=\begin{bmatrix} p_{1,2} \\ p_{2,2} \\ \vdots \\ p_{n-m,2} \end{bmatrix}, p_{r_1,1}\in[0,1], m=1,2,3,\ldots,r_1.$$

Now, construct a homotopy $v_h=\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ such that

$$\vartheta(v_h,p)=\begin{bmatrix} H_1(v_1,p_1) \\ H_2(v_2,p_2) \end{bmatrix}=\begin{bmatrix} (I-p_1)\left[\varphi_{1,\mu}(v_1)-\varphi_{1,\mu}(u_{1,0})+p_1\left[\mu(v_1)-g_1(t)\right]\right] \\ (I-p_2)\left[\varphi_{2,\mu}(v_2)-\varphi_{2,\mu}(u_{2,0})+p_2\left[\mu(v_2)-g_2(t)\right]\right] \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{6}$$

The homotopy that formed $v_h(x,p)$ achieves the following equation

$$\begin{aligned}\begin{bmatrix} H_1(v_1,p_1) \\ H_2(v_2,p_2) \end{bmatrix}&=\vartheta(v,1)=\begin{bmatrix} H_1(v_1,I) \\ H_2(v_2,I) \end{bmatrix}\\ &=\begin{bmatrix} (I-p_1)\left[D^\delta v_1(t)-D^\delta u_{1,0}(t)\right] \\ \quad +p_1\left[D^\delta v_1(t)-D^\gamma v_1(t)-E_{11}{}^{-1}B_{11}g_1(t)+E_{11}{}^{-1}B_{12}g_2(t)-E_{11}{}^{-1}C_{11}I^\alpha M_1(u_1(t))\right) \\ \quad -E_{11}{}^{-1}C_{12}I^\alpha M_2\left(u_2(t)-E_{11}{}^{-1}F_{11}D^\beta u_1(t)-E_{11}{}^{-1}F_{12}D^\beta u_2(t)\right] \\ (I-p_2)\left[D^\beta v_2(t)-D^\beta u_{2,0}(t)\right] \\ \quad +p_2\left[D^\beta v_2(t)+F_{22}{}^{-1}F_{21}D^\beta v_1(t)-F_{22}{}^{-1}B_{21}g_1(t)-F_{22}{}^{-1}B_{22}g_2(t)\right] \\ \quad -F_{22}{}^{-1}C_{21}I^\alpha M_1(u_1(t),u_2(t)))-F_{22}{}^{-1}C_{22}I^\alpha M_2(u_2(t)) \end{bmatrix}\\ &=\begin{bmatrix} 0 \\ 0 \end{bmatrix}\end{aligned} \tag{7}$$

Where $u_{1,0}$, $u_{2,0}$ can be obtained from an equation (4), so $\vartheta(v,0)=\begin{bmatrix} H_1(v_1,0) \\ H_2(v_2,0) \end{bmatrix}=\begin{bmatrix} \left[D^\delta v_1(t)-D^\delta u_{1,0}(t)\right] \\ \left[D^\delta v_2(t)-D^\delta u_{2,0}(t)\right] \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{aligned}\vartheta(v,1)&=\begin{bmatrix} H_1(v_1,I) \\ H_2(v_2,I) \end{bmatrix}\\ &=\begin{bmatrix} D^\delta v_1(t)-D^\gamma v_1(t)-E_{11}{}^{-1}B_{11}g_1(t)+E_{11}{}^{-1}B_{12}g_2(t) \\ -E_{11}{}^{-1}C_{11}I^\alpha M_1(u_1(t))-E_{11}{}^{-1}C_{12}I^\alpha M_2(u_2(t)) \\ +E_{11}{}^{-1}F_{11}D^\beta v_1(t)+E_{11}{}^{-1}F_{12}D^\beta v_2(t) \end{bmatrix}\\ &\quad\times\begin{bmatrix} D^\beta v_2(t)+F_{22}{}^{-1}F_{21}D^\beta v_1(t)-F_{22}{}^{-1}B_{21}g_1(t)-F_{22}{}^{-1}B_{22}g_2(t) \\ -F_{22}{}^{-1}C_{21}I^\alpha M_1(u_1(t))-F_{22}{}^{-1}C_{22}I^\alpha M_2(u_2(t)) \end{bmatrix}\\ &=\begin{bmatrix} 0 \\ 0 \end{bmatrix}\end{aligned} \tag{8}$$

It is noticeable that the processes of change that p from zero to unity is just that of $\vartheta(v_h, p)$ from $\begin{bmatrix} D^\delta v_1(t) - D^\delta u_{1,0}(t) \\ D^\delta v_2(t) - D^\delta u_{2,0}(t) \end{bmatrix}$ to

$$\begin{bmatrix} D^\delta v_1(t) - D^\gamma v_1(t) - E_{11}^{-1} B_{11} g_1(t) + E_{11}^{-1} B_{12} g_2(t) \\ -E_{11}^{-1} C_{11} I^\alpha M_1(u_1(t)) - E_{11}^{-1} C_{12} I^\alpha M_2(u_2(t)) \\ +E_{11}^{-1} F_{11} D^\beta u_1(t) + E_{11}^{-1} F_{12} D^\beta u_2(t) \\ D^\beta v_2(t) + F_{22}^{-1} F_{21} D^\beta v_1(t) - F_{22}^{-1} B_{21} g_1(t) - F_{22}^{-1} B_{22} g_2(t) \\ -F_{22}^{-1} C_{21} I^\alpha M_1(u_1(t)) - F_{22}^{-1} C_{22} I^\alpha M_2(u_2(t)) \end{bmatrix} \tag{9}$$

We can take advantage of a topic in topology where we call the amount deformation $\begin{bmatrix} D^\delta v_1(t) - D^\delta u_{1,0}(t) \\ D^\delta v_2(t) - D^\delta u_{2,0}(t) + F_{22}^{-1} F_{21}\left(D^\delta v_1(t) - D^\delta u_{1,0}(t)\right) \end{bmatrix}$ and the other part

$$\begin{bmatrix} D^\delta v_1(t) - D^\gamma v_1(t) - E_{11}^{-1} B_{11} g_1(t) + E_{11}^{-1} B_{12} g_2(t) \\ - E_{11}^{-1} C_{11} I^\alpha M_1(u_1(t)) - E_{11}^{-1} C_{12} I^\alpha M_2(u_2(t)) \\ + E_{11}^{-1} F_{11} D^\beta u_1(t) + E_{11}^{-1} F_{12} D^\beta u_2(t) \\ D^\beta v_2(t) + F_{22}^{-1} F_{21} D^\beta v_1(t) - F_{22}^{-1} B_{21} g_1(t) - F_{22}^{-1} B_{22} g_2(t) \\ - F_{22}^{-1} C_{21} I^\alpha M_1(u_1(t)) - F_{22}^{-1} C_{22} I^\alpha M_2(u_2(t)) \end{bmatrix}$$

can be called a homotopic equation.

