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Algebraic properties and operator analysis of penta-partitioned intuitionistic neutrosophic soft sets for pattern recognition applications

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Abstract

Complex decision-making problems are uncertain, contradictory and partially ignorant conditions that surpass the representational limits of traditional fuzzy, intuitionistic and neutrosophic soft set models. To overcome these limitations, this paper introduces the penta partitioned intuitionistic neutrosophic soft set (PPINSS), a five component extension that distinctly represents truth, indeterminacy, contradiction, falsity and ignorance. Unlike previous models, PPINSS incorporates an intuitionistic dependency between truth and falsity through the balance relation $\pi = 1 - \tau - \phi$, ensuring a coherent and bounded characterization of uncertainty. Fundamental set-theoretic operations such as complement, subset and equality are formally defined and their algebraic properties are rigorously established. Moreover, a comprehensive family of operators—including necessity (◎) and possibility (⊛) transformations, aggregation operators (⊕, ⊖) and parametric mappings ($\Delta_{\mu,\nu}, \Phi_{\mu,\nu}$) is introduced to model dynamic uncertainty and interdependent reasoning. These operators are shown to satisfy algebraic consistency, duality and closure within the PPINSS domain. By integrating the intuitionistic interdependence of truth and falsity with a fifth component representing latent ignorance, PPINSS offers a unified, logically coherent and semantically rich framework for modeling complex real-world uncertainties. It serves as an effective analytical foundation for multi-criteria decision analysis, distributed intelligence and cognitive reasoning systems.

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Key words and Phrases: penta-partitioned neutrosophic soft set; intuitionistic neutrosophic logic; algebraic operators; parametric transformations; multi-criteria decision-making.

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1. Introduction

Uncertainty, vagueness and indeterminacy are essential characteristics of real-world systems. Traditional mathematical models such as fuzzy sets [22] and intuitionistic fuzzy sets [2] have effectively modelled partial truth and hesitation but remain limited when information exhibits contradiction, incompleteness, or ignorance. To overcome these limitations, more expressive frameworks such as neutrosophic sets [20] and their extensions have emerged providing independent quantification of truth, falsity and indeterminacy.

Soft set theory introduced by Molodtsov [15] offers a parameterized approach to uncertainty modelling without relying on complex membership functions. When integrated with neutrosophic logic [20] it yields the neutrosophic soft set (NSS) [13] a versatile framework widely applied in decision-making, data analysis and information fusion. Several advanced versions including possibility neutrosophic soft sets [10], effective neutrosophic soft sets [11], neutrosophic soft open sets [18] and block matrix-based neutrosophic soft sets [8] have extended its scope and applicability.

In recent years, partitioned neutrosophic models have gained prominence for their enhanced expressivity. The quadri-partitioned neutrosophic soft set [12] introduced an explicit representation of contradiction alongside truth, indeterminacyand falsity. However, these four components do not capture the notion of ignorance a form of latent knowledge inherent in real-world systems such as distributed sensing, medical diagnostics and intelligent decision environments. To address this limitation, penta-partitioned neutrosophic sets were proposed [7, 14, 16, 17], providing a fifth dimension to represent unawareness.

Building upon this foundation, the present work introduces the Penta-Partitioned Intuitionistic Neutrosophic Soft Set (PPINSS), which generalizes existing soft and neutrosophic frameworks while preserving the dependency between truth and falsity. Unlike previous models, the PPINSS integrates intuitionistic principles [5, 6] with penta-partitioned neutrosophic logic, ensuring a balanced representation of dependent truth–falsity values and independent indeterminacy, contradiction and contextual ignorance measures.

The development of algebraic operators has played a vital role in enhancing the analytical depth of soft computing models. Recent contributions such as aggregation and similarity based extensions [1, 9, 19] lattice and ordering structures [21] and multi-polar or interval-valued systems [3, 23], have strengthened the theoretical foundation for multi-criteria decision-making (MCDM). These studies collectively highlight the growing demand for flexible, algebraically consistent models capable of representing layered uncertainty.