Now, we consider that $\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$ is the solution of the equation (7),

$$\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} p_1^0 v_{1,0}(t) + p_1^1 v_{1,1}(t) + p_1^2 v_{1,2}(t) + \dots \\ p_2^0 v_{2,0}(t) + p_2^1 v_{2,1}(t) + p_2^2 v_{2,2}(t) + \dots \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty} p_1^k v_{1,k}(t) \\ \sum_{k=0}^{\infty} p_2^k v_{2,k}(t) \end{bmatrix} \tag{10}$$

We can get the approximate solution to the descriptor equation (4), as follows,

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} \lim\limits_{p \to 1} v_1(t) \\ \lim\limits_{p \to 1} v_2(t) \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty} v_{1,k}(t) \\ \sum_{k=0}^{\infty} v_{2,k}(t) \end{bmatrix} \tag{11}$$

by substituting equation (10) into equation (7), we get

$$\begin{bmatrix} H_1(v_1,p_1) \\ H_2(v_2,p_2) \end{bmatrix} = \begin{bmatrix} \begin{array}{c} \left[\sum_{k=0}^{\infty} p_1^k D^{\delta} v_{1,k}(t)\right] - D^{\delta} u_{1,0}(t) + p_1 D^{\delta} u_{1,0}(t) \\ +p_1 \left[-E_{11}^{-1} C_{11} I^{\alpha} M_1 \left(\sum_{k=0}^{\infty} p_1^k v_{1,k}(t)\right) \right. \\ -E_{11}^{-1} C_{12} I^{\alpha} M_2 \left(\sum_{k=0}^{\infty} p_2^k v_{2,k}(t)\right) \\ \left. -E_{11}^{-1} B_{11} g_1(t) + E_{11}^{-1} B_{12} g_2(t) \right] \\ +E_{11}^{-1} F_{11} \left[\sum_{k=0}^{\infty} p_1^k D^{\beta} v_{1,k}(t)\right] + E_{11}^{-1} F_{12} \left[\sum_{k=0}^{\infty} p_2^k D^{\beta} v_{2,k}(t)\right] \\ -\left[\sum_{k=0}^{\infty} p_1^k D^{\gamma} v_{1,k}(t)\right]\Bigg] \end{array} \\ \left[\begin{array}{l} \left[\sum_{k=0}^{\infty} p_2^k D^{\beta} v_{2,k}(t)\right] - D^{\beta} u_{2,0}(t) + p_2 D^{\beta} u_{2,0}(t) \\ +p_2 \left\{ \left[-F_{22}^{-1} C_{22} I^{\alpha} M_2 \left(\sum_{k=0}^{\infty} p_2^k v_{2,k}(t)\right) \right] \right. \\ -F_{22}^{-1} C_{21} I^{\alpha} M_1 \left(\sum_{k=0}^{\infty} p_1^k v_{1,k}(x)\right) - F_{22}^{-1} \\ \left. B_{21} g_1(t) - F_{22}^{-1} B_{22} g_2(t) + F_{22}^{-1} F_{21} \left[\sum_{k=0}^{\infty} p_1^k v_{1,k}(t)\right] \right\} \end{array} \right] \end{bmatrix}$$

By the equation (9), we get

$$= \begin{bmatrix} D^{\delta}\left(p_1^0 v_{1,0}(t) + p_1^1 v_{1,1}(t) + p_1^2 v_{1,2}(t) + \cdots\right) \\ -D^{\delta} u_{1,0}(t) + p_1 D^{\delta} u_{1,0}(t) \\ +p_1 \left[-E_{11}^{-1} C_{11} I^{\alpha} M_1 \left(\sum_{k=0}^{\infty} p_1^k v_{1,k}(t)\right) - E_{11}^{-1} C_{12} I^{\alpha} M_2 \left(\sum_{k=0}^{\infty} p_2^k D^{\beta} v_{2,k}(t)\right) \right. \\ \left. -E_{11}^{-1} B_{11} g_1(t) + E_{11}^{-1} B_{12} g_2(t) \right] \\ +E_{11}^{-1} F_{11} D^{\beta}\left(p_1^0 v_{1,0}(t) + p_1^1 v_{1,1}(t) + p_1^2 v_{1,2}(t) + \cdots\right) + E_{11}^{-1} F_{12} \\ D^{\beta}\left(p_2^0 v_{2,0}(t) + p_2^1 v_{2,1}(t) + p_2^2 v_{2,2}(t) + \cdots\right)\Big] = 0 \\ +D^{\gamma}\left(p_1^0 v_{1,0}(t) + p_1^1 v_{1,1}(t) + p_1^2 v_{1,2}(t) + \cdots\right)\Big] \\ D^{\beta}\left(p_2^0 v_{2,0}(t) + p_2^1 v_{2,1}(t) + p_2^2 v_{2,2}(t) + \cdots\right) - D^{\beta} u_{2,0}(t) + p_2 D^{\beta} u_{2,0}(t) \\ +p_2 \left[-F_{22}^{-1} C_{22} I^{\alpha} M_2 \left(\sum_{k=0}^{\infty} p_2^k v_{2,k}(t)\right) \right] \\ -F_{22}^{-1} C_{21} I^{\alpha} M_1 \left(\sum_{k=0}^{\infty} p_1^k v_{1,k}(t)\right) - F_{22}^{-1} B_{21} g_1(t) - F_{22}^{-1} B_{22} g_2(t) \\ +F_{22}^{-1} F_{21}\left[D^{\beta}\left(p_1^0 v_{1,0}(t) + p_1^1 v_{1,1}(t) + p_1^2 v_{1,2}(t) + \cdots\right)\right] = 0 \end{bmatrix} \tag{12}$$

Now equating same power of p of equation (12), we get

$$\begin{bmatrix} p_1^0 \\ p_2^0 \end{bmatrix} = \begin{bmatrix} D^{\delta} v_{1,0}(t) - D^{\delta} u_{1,0}(t) = 0 \\ D^{\beta} v_{2,0}(t) - D^{\beta} u_{2,0}(t) = 0 \end{bmatrix} \tag{13}$$

The first, we find solution $\begin{bmatrix} v_{1,0} \\ v_{2,0} \end{bmatrix}$, using multiply both side, of the equation (13), by the Riemann-Liouville fractional integral operator $\begin{bmatrix} I^{\delta} \\ I^{\beta} \end{bmatrix}$

$$\begin{bmatrix} p_1^1 \\ p_2^1 \end{bmatrix} = \begin{bmatrix} D^{\delta} v_{1,1}(t) + D^{\gamma} v_{1,0}(t) + D^{\delta} u_{1,0}(t) + D^{\gamma} u_{1,0}(t) - E_{11}{}^{-1} B_{11} g_1(t) + E_{11}{}^{-1} B_{12} g_2(t) \\ -E_{11}{}^{-1} C_{11} I^{\alpha} M_1\left(v_{1,0}(t)\right) - E_{11}{}^{-1} C_{12} I^{\alpha} M_2\left(\left(v_{2,0}(t)\right)\right. \\ +E_{11}{}^{-1} F_{11} D^{\beta} v_{1,0}(t) + E_{11}{}^{-1} F_{12} D^{\beta} v_{2,0}(t) = 0 \\ D^{\beta} v_{2,1}(t) + D^{\beta} u_{2,0}(t) + F_{22}{}^{-1} F_{21}\left[D^{\beta} v_{1,1}(t) - D^{\beta} u_{1,0}(t)\right] - F_{22}{}^{-1} B_{21} g_1(t) - F_{22}{}^{-1} B_{22} g_2(t) \\ -F_{22}{}^{-1} C_{21} I^{\alpha} M_1\left(v_{1,0}(t)\right) - F_{22}{}^{-1} C_{22} I^{\alpha} M_2(t)\big) = 0 \end{bmatrix} \tag{14}$$

Second, to calculate $\begin{bmatrix} v_{1,1} \\ v_{2,1} \end{bmatrix}$, using multiply both sides, of the equation (14), by $\begin{bmatrix} I^{\delta} \\ I^{\beta} \end{bmatrix}$,

$$\begin{bmatrix} p_1^2 \\ p_2^2 \end{bmatrix} = \begin{bmatrix} D^{\delta} v_{1,2}(t) + D^{\gamma} v_{1,1}(t) + E_{11}{}^{-1} F_{11} D^{\beta} v_{1,1}(t) + E_{11}{}^{-1} F_{12} D^{\beta} v_{2,1}(t) = 0 \\ D^{\beta} v_{2,2}(t) + F_{22}{}^{-1} F_{21} [D^{\beta} v_{1,1}(t) = 0 \end{bmatrix} \tag{15}$$

Third, in the same way above we will find solution $\begin{bmatrix} v_{1,2} \\ v_{2,2} \end{bmatrix}$, depending on equation (15),

And the other term of power p can be written according to the following formula:

$$\begin{bmatrix} p_1^j \\ p_2^j \end{bmatrix} = \begin{bmatrix} D^{\delta} v_{1,j}(t) + D^{\gamma} v_{1,j-1}(t) - +E_{11}{}^{-1} F_{11} D^{\beta} v_{1,j-1}(t) + E_{11}{}^{-1} F_{12} D^{\beta} v_{2,j-1} = 0 \\ D^{\beta} v_{2,j}(t) + F_{22}{}^{-1} F_{21} [D^{\beta} v_{1,j-1}(t) = 0 \end{bmatrix}, j = 3, 4, \ldots \tag{16}$$

Finally, we calculate solution $\begin{bmatrix} v_{1,j} \\ v_{2,j} \end{bmatrix}$, , $j = 3, 4, \ldots$, using multiply both side of the equation (16), by the Riemann-Liouville fractional integral operator $\begin{bmatrix} I^{\delta} \\ I^{\beta} \end{bmatrix}$, through the power p we will get the results of v that are substituting in u and thus lead to the solution of the system(4), that means (by substituting solution of equation (14), (15) and (16) into equation (11)). then we obtain the approximate solution to the descriptor equation (4).

Example (3.1): Consider the descriptor multi fractional integro – differential nonlinear system

$$E\left[D^{\delta} u(t) + D^{\gamma} u(t)\right] + F D^{\beta} u(t) = B g(t) + C I^{\alpha} M(t)_{,}\ t \in [0, 4], \tag{17}$$

$\begin{bmatrix} u_{1,0}(0) \\ u_{2,0}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} D^{0.75} u_{1,0}(0) \\ D^{0.75} u_{2,0}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} D^{0.3} u_{1,0}(0) \\ D^{0.3} u_{2,0}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, the exact solution of system (17) is given by

$$u(t)=\begin{bmatrix}\frac{2}{\mathcal{3}(2.3)}t^{1.3}\\ \frac{6}{\mathcal{3}(3.5)}t^{2.5}\end{bmatrix},\ \text{Such that}\ E=\begin{bmatrix}1&0\\0&0\end{bmatrix},\quad F=\begin{bmatrix}0&0\\0&1\end{bmatrix},\quad B=\begin{bmatrix}1&0\\0&1\end{bmatrix},\ C=\begin{bmatrix}1&0\\0&1\end{bmatrix},\delta=0.75,\gamma=0.5,$$

$$\beta=0.3,\alpha=0.2,\begin{bmatrix}g_1(t)\\g_2(t)\end{bmatrix}=\begin{bmatrix}\frac{2}{\mathcal{3}(2.25)}t^{1.25}\\0\end{bmatrix},\begin{bmatrix}M_1(t)\\M_2(t)\end{bmatrix}=\begin{bmatrix}\frac{2}{\mathcal{3}(2.3)}t^{1.3}\\ \frac{6}{\mathcal{3}(3.5)}t^{2.5}\end{bmatrix}$$

$$\begin{aligned}&\begin{bmatrix}1&0\\0&0\end{bmatrix}\left[\begin{bmatrix}D^{0.75}u_1(t)\\D^{0.75}u_2(t)\end{bmatrix}+\begin{bmatrix}D^{0.5}u_1(t)\\D^{0.5}u_2(t)\end{bmatrix}\right]+\begin{bmatrix}0&0\\0&1\end{bmatrix}\begin{bmatrix}D^{0.3}u_1(t)\\D^{0.3}u_2(t)\end{bmatrix}\\&=\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}\frac{2}{\mathcal{3}(2.3)}t_1^{1.3}\\ \frac{6}{\mathcal{3}(3.5)}t_1^{2.5}\end{bmatrix}+\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}I^{0.2}M_1(t)\\I^{0.2}M_2(t)\end{bmatrix}\end{aligned}\tag{18}$$

$u(0)=u_0=\begin{bmatrix}u_{1,0}\\u_{2,0}\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$, then,

$$\begin{bmatrix}D^{0.75}u_1(t)+D^{0.5}u_1(t)\\D^{0.3}u_2(t)\end{bmatrix}=\begin{bmatrix}\frac{2}{\mathcal{3}(2.25)}t^{1.25}+\frac{2}{\mathcal{3}(2.5)}t^{1.5}\\ \frac{6}{\mathcal{3}(3.7)}t^{2.7}\end{bmatrix}$$