In this context, the proposed PPINSS framework unifies the essential features of these models by:

1. Introducing a five-dimensional intuitionistic neutrosophic soft structure that captures truth, falsity, indeterminacy, contradiction and contextual ignorance. The truth and falsity components are dependent on each other, while the other components are independent of each other;
2. Defining comprehensive operator families including necessity $(\circledcirc)$, possibility $(\circledast)$, additive $(\oplus)$, difference $(\ominus)$ and parametric transformations $(\Delta_{\mu,\nu}, \Phi_{\mu,\nu})$ and establishing their algebraic properties;
3. Demonstrating that traditional neutrosophic and quadri-partitioned systems arise as special cases within the PPINSS framework;

Thus, PPINSS represents a significant theoretical advancement in neutrosophic soft set theory bridging intuitionistic and penta-partitioned reasoning within a unified algebraic framework that accommodates both dependency and contextual unawareness in uncertainty modeling.

2. Preliminaries

This part describes some of the basic mathematical concepts underlying the envisioned framework, viz. fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets and quadri-partitioned neutrosophic sets.

Let $\mathcal{U}$ denote the universal set of discourse consisting of all possible objects under consideration and Λ a set of parameters describing attributes, features, or decision criteria associated with the elements of $\mathcal{U}$. A neutrosophic structure is characterized by a set of membership functions that express multiple facets of uncertainty such as truth, indeterminacy, contradiction, falsity, and other context-dependent components. For any $x \in \mathcal{U}$, the functions $T(x)$, $I(x)$, $C(x)$, $F(x)$, and $U(x)$ represent, respectively, the degrees of truth, indeterminacy, contradiction, falsity and ignorance. Each function takes values in [0,1]. These components together enable neutrosophic based models to accommodate both conflicting and incomplete information, offering a comprehensive representation of uncertainty.

Definition 2.1: [22] *A fuzzy set $\mathcal{F}$ on $\mathcal{U}$ is characterized by a membership function $\mu_{\mathcal{F}} : \mathcal{U} \to [0,1]$, which assigns to each element $x \in \mathcal{U}$ a real value $\mu_{\mathcal{F}}(x)$ representing its grade of belongingness to $\mathcal{F}$. The fuzzy set is thus represented as $\mathcal{F} = \{(x, \mu_{\mathcal{F}}(x)) \mid x \in \mathcal{U}\}$.*

Definition 2.2: [2] *An intuitionistic fuzzy set (IFS) $\mathcal{I}$ on $\mathcal{U}$ extends the fuzzy concept by introducing an explicit representation of non-membership. It is defined as $\mathcal{I} = \{(x, \mu_{\mathcal{I}}(x), \nu_{\mathcal{I}}(x)) \mid x \in \mathcal{U}\}$, where $\mu_{\mathcal{F}}(x)$ and $\nu_{\mathcal{I}}(x)$ represent, respectively, the degrees of membership and non-membership of x in $\mathcal{I}$, subject to the constraint $0 \le \mu_{\mathcal{I}}(x) + \nu_{\mathcal{I}}(x) \le 1, \quad \forall x \in \mathcal{U}$. The residual uncertainty, referred to as the* hesitation degree*, is quantified by $\pi_{\mathcal{I}}(x) = 1 - \mu_{\mathcal{I}}(x) - \nu_{\mathcal{I}}(x)$, which captures the extent of indeterminacy in assessing the membership of x.*

Definition 2.3: [20] *A neutrosophic set $\mathcal{N}$ generalizes the intuitionistic fuzzy framework by independently quantifying the degrees of truth, indeterminacy and falsity for each element. Represented as $\mathcal{N} = \{(x, T_{\mathcal{N}}(x), I_{\mathcal{N}}(x), F_{\mathcal{N}}(x)) \mid x \in \mathcal{U}\}$, where $T_{\mathcal{N}}(x), I_{\mathcal{N}}(x), F_{\mathcal{N}}(x) \in [0,1], \quad \forall x \in \mathcal{U}$.*