By equation (6), we can construct homotopy $v_h=\begin{bmatrix}v_1\\v_2\end{bmatrix}$ such that

$$\begin{aligned}\vartheta(v_h,p)&=\begin{bmatrix}H_1(v_1,p_1)\\H_2(v_2,p_2)\end{bmatrix}\\&=\begin{bmatrix}\left[p_1^0D^{0.75}v_{1,0}(t)+p_1^1D^{0.75}v_{1,1}(t)+p_1^2D^{0.75}v_{1,2}(t)+\cdots\right]-D^{0.75}u_{1,0}(t)\\+p_1^1\left\{\left[D^{0.75}u_{1,0}(t)-\frac{2}{\Gamma(2.25)}t^{1.25}-\frac{2}{\Gamma(2.5)}t^{1.5}\right]\right.\\ \left.+\left[p_1^0D^{0.5}v_{1,0}(t)+p_1^1D^{0.5}v_{1,1}(t)+p_1^2D^{0.5}v_{1,2}(t)+\cdots\right]\right\}=0\\ \left[p_2^0D^{0.3}v_{2,0}(t)+p_2^1D^{0.3}v_{2,1}(t)+p_2^2D^{0.3}v_{2,2}(t)+\cdots\right]\\-D^{0.3}u_{2,0}(t)+p_2^1D^{0.3}u_{2,0}(t)+p_2^1\left[-\frac{6}{\Gamma(3.7)}t^{2.7}\right]=0\end{bmatrix}\end{aligned}\tag{19}$$

Now, then, $\begin{bmatrix}v_{1,0}(t)\\v_{2,0}(t)\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}v_{1,1}(t)\\v_{2,1}(t)\end{bmatrix}=\begin{bmatrix}t^2+\frac{2}{\mathcal{3}(3.25)}t^{2.25}\\t^3\end{bmatrix}$, then

$$\begin{bmatrix}v_{1,2}(t)\\v_{2,2}(t)\end{bmatrix}=\begin{bmatrix}-\left[\frac{2}{\mathcal{3}(3.25)}t^{2.25}+\frac{2}{\mathcal{3}(3.5)}t^{2.5}\right]\\0\end{bmatrix},\quad\begin{bmatrix}v_{1,3}(t)\\v_{2,3}(t)\end{bmatrix}=\begin{bmatrix}\left[\frac{2}{\mathcal{3}(3.5)}t^{2.5}+\frac{2}{\mathcal{3}(3.75)}t^{2.75}\right]\\0\end{bmatrix},$$

then, $\begin{bmatrix} v_{1,4}(t) \\ v_{2,4}(t) \end{bmatrix} = \begin{bmatrix} -\left[\frac{2}{3(3.75)}t^{2.75} + \frac{1}{3}t^{3}\right] \\ 0 \end{bmatrix}$, $\begin{bmatrix} v_{1,5}(t) \\ v_{2,5}(t) \end{bmatrix} = \begin{bmatrix} \left[\frac{2}{3(4)}t^{3} + \frac{1}{3(4.25)}t^{3.25}\right] \\ 0 \end{bmatrix}$, We can get the approximate solution to the descriptor equation (17), as follows,

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{11} v_{1,k}(t) \\ \sum_{k=0}^{11} v_{2,k}(t) \end{bmatrix}$$

$$= \begin{bmatrix} [t^2 + \frac{2}{3(3.25)}t^{2.25}] - \left[\frac{2}{3(3.25)}t^{2.25} + \frac{2}{3(3.5)}t^{2.5}\right] \\ + \left[\frac{2}{3(3.5)}t^{2.5} + \frac{2}{3(3.75)}t^{2.75}\right] - \left[\frac{2}{3(3.75)}t^{2.75} + \frac{1}{3}t^{3}\right] \\ + \left[\frac{2}{3(4)}t^{3} + \frac{1}{3(4.25)}t^{3.25}\right] + \ldots - \ldots \\ t^3 \end{bmatrix} \tag{20}$$

$$= \begin{bmatrix} t^2 \\ t^3 \end{bmatrix}$$

the equation (20) is approximate solution of equation (17).

We can observe that the behavior of the curve in the a (HPM) approach approximates the behavior of the exact solution in Figure (1).

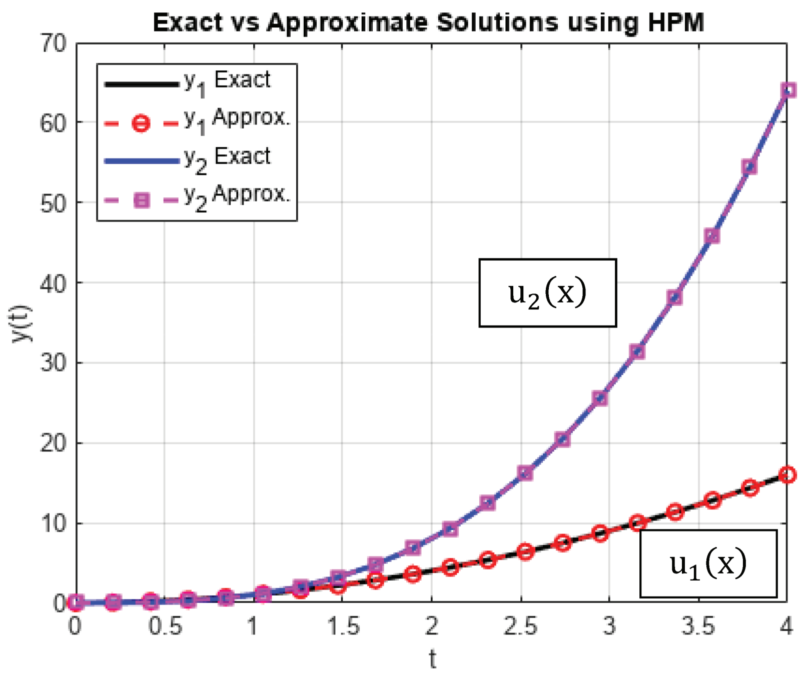

Figure 1. The figure gives us a comparison between the approximate solution and the exact solution

Example (3.2): Consider the descriptor multi fractional integro – differential nonlinear system

$$E\left[D^{\delta}u(t)+D^{\gamma}u(t)\right]+FD^{\beta}u(t)=Bg(t)+CI^{\alpha}M(t),\ t\in[0,4], \tag{21}$$

$$\begin{bmatrix} u_{1,0}(0) \\ u_{2,0}(0) \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix},\begin{bmatrix} D^{0.8}u_{1,0}(0) \\ D^{0.8}u_{2,0}(0) \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix},\begin{bmatrix} D^{0.4}u_{1,0}(0) \\ D^{0.4}u_{2,0}(0) \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

the exact solution of system (21) is given by $u(t)=\begin{bmatrix} t^{3.1} \\ t^{0.85} \end{bmatrix}$