Definition 2.4: [4] *A quadri-partitioned neutrosophic set (QPNS) $\mathcal{Q}$ further enhances the expressive capacity of the neutrosophic framework by introducing an additional component that quantifies* contradiction. *It is defined as $\mathcal{Q} = \{(x, T_{\mathcal{Q}}(x), I_{\mathcal{Q}}(x), C_{\mathcal{Q}}(x), F_{\mathcal{Q}}(x)) \mid x \in \mathcal{U}\}$, where each function $T_{\mathcal{Q}}(x), I_{\mathcal{Q}}(x), C_{\mathcal{Q}}(x), F_{\mathcal{Q}}(x) \in [0,1], \quad \forall x \in \mathcal{U}$.*

Definition 2.5: [4] *A penta-partitioned neutrosophic set (PPNS) $\mathcal{P}$ extends the quadri-partitioned neutrosophic framework. It is represented as $\mathcal{P} = \{(x, T_{\mathcal{P}}(x), I_{\mathcal{P}}(x), C_{\mathcal{P}}(x), F_{\mathcal{P}}(x), U_{\mathcal{P}}(x)) \mid x \in \mathcal{U}\}$, where each function $T_{\mathcal{P}}(x), I_{\mathcal{P}}(x), C_{\mathcal{P}}(x), F_{\mathcal{P}}(x), U_{\mathcal{P}}(x) \in [0,1], \quad \forall x \in \mathcal{U}$.*

3. Main Results

Definition 3.1: *PPINS A on $\mathcal{U}$ is defined as $A = \left\{\langle u, \tau_A(u), \iota_A(u), \kappa_A(u), \phi_A(u), \upsilon_A(u)\rangle : u \in \mathcal{U}\right\}$, where $\tau_A(u), \iota_A(u), \kappa_A(u), \phi_A(u)$ and $\upsilon_A(u)$ are real-valued functions mapping $\mathcal{U}$ into [0,1], representing respectively the degrees of truth, indeterminacy, contradiction, falsity and unawareness. These components satisfy the following conditions: $0 \le \tau_A(u), \iota_A(u), \kappa_A(u), \phi_A(u), \upsilon_A(u) \le 1$, and $\tau_A(u) + \phi_A(u) \le 1$. The inequality $\tau_A(u) + \phi_A(u) \le 1$ captures the* intuitionistic dependency *between truth and falsity memberships, while $\iota_A(u)$, $\kappa_A(u)$ and $\upsilon_A(u)$ remain independent uncertainty descriptors.*

Example 3.2: *Consider a weather forecasting system equipped with three sensors S_1, S_2 and S_3, each providing evidence about the event "Rain" for the following day.*

(1) S_1 predicts a high chance of rain (0.9)*,*
(2) S_2 predicts a low chance of rain (0.1) *and*
(3) S_3 fails to transmit data (sensor offline).

Each sensor's observation is represented by a PPINS

$$A_{S_k}(\text{Rain}) = \left(\tau_{S_k}, \iota_{S_k}, \kappa_{S_k}, \phi_{S_k}, \upsilon_{S_k}\right),$$

as follows:

$$A_{S_1}(\text{Rain}) = (0.9, 0.1, 0.0, 0.1, 0.0),$$
$$A_{S_2}(\text{Rain}) = (0.1, 0.1, 0.0, 0.8, 0.0),$$
$$A_{S_3}(\text{Rain}) = (0.0, 0.0, 0.0, 0.0, 1.0).$$

Definition 3.3: *A pair (δ, Λ) is called a penta-partitioned intuitionistic neutrosophic soft set (PPINSS) over $\mathcal{U}$ if $\delta : \Lambda \to \mathcal{P}^5$, where $\mathcal{P}5$ denotes the collection of all PPINS. For each parameter $e \in \Lambda$, the image $\delta(e)$ is a PPINS over $\mathcal{U}$, defined as $\delta(e) = \left\{\langle u, \tau_{\delta(e)}(u), \iota_{\delta(e)}(u), \kappa_{\delta(e)}(u), \phi_{\delta(e)}(u), \upsilon_{\delta(e)}(u)\rangle : u \in \mathcal{U}\right\}$. The truth and falsity components are considered dependent. That is, for each $u \in \mathcal{U}$ and $e \in \Lambda$, $\tau_{\delta(e)}(u) + \phi_{\delta(e)}(u) \le 1$, and each component satisfies*

$$0 \le \tau_{\delta(e)}(u), \iota_{\delta(e)}(u), \kappa_{\delta(e)}(u), \phi_{\delta(e)}(u), \upsilon_{\delta(e)}(u) \le 1.$$