Such that $E=\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$, $F=\begin{bmatrix} 0 & 0 \\ 3 & 4 \end{bmatrix}$, $B=\begin{bmatrix} 3\Gamma(4.1) & 0 \\ 0 & 4\Gamma(1.85) \end{bmatrix}$, $C=\begin{bmatrix} \frac{\Gamma(4.1)}{\Gamma(3.2)} & 0 \\ 0 & 3\Gamma(4.1) \end{bmatrix}$, $\delta=0.8, \gamma=0.6,$ $\beta=0.4, \alpha=0.3$, $\begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix}=\begin{bmatrix} \frac{1}{\Gamma(3.3)}t^{2.3} \\ \frac{1}{\Gamma(1.45)}t^{0.45} \end{bmatrix}$, $\begin{bmatrix} M_1(t) \\ M_2(t) \end{bmatrix}=\begin{bmatrix} 3t^{2.2} \\ \frac{1}{\Gamma(3.4)}t^{2.4} \end{bmatrix}$ then

$$u(0)=u_0=\begin{bmatrix} u_{1,0} \\ u_{2,0} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ hence,}$$

$$\begin{bmatrix} D^{0.8}u_1(t)+D^{0.6}u_1(t) \\ D^{0.4}u_2(t)+\frac{3}{4}D^{0.4}u_1(t) \end{bmatrix}=\begin{bmatrix} \frac{\Gamma(4.1)}{\Gamma(3.3)}t^{2.3}+\frac{\Gamma(4.1)}{\Gamma(3.5)}t^{2.5} \\ \frac{\Gamma(1.85)}{\Gamma(1.45)}t^{0.45}+\frac{3\Gamma(4.1)}{4\Gamma(3.7)}t^{2.7} \end{bmatrix}, \text{ we can get}$$

$$\begin{bmatrix} D^{0.8}u_1(t) \\ D^{0.4}u_2(t) \end{bmatrix}=\begin{bmatrix} \frac{\Gamma(4.1)}{\Gamma(3.3)}t^{2.3}+\frac{\Gamma(4.1)}{\Gamma(3.5)}t^{2.5}-D^{0.6}u_1(t) \\ \frac{\Gamma(1.85)}{\Gamma(1.45)}t^{0.45}+\frac{3\Gamma(4.1)}{4\Gamma(3.7)}t^{2.7}-\frac{3}{4}D^{0.4}u_1(t) \end{bmatrix} \tag{22}$$

By equation (6), we can construct homotopy $v_h=\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ such that

$$\vartheta(v_h,p)=\begin{bmatrix} H_1(v_1,p_1) \\ H_2(v_2,p_2) \end{bmatrix}$$

$$=\begin{bmatrix} \begin{gathered} \left[p_1^0D^{0.8}v_{1,0}(t)+p_1^1D^{0.8}v_{1,1}(t)+p_1^2D^{0.8}v_{1,2}(t)+\cdots\right]-D^{0.8}u_{1,0}(t)+p_1^1D^{0.8}u_{1,0}(t) \\ +p_1^1\left\{\left[-\frac{\Gamma(4.1)}{\Gamma(3.3)}t^{2.3}-\frac{\Gamma(4.1)}{\Gamma(3.5)}t^{2.5}\right]\right. \\ \left.+\left[p_1^0D^{0.6}v_{1,0}(t)+p_1^1D^{0.6}v_{1,1}(t)+p_1^2D^{0.6}v_{1,2}(t)+\cdots\right]\right\}=0 \\ \left[p_2^0D^{0.4}v_{2,0}(t)+p_2^1D^{0.4}v_{2,1}(t)+p_2^2D^{0.4}v_{2,2}(t)+\cdots\right] \\ -D^{0.4}u_{2,0}(t)+p_2^1D^{0.4}u_{2,0}(t)+p_2^1\left\{\left[-\left(\frac{\Gamma(1.85)}{\Gamma(1.45)}t^{0.45}+\frac{3\Gamma(4.1)}{4\Gamma(3.7)}t^{2.7}\right]\right.\right. \\ \left.+\frac{3}{4}\left[p_1^0D^{0.4}v_{1,0}(t)+p_1^1D^{0.4}v_{1,1}(t)+p_1^2D^{0.4}v_{1,2}(t)+\cdots\right]\right\}=0 \end{gathered} \end{bmatrix} \tag{23}$$

Now, depending on the power of p, we find the solutions of $\begin{bmatrix} \mathrm{v}_{1,j} \\ \mathrm{v}_{2,j} \end{bmatrix}$, $j = 1, 2, \ldots$, we get

$$\begin{bmatrix} \mathrm{v}_{1,0}(\mathrm{t}) \\ \mathrm{v}_{2,0}(\mathrm{t}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \mathrm{v}_{1,1}(\mathrm{t}) \\ \mathrm{v}_{2,1}(\mathrm{t}) \end{bmatrix} = \begin{bmatrix} \mathrm{t}^{3.1} + \dfrac{3(4.1)}{3(4.3)} \mathrm{t}^{3.3} \\ \mathrm{t}^{0.85} + \dfrac{3}{4} \mathrm{t}^{3.1} \end{bmatrix},$$

then

$$\begin{bmatrix} \mathrm{v}_{1,2}(\mathrm{t}) \\ \mathrm{v}_{2,2}(\mathrm{t}) \end{bmatrix} = \begin{bmatrix} -\left[\dfrac{3(4.1)}{3(4.3)} \mathrm{t}^{3.3} + \dfrac{3(4.1)}{3(4.5)} \mathrm{t}^{3.5} \right] \\ -\dfrac{3}{4} \left(\mathrm{t}^{3.1} + \dfrac{3(4.1)}{3(4.3)} \mathrm{t}^{3.3} \right) \end{bmatrix}, \begin{bmatrix} \mathrm{v}_{1,3}(\mathrm{t}) \\ \mathrm{v}_{2,3}(\mathrm{t}) \end{bmatrix} = \begin{bmatrix} \left[\dfrac{3(4.1)}{3(4.5)} \mathrm{t}^{3.5} + \dfrac{3(4.1)}{3(4.7)} \mathrm{t}^{3.7} \right] \\ \dfrac{3}{4} \left[\dfrac{3(4.1)}{3(4.3)} \mathrm{t}^{3.3} + \dfrac{3(4.1)}{3(4.5)} \mathrm{t}^{3.5} \right] \end{bmatrix},$$

$$\begin{bmatrix} \mathrm{v}_{1,4}(\mathrm{t}) \\ \mathrm{v}_{2,4}(\mathrm{t}) \end{bmatrix} = \begin{bmatrix} -\left[\dfrac{3(4.1)}{3(4.7)} \mathrm{t}^{3.7} + \dfrac{3(4.1)}{3(4.9)} \mathrm{t}^{3.9} \right] \\ -\dfrac{3}{4} \left[\dfrac{3(4.1)}{3(4.5)} \mathrm{t}^{3.5} + \dfrac{3(4.1)}{3(4.7)} \mathrm{t}^{3.7} \right] \end{bmatrix}, \begin{bmatrix} \mathrm{v}_{1,5}(\mathrm{t}) \\ \mathrm{v}_{2,5}(\mathrm{t}) \end{bmatrix} = \begin{bmatrix} \left[\dfrac{3(4.1)}{3(4.9)} \mathrm{t}^{3.9} + \dfrac{3(4.1)}{3(5.1)} \mathrm{t}^{4.1} \right] \\ \dfrac{3}{4} \left[\dfrac{3(4.1)}{3(4.7)} \mathrm{t}^{3.7} + \dfrac{3(4.1)}{3(4.9)} \mathrm{t}^{3.9} \right] \end{bmatrix},$$