Example 3.4: *Let $\mathcal{U} = \{u_1, u_2, u_3\}$ and $\Lambda = \{e_1, e_2\}$. A PPINSS (δ, Λ) is defined by the parameter-wise mappings $\delta(e_1)$ and $\delta(e_2)$ given below. Each entry is a tuple $(\tau, \iota, \kappa, \phi, \upsilon)$ with values in [0,1] satisfying the dependency condition $\tau + \phi \le 1$.*

Parameter e_2:

u	$\tau_{\delta(e_1)}(u)$	$\iota_{\delta(e_1)}(u)$	$\kappa_{\delta(e_1)}(u)$	$\phi_{\delta(e_1)}(u)$	$\upsilon_{\delta(e_1)}(u)$
u_1	0.80	0.10	0.05	0.15	0.10
u_2	0.40	0.20	0.10	0.50	0.10
u_3	0.10	0.05	0.00	0.10	0.85

Equivalently,

$$\delta(e_1) = \{\langle u_1, (0.80, 0.10, 0.05, 0.15, 0.10)\rangle, \\ \langle u_2, (0.40, 0.20, 0.10, 0.50, 0.10)\rangle, \\ \langle u_3, (0.10, 0.05, 0.00, 0.10, 0.85)\rangle\}$$

Parameter e_2:

u	$\tau_{\delta(e_2)}(u)$	$\iota_{\delta(e_2)}(u)$	$\kappa_{\delta(e_2)}(u)$	$\phi_{\delta(e_2)}(u)$	$\upsilon_{\delta(e_2)}(u)$
u_1	0.30	0.30	0.10	0.60	0.10
u_2	0.60	0.10	0.05	0.40	0.10
u_3	0.50	0.20	0.20	0.50	0.10

so

$$\delta(e_2) = \{\langle u_1, (0.30, 0.30, 0.10, 0.60, 0.10)\rangle, \\ \langle u_2, (0.60, 0.10, 0.05, 0.40, 0.10)\rangle, \\ \langle u_3, (0.50, 0.20, 0.20, 0.50, 0.10)\rangle\}.$$

Definition 3.5: *Let (δ_1, Λ_1) and (δ_2, Λ_2) be two PPINSS defined over a common universe $\mathcal{U}$. Then the logical operations "AND" and "OR" between them are defined as follows:*

(i) *The **AND** operation between (δ_1, Λ_1) and (δ_2, Λ_2) is denoted by*

$$(\delta_1, \Lambda_1) \wedge (\delta_2, \Lambda_2) = (\delta_\wedge, \Lambda_1 \times \Lambda_2),$$

where, for every $(e_1, e_2) \in \Lambda_1 \times \Lambda_2$,

$$\delta_{\wedge}(e_1, e_2) = \delta_1(e_1) \cap \delta_2(e_2).$$

The corresponding PPINS of each element $u \in \mathcal{U}$ *under this operation is given by*

$$\begin{aligned}\delta_{\wedge}(e_1, e_2)(u) = \Big\langle &\min(\tau_{\delta_1(e_1)}(u), \tau_{\delta_2(e_2)}(u)), \min(\iota_{\delta_1(e_1)}(u), \iota_{\delta_2(e_2)}(u)),\\ &\min(\kappa_{\delta_1(e_1)}(u), \kappa_{\delta_2(e_2)}(u)), \max(\phi_{\delta_1(e_1)}(u), \phi_{\delta_2(e_2)}(u)),\\ &\max(\upsilon_{\delta_1(e_1)}(u), \upsilon_{\delta_2(e_2)}(u))\Big\rangle.\end{aligned}$$