We can get the approximate solution to the descriptor equation (21), as follows

$$\begin{aligned} u(t) &= \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \\ &= \begin{bmatrix} \sum_{k=0}^{11} v_{1,k}(t) \\ \sum_{k=0}^{11} v_{2,k}(t) \end{bmatrix} = \begin{bmatrix} \left[t^{3.1} + \dfrac{\Gamma(4.1)}{\Gamma(4.3)} t^{3.3} \right] - \left[\dfrac{\Gamma(4.1)}{\Gamma(4.3)} t^{3.3} + \dfrac{\Gamma(4.1)}{\Gamma(4.5)} t^{3.5} \right] \\ + \left[\dfrac{\Gamma(4.1)}{\Gamma(4.5)} t^{3.5} + \dfrac{\Gamma(4.1)}{\Gamma(4.7)} t^{3.7} \right] - \left[\dfrac{\Gamma(4.1)}{\Gamma(4.7)} t^{3.7} + \dfrac{\Gamma(4.1)}{\Gamma(4.9)} t^{3.9} \right] \\ + \left[\dfrac{\Gamma(4.1)}{\Gamma(4.9)} t^{3.9} + \dfrac{\Gamma(4.1)}{\Gamma(5.1)} t^{4.1} \right] + \cdots - \cdots \\ \left[t^{0.85} + \dfrac{3}{4} t^{3.1} \right] - \left[\dfrac{3}{4} \left[t^{3.1} + \dfrac{\Gamma(4.1)}{\Gamma(4.3)} t^{3.3} \right] \right] \\ + \left[\dfrac{3}{4} \left[\dfrac{\Gamma(4.1)}{\Gamma(4.3)} t^{3.3} + \dfrac{\Gamma(4.1)}{\Gamma(4.5)} t^{3.5} \right] \right] - \left[\dfrac{3}{4} \left[\dfrac{\Gamma(4.1)}{\Gamma(4.5)} t^{3.5} + \dfrac{\Gamma(4.1)}{\Gamma(4.7)} t^{3.7} \right] \right] \\ + \left[\dfrac{3}{4} \left[\dfrac{\Gamma(4.1)}{\Gamma(4.7)} t^{3.7} + \dfrac{\Gamma(4.1)}{\Gamma(4.9)} t^{3.9} \right] \right] + \cdots - \cdots \end{bmatrix} \\ &= \begin{bmatrix} t^{3.1} \\ t^{0.85} \end{bmatrix} \end{aligned} \tag{24}$$

The equation (24) is approximate solution of equation (21).

We can observe that the behavior of the curve in the a (HPM) approach approximates the behavior of the exact solution in Figure 2.

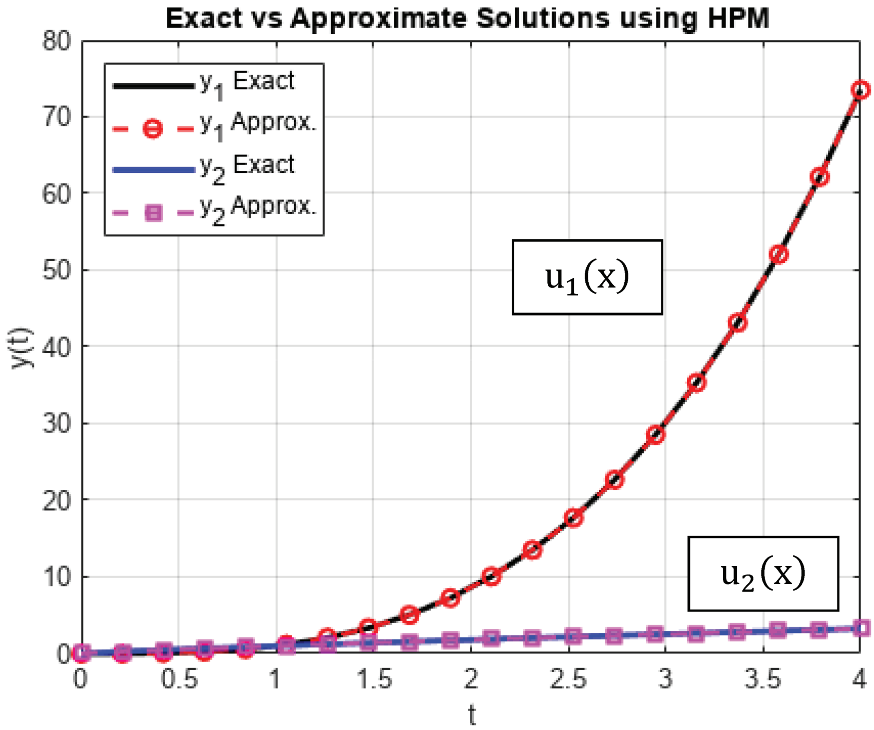

Figure 2. The following figure gives us a comparison between the approximate solution and the exact solution

Example (3.3): Consider the descriptor multi fractional integro - differential nonlinear system

$$E\left[D^{\delta}u(t)+D^{\gamma}u(t)\right]+FD^{\beta}u(t)=Bg(t)+CI^{\alpha}M(t),\ t\in[0,4], \tag{25}$$