(ii) The ***OR*** operation between (δ_1, Λ_1) *and* (δ_2, Λ_2) is denoted by

$$(\delta_1, \Lambda_1) \vee (\delta_2, \Lambda_2) = (\delta_{\vee}, \Lambda_1 \times \Lambda_2),$$

where, for every $(e_1, e_2) \in \Lambda_1 \times \Lambda_2$,

$$\delta_{\vee}(e_1, e_2) = \delta_1(e_1) \cup \delta_2(e_2).$$

The corresponding PPINS of each element $u \in \mathcal{U}$ is defined as

$$\begin{aligned}\delta_{\vee}(e_1, e_2)(u) = \Big\langle &\max(\tau_{\delta_1(e_1)}(u), \tau_{\delta_2(e_2)}(u)), \max(\iota_{\delta_1(e_1)}(u), \iota_{\delta_2(e_2)}(u)),\\ &\max(\kappa_{\delta_1(e_1)}(u), \kappa_{\delta_2(e_2)}(u)), \min(\phi_{\delta_1(e_1)}(u), \phi_{\delta_2(e_2)}(u)),\\ &\min(\upsilon_{\delta_1(e_1)}(u), \upsilon_{\delta_2(e_2)}(u))\Big\rangle.\end{aligned}$$

Example 3.6: *Let* $\mathcal{U} = \{u_1, u_2\}$. *Consider two PPINSS* $(\delta_1, \{e\})$ *and* $(\delta_2, \{f\})$ *defined over* $\mathcal{U}$ *as*

$$\begin{aligned}\delta_1(e) &= \Big\{\langle u_1, (0.80, 0.10, 0.00, 0.10, 0.05)\rangle, \langle u_2, (0.40, 0.20, 0.10, 0.30, 0.10)\rangle\Big\},\\ \delta_2(f) &= \Big\{\langle u_1, (0.60, 0.05, 0.20, 0.20, 0.10)\rangle, \langle u_2, (0.20, 0.30, 0.00, 0.50, 0.05)\rangle\Big\}.\end{aligned}$$

Applying the definitions of $\wedge$ *and* $\vee$:

AND: $(\delta_1, \{e\}) \wedge (\delta_2, \{f\}) = (\delta_{\wedge}, \{e\} \times \{f\})$

$$\begin{aligned}\delta_{\wedge}(e, f)(u_1) &= (0.60, 0.05, 0.00, 0.20, 0.10),\\ \delta_{\wedge}(e, f)(u_2) &= (0.20, 0.20, 0.00, 0.50, 0.10).\end{aligned}$$

OR: $(\delta_1, \{e\}) \vee (\delta_2, \{f\}) = (\delta_{\vee}, \{e\} \times \{f\})$

$$\begin{aligned}\delta_{\vee}(e, f)(u_1) &= (0.80, 0.10, 0.20, 0.10, 0.05),\\ \delta_{\vee}(e, f)(u_2) &= (0.40, 0.30, 0.10, 0.30, 0.05).\end{aligned}$$

Definition 3.7: *Let* (δ_1, Λ_1) *and* (δ_2, Λ_2) *be two PPINSS over a common universe* $\mathcal{U}$.

(i) The union of (δ_1, Λ_1) *and* (δ_2, Λ_2) *is denoted by*

$$(\delta_1, \Lambda_1) \Cup (\delta_2, \Lambda_2) = (\delta_{\Cup}, \Lambda_{\Cup}),$$

where $\Lambda_{\Cup} = \Lambda_1 \cup \Lambda_2$ *and for every parameter* $\lambda \in \Lambda_{\Cup}$,

$$\delta_{\Cup}(\lambda)=\begin{cases}\delta_1(\lambda), & \text{if } \lambda\in \Lambda_1\setminus \Lambda_2,\\ \delta_2(\lambda), & \text{if } \lambda\in \Lambda_2\setminus \Lambda_1,\\ \{\langle u, \max\left(\tau_{\delta_1(\lambda)}(u),\tau_{\delta_2(\lambda)}(u)\right), \max\left(\iota_{\delta_1(\lambda)}(u),\iota_{\delta_2(\lambda)}(u)\right), & \\ \quad \max\left(\kappa_{\delta_1(\lambda)}(u),\kappa_{\delta_2(\lambda)}(u)\right), \min\left(\phi_{\delta_1(\lambda)}(u),\phi_{\delta_2(\lambda)}(u)\right), & \\ \quad \min\left(\upsilon_{\delta_1(\lambda)}(u),\upsilon_{\delta_2(\lambda)}(u)\right)\rangle : u\in \mathcal{U}\}, & \text{if } \lambda\in \Lambda_1\cap \Lambda_2.\end{cases}$$

(ii) The intersection of (δ_1,Λ_1) *and* (δ_2,Λ_2) *is denoted by*

$$(\delta_1,\Lambda_1)\Cap(\delta_2,\Lambda_2)=(\delta_{\Cap},\Lambda_{\Cap}),$$

where $\Lambda_{\Cap}=\Lambda_1\cap\Lambda_2$ *and for every parameter* $\lambda\in\Lambda_{\Cap}$,

$$\delta_{\Cap}(\lambda)=\begin{cases}\delta_1(\lambda), & \text{if } \lambda\in \Lambda_1\setminus \Lambda_2,\\ \delta_2(\lambda), & \text{if } \lambda\in \Lambda_2\setminus \Lambda_1,\\ \{\langle u, \min(\tau_{\delta_1(\lambda)}(u),\tau_{\delta_2(\lambda)}(u)), \min(\iota_{\delta_1(\lambda)}(u),\iota_{\delta_2(\lambda)}(u)), & \\ \quad \min(\kappa_{\delta_1(\lambda)}(u),\kappa_{\delta_2(\lambda)}(u)), \max(\phi_{\delta_1(\lambda)}(u),\phi_{\delta_2(\lambda)}(u)), & \\ \quad \max(\upsilon_{\delta_1(\lambda)}(u),\upsilon_{\delta_2(\lambda)}(u))\rangle : u\in \mathcal{U}\}, & \text{if } \lambda\in \Lambda_1\cap \Lambda_2.\end{cases}$$

Definition 3.8: *Let* (δ,Λ) *be a penta-partitioned intuitionistic neutrosophic soft set (PPINSS) over a universe* $\mathcal{U}$*, where*

$$\delta(\lambda)=\left\{\langle u,(\tau_{\delta(\lambda)}(u),\iota_{\delta(\lambda)}(u),\kappa_{\delta(\lambda)}(u),\phi_{\delta(\lambda)}(u),\upsilon_{\delta(\lambda)}(u))\rangle : u\in\mathcal{U}\right\},\quad \lambda\in\Lambda.$$

Then, the complement of (δ,Λ)*, denoted by* $(\delta,\Lambda)^c$*, is defined as*

$$(\delta,\Lambda)^c=(\delta^c,\Lambda),$$

where for each parameter $\lambda\in\Lambda$ *and for all* $u\in\mathcal{U}$,

$$\delta^c(\lambda)(u)=\left\langle \phi_{\delta(\lambda)}(u),1-\iota_{\delta(\lambda)}(u),1-\kappa_{\delta(\lambda)}(u),\tau_{\delta(\lambda)}(u),1-\upsilon_{\delta(\lambda)}(u)\right\rangle.$$

Example 3.9: *Let* $\mathcal{U}=\{u_1,u_2,u_3\}$ *and consider a PPINSS* $(\delta,\{e\})$ *defined as*

$$\begin{aligned}\delta(e)=\{&\langle u_1,(0.70,0.20,0.10,0.10,0.05)\rangle,\\ &\langle u_2,(0.30,0.40,0.20,0.50,0.10)\rangle,\\ &\langle u_3,(0.10,0.10,0.10,0.20,0.50)\rangle\}.\end{aligned}$$