$\begin{bmatrix} u_{1,0}(0) \\ u_{2,0}(0) \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} D^{0.85}u_{1,0}(0) \\ D^{0.85}u_{2,0}(0) \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} D^{0.45}u_{1,0}(0) \\ D^{0.45}u_{2,0}(0) \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, the exact solution of system (25) is given by

$$u(t)=\begin{bmatrix} t \\ t^{1.55}+t^{0.45} \end{bmatrix}$$

Such that $E=\begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}, F=\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}, B=\begin{bmatrix} 4 & 0 \\ 0 & 3(1.45) \end{bmatrix},$

$$C=\begin{bmatrix} \frac{4}{3(1.2)} & 0 \\ 0 & 23(2.55) \end{bmatrix}, \delta=0.85, \gamma=0.7, \beta=0.45, \alpha=0.35,$$

$$\begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix}=\begin{bmatrix} \frac{1}{3(1.15)}t^{0.15}+\frac{1}{3(1.3)}t^{0.3} \\ 2 \end{bmatrix}, \begin{bmatrix} M_1(t) \\ M_2(t) \end{bmatrix}=\begin{bmatrix} t^{0.3} \\ \frac{1}{3(1.75)}t^{0.75} \end{bmatrix}$$

$$\begin{aligned}&\begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}\left[\begin{bmatrix} D^{0.85}u_1(t) \\ D^{0.85}u_2(t) \end{bmatrix}+\begin{bmatrix} D^{0.7}u_1(t) \\ D^{0.7}u_2(t) \end{bmatrix}\right]+\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} D^{0.45}u_1(t) \\ D^{0.45}u_2(t) \end{bmatrix} \\ &=\begin{bmatrix} 4 & 0 \\ 0 & 3(1.45) \end{bmatrix}\begin{bmatrix} \frac{1}{3(1.15)}t^{0.15}+\frac{1}{3(1.3)}t^{0.3} \\ 2 \end{bmatrix}+\begin{bmatrix} \frac{4}{3(1.2)} & 0 \\ 0 & 23(2.55) \end{bmatrix}\begin{bmatrix} I^{0.35}M_1(t) \\ I^{0.35}M_2(t) \end{bmatrix}\end{aligned} \tag{26}$$

$u(0) = u_0 = \begin{bmatrix} u_{1,0} \\ u_{2,0} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then,

$$\begin{bmatrix} D^{0.85}u_1(t) + D^{0.7}u_1(t) + D^{0.45}u_1(t) \\ D^{0.45}u_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{\mathit{3}(1.15)}t^{0.15} + \frac{1}{\mathit{3}(1.3)}t^{0.3} + \frac{1}{\mathit{3}(1.55)}t^{0.55} \\ \mathit{3}(1.45) + \frac{\mathit{3}(2.55)}{\mathit{3}(2.1)}t^{1.1} \end{bmatrix},$$

we can get

$$\begin{bmatrix} D^{0.85}u_1(t) \\ D^{0.45}u_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{\mathit{3}(1.15)}t^{0.15} + \frac{1}{\mathit{3}(1.3)}t^{0.3} + \frac{1}{\mathit{3}(1.55)}t^{0.55} - D^{0.7}u_1(t) - D^{0.45}u_1(t) \\ \mathit{3}(1.45) + \frac{\mathit{3}(2.55)}{\mathit{3}(2.1)}t^{1.1} \end{bmatrix}$$

By equation (6), we can construct homotopy $v_h = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ such that

$$\vartheta(v_h, p) = \begin{bmatrix} H_1(v_1, p_1) \\ H_2(v_2, p_2) \end{bmatrix}$$

$$= \begin{bmatrix} \left[p_1^0 D^{0.85}v_{1,0}(t) + p_1^1 D^{0.85}v_{1,1}(t) + p_1^2 D^{0.85}v_{1,2}(t) + \cdots\right] - D^{0.85}u_{1,0}(t) + p_1^1 D^{0.85}u_{1,0}(t) \\ +p_1^1 \left\{ \left[-\frac{1}{\Gamma(1.15)}t^{0.15} - \frac{1}{\Gamma(1.3)}t^{0.3} - \frac{1}{\Gamma(1.55)}t^{0.55} \right] \right. \\ +\left[p_1^0 D^{0.7}v_{1,0}(t) + p_1^1 D^{0.7}v_{1,1}(t) + p_1^2 D^{0.7}v_{1,2}(t) + \cdots\right] \\ \left. +\left[p_1^0 D^{0.45}v_{1,0}(t) + p_1^1 D^{0.45}v_{1,1}(t) + p_1^2 D^{0.45}v_{1,2}(t) + \cdots\right] \right\} = 0 \\ \left[p_2^0 D^{0.45}v_{2,0}(t) + p_2^1 D^{0.45}v_{2,1}(t) + p_2^2 D^{0.45}v_{2,2}(t) + \cdots\right] \\ -D^{0.45}u_{2,0}(t) + p_2^1 D^{0.45}u_{2,0}(t) + p_2^1 \left[-\Gamma(1.45) - \frac{\Gamma(2.55)}{\Gamma(2.1)}t^{1.1} \right] = 0 \end{bmatrix} \tag{27}$$

Now, depending on the power of p, we find the solutions of $\begin{bmatrix} v_{1,j} \\ v_{2,j} \end{bmatrix}$, $j = 1, 2, \ldots$

$\begin{bmatrix} p_1^0 \\ p_2^0 \end{bmatrix} = \begin{bmatrix} D^{0.85}v_{1,0}(t) - D^{0.85}u_{1,0}(t) = 0 \\ D^{0.45}v_{2,0}(t) - D^{0.45}u_{2,0}(t) = 0 \end{bmatrix}$, we get $\begin{bmatrix} v_{1,0}(t) \\ v_{2,0}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

then, $\begin{bmatrix} v_{1,1}(t) \\ v_{2,1}(t) \end{bmatrix} = \begin{bmatrix} t^1 + \frac{1}{\mathit{3}(2.15)}t^{1.15} + \frac{1}{\mathit{3}(2.4)}t^{1.4} \\ t^{0.45} + t^{1.55} \end{bmatrix}$

$$\begin{bmatrix} v_{1,2}(t) \\ v_{2,2}(t) \end{bmatrix} = \begin{bmatrix} -\left[\frac{1}{\mathit{3}(2.15)}t^{1.15} + \frac{1}{\mathit{3}(2.30)}t^{1.30} + \frac{2}{\mathit{3}(2.55)}t^{1.55} + \frac{1}{\mathit{3}(2.4)}t^{1.4} + \frac{1}{\mathit{3}(2.8)}t^{1.8} \right] \\ 0 \end{bmatrix}, \text{ then}$$