The complement $(\delta,\{e\})\,c$ *becomes*

$$\begin{aligned}\delta^c(e)=\{&\langle u_1,(0.10,0.80,0.90,0.70,0.95)\rangle,\\ &\langle u_2,(0.50,0.60,0.80,0.30,0.90)\rangle,\\ &\langle u_3,(0.20,0.90,0.90,0.10,0.50)\rangle\}.\end{aligned}$$

Theorem 3.10: *Let* (δ_1,Λ_1) *and* (δ_2,Λ_2) *be two PPINSS over a common universe* $\mathcal{U}$*. Then,*

(i) $\left[(\delta_1,\Lambda_1)\Cap(\delta_2,\Lambda_2)\right]^c=(\delta_1,\Lambda_1)^c\Cup(\delta_2,\Lambda_2)^c$,

(ii) $\left[(\delta_1,\Lambda_1)\Cup(\delta_2,\Lambda_2)\right]^c=(\delta_1,\Lambda_1)^c\Cap(\delta_2,\Lambda_2)^c$.

Proof. We prove (i); the second follows by duality.

Let

$$(\delta_1, \Lambda_1) \Cap (\delta_2, \Lambda_2) = (\delta_\Cap, \Lambda_\Cap), \qquad \Lambda_\Cap = \Lambda_1 \cap \Lambda_2.$$

For each $\lambda \in \Lambda_\Cap$ and $u \in \mathcal{U}$,

$$\delta_\Cap(\lambda)(u) = \langle \min(\tau_1, \tau_2),\ \min(\iota_1, \iota_2),\ \min(\kappa_1, \kappa_2),\ \max(\phi_1, \phi_2),\ \max(\upsilon_1, \upsilon_2) \rangle.$$

Applying the updated complement rule

$$(\tau, \iota, \kappa, \phi, \upsilon)^c = (\phi, 1-\iota, 1-\kappa, \tau, 1-\upsilon),$$

we obtain

$$\delta^c_\Cap(\lambda)(u) = \langle \max(\phi_1, \phi_2),\ \max(1-\iota_1, 1-\iota_2),\ \max(1-\kappa_1, 1-\kappa_2),\ \min(\tau_1, \tau_2),\ \min(1-\upsilon_1, 1-\upsilon_2) \rangle.$$

Next, consider the union of the complements:

$$(\delta_1, \Lambda_1)^c \Cup (\delta_2, \Lambda_2)^c = (\delta_\Cup, \Lambda_\Cup), \qquad \Lambda_\Cup = \Lambda_1 \cup \Lambda_2.$$

For $\lambda \in \Lambda_1 \cap \Lambda_2$ and $u \in \mathcal{U}$,

$$\delta_\Cup(\lambda)(u) = \langle \max(\phi_1, \phi_2),\ \max(1-\iota_1, 1-\iota_2),\ \max(1-\kappa_1, 1-\kappa_2),\ \min(\tau_1, \tau_2),\ \min(1-\upsilon_1, 1-\upsilon_2) \rangle.$$

Comparing the two expressions component-wise, we find

$$\delta^c_\Cap(\lambda)(u) = \delta_\Cup(\lambda)(u), \quad \forall u \in \mathcal{U}, \lambda \in \Lambda_1 \cap \Lambda_2.$$

Hence,

$$\left[(\delta_1, \Lambda_1) \Cap (\delta_2, \Lambda_2)\right]^c = (\delta_1, \Lambda_1)^c \Cup (\delta_2, \Lambda_2)^c.$$

By the duality of min and max, the second law follows directly:

$$\left[(\delta_1, \Lambda_1) \Cup (\delta_2, \Lambda_2)\right]^c = (\delta_1, \Lambda_1)^c \Cap (\delta_2, \Lambda_2)^c.$$

□

Definition 3.11: Let Λ_1 and Λ_2 be subsets of a common parameter set Λ.

The pair (δ_1, Λ_1) is said to be a penta-partitioned intuitionistic neutrosophic soft subset (PPINSS-subset) of (δ_2, Λ_2), denoted by $(\delta_1, \Lambda_1) \Subset (\delta_2, \Lambda_2)$, if and only if the following two conditions hold:

(i) $\Lambda_1 \subseteq \Lambda_2$;

(ii) For every $\lambda \in \Lambda_1$ and each $u \in \mathcal{U}$,

$$\tau_{\delta_1(\lambda)}(u) \le \tau_{\delta_2(\lambda)}(u), \iota_{\delta_1(\lambda)}(u) \ge \iota_{\delta_2(\lambda)}(u),$$
$$\kappa_{\delta_1(\lambda)}(u) \ge \kappa_{\delta_2(\lambda)}(u), \phi_{\delta_1(\lambda)}(u) \ge \phi_{\delta_2(\lambda)}(u),\ \upsilon_{\delta_1(\lambda)}(u) \ge \upsilon_{\delta_2(\lambda)}(u).$$

Furthermore, both soft sets respect the intuitionistic dependency constraint

$$\tau_{\delta_i(\lambda)}(u) + \phi_{\delta_i(\lambda)}(u) \le 1, \quad \text{for } i = 1, 2,$$

In this case, (δ_2, Λ_2) is called the penta-partitioned intuitionistic neutrosophic soft superset (PPINSS-superset) of (δ_1, Λ_1), denoted by

$$(\delta_2, \Lambda_2) \Supset (\delta_1, \Lambda_1).$$

Definition 3.12: *Let* (δ_1, Λ_1) *and* (δ_2, Λ_2) *be two* Penta-Partitioned Intuitionistic Neutrosophic Soft Sets *over the same universe* $\mathcal{U}$. *They are said to be equal, written as* $(\delta_1, \Lambda_1) = (\delta_2, \Lambda_2)$, *if and only if* $(\delta_1, \Lambda_1) \Subset (\delta_2, \Lambda_2)$ *and* $(\delta_2, \Lambda_2) \Subset (\delta_1, \Lambda_1)$.

4. Necessity (⊚) and Possibility (⊛) Operators on PPINSS

We introduce the necessity and possibility operators on PPINSS and describe their structural transformations. These operators provide dual perspectives of certainty and potentiality in neutrosophic information modelling.

Definition 4.1: *Let* (δ, Λ) *be a PPINSS over a universe* $\mathcal{U}$, *where* $\delta(\lambda) = \{\langle u, (\tau_{\delta(\lambda)}(u), \iota_{\delta(\lambda)}(u), \kappa_{\delta(\lambda)}(u), \phi_{\delta(\lambda)}(u), \upsilon_{\delta(\lambda)}(u))\rangle : u \in \mathcal{U}\}$, $\lambda \in \Lambda$, *and satisfies the intuitionistic constraint* $\tau_{\delta(\lambda)}(u) + \phi_{\delta(\lambda)}(u) \leq 1$.

(i) The necessity operator, denoted by $(\circledcirc)$, *is defined as* $\circledcirc(\delta, \Lambda) = (\delta_{\circledcirc}, \Lambda)$, *where for each* $\lambda \in \Lambda$ *and* $u \in \mathcal{U}$,

$$\delta_{\circledcirc}(\lambda)(u) = \left\langle \tau_{\delta(\lambda)}(u), \iota_{\delta(\lambda)}(u), \kappa_{\delta(\lambda)}(u), 1 - \tau_{\delta(\lambda)}(u), \upsilon_{\delta(\lambda)}(u) \right\rangle.$$

(ii) The possibility operator, denoted by $(\circledast)$, *is defined as* $\circledast(\delta, \Lambda) = (\delta_{\circledast}, \Lambda)$, *where for each* $\lambda \in \Lambda$ *and* $u \in \mathcal{U}$,

$$\delta_{\circledast}(\lambda)(u) = \left\langle 1 - \phi_{\delta(\lambda)}(u), \iota_{\delta(\lambda)}(u), \kappa_{\delta(\lambda)}(u), \phi_{\delta(\lambda)}(u), \upsilon_{\delta(\lambda)}(u) \right\rangle.$$

Theorem 4.2: *Let* (δ_1, Λ_1) *and* (δ_2, Λ_2) *be two PPINSS over a universe* $\mathcal{U}$.

Then:

(i) $\circledcirc\left[(\delta_1, \Lambda