$$\begin{bmatrix} v_{1,3}(t) \\ \\ v_{2,3}(t) \end{bmatrix} = \begin{bmatrix} \left[\frac{1}{3(2.30)}t^{1.30} + \frac{1}{3(2.45)}t^{1.45} + \frac{3}{3(2.70)}t^{1.70}\right] \\ + \left[\frac{2}{3(2.55)}t^{1.55} + \frac{3}{3(2.95)}t^{1.95} \frac{1}{3(2.8)}t^{1.80} + \frac{1}{3(3.2)}t^{2.2}\right] \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} v_{1,4}(t) \\ \\ v_{2,4}(t) \end{bmatrix} = \begin{bmatrix} -\{\left[\frac{1}{3(2.45)}t^{1.45} \frac{1}{3(2.6)}t^{1.60} + \frac{4}{3(2.85)}t^{1.85}\right] \\ + \left[\frac{3}{3(2.70)}t^{1.70} + \frac{6}{3(3.1)}t^{2.1} + \frac{3}{3(2.95)}t^{1.95}\right] \\ + \left[\frac{4}{3(3.35)}t^{2.35} + \frac{1}{3(3.2)}t^{2.2} + \frac{1}{3(3.6)}t^{2.6}\right]\} \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} v_{1,5}(t) \\ \\ v_{2,5}(t) \end{bmatrix} = \begin{bmatrix} \left[\frac{1}{3(2.6)}t^{1.6} + \frac{1}{3(2.75)}t^{1.75} + \frac{5}{2}t^{2} + \frac{4}{3(2.85)}t^{1.85}\right] \\ + \left[\frac{10}{3(3.25)}t^{2.25} + \frac{6}{3(3.1)}t^{2.1} + \frac{10}{3(3.5)}t^{2.5}\right] \\ + \left[\frac{4}{3(3.35)}t^{2.35} + \frac{5}{3(3.75)}t^{2.75} + \frac{1}{3(3.6)}t^{2.6} + \frac{1}{6}t^{3}\right] \\ 0 \end{bmatrix},$$

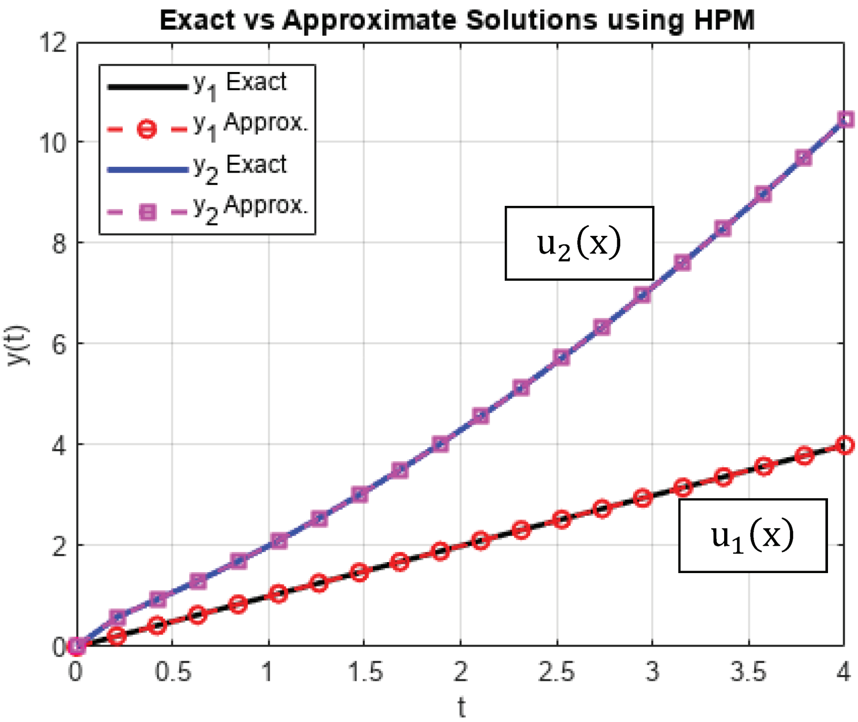

Figure 3. The following figure gives us a comparison between the approximate solution and the exact solution

We can get the approximate solution to the descriptor equation (25), as follows,

$$u(t)=\begin{bmatrix}u_1(t)\\u_2(t)\end{bmatrix}=\begin{bmatrix}\sum_{k=0}^{11}v_{1,k}(t)\\\sum_{k=0}^{11}v_{2,k}(t)\end{bmatrix}$$

$$=n\begin{bmatrix}\begin{gathered}\left[t^1+\frac{1}{r(2.15)}t^{1.15}+\frac{1}{r(2.4)}t^{1.4}\right]-\left[\frac{1}{r(2.15)}t^{1.15}+\frac{1}{r(2.30)}t^{1.30}+\frac{2}{r(2.55)}t^{1.55}+\frac{1}{r(2.4)}t^{1.4}+\frac{1}{r(2.8)}t^{1.8}\right.\\ \left.+\left[\frac{1}{r(2.30)}t^{1.30}+\frac{1}{r(2.45)}t^{1.45}+\frac{3}{r(2.70)}t^{1.70}\right]\right]\\ +\left[\frac{2}{r(2.55)}t^{1.55}+\frac{3}{r(2.95)}t^{1.95}+\frac{1}{r(2.8)}t^{1.80}+\frac{1}{r(3.2)}t^{2.2}\right]\\ -\left\{\left[\frac{1}{r(2.45)}t^{1.45}+\frac{1}{r(2.6)}t^{1.60}+\frac{4}{r(2.70)}t^{1.85}\right]\right.\\ +\left[\frac{3}{r(2.70)}t^{1.70}+\frac{6}{r(3.1)}t^{2.1}+\frac{3}{r(2.95)}t^{1.95}\right]\\ \left.+\left[\frac{4}{r(3.35)}t^{2.35}+\frac{1}{r(3.2)}t^{2.2}+\frac{1}{r(3.6)}t^{2.6}\right]\right\}\\ \left[\frac{1}{r(2.6)}t^{1.6}+\frac{1}{r(2.75)}t^{1.75}+\frac{5}{2}t^2+\frac{4}{r(2.85)}t^{1.85}\right]\\ +\left[\frac{10}{r(3.25)}t^{2.25}+\frac{6}{r(3.1)}t^{2.1}+\frac{10}{r(3.5)}t^{2.5}\right]\\ +\left[\frac{4}{r(3.35)}t^{2.35}+\frac{5}{r(3.75)}t^{2.75}+\frac{1}{r(3.6)}t^{2.6}+\frac{1}{6}t^3\right]+\cdots\end{gathered}\\ t^{0.45}+t^{1.55}\end{bmatrix} \tag{28}$$

$$=\begin{bmatrix}\mathrm{t}\\ \mathrm{t}^{0.45}+\mathrm{t}^{1.55}\end{bmatrix} \tag{29}$$

the equation (29) is approximate solution of equation (25).

We can observe that the behavior of the curve in the a (HPM) approach approximates the behavior of the exact solution in Figure (3).

4. Conclusion

In this work we were able to find an approximate solution using the (HPM). We got very accurate approximate solutions compared to the exact solution, whose results were largely identical. In addition, the drawing gave us a clear view of the convergent between the approximate solution and the exact solution. We can notice the diversity in examples, in the first example, we found that the system contains two fractional derivatives for the first component, u1, and one fractional derivative for the second component, u2. In the second example, we found that the system contains two fractional derivatives for the first component, u1, and two fractional derivatives for the second component, u2 and u1. In the third example, we found that the system contains three fractional derivatives for the first component, u1, and one fractional derivative for the second component, u2. Using the iterative method, HPM, we arrived at an approximate solution.

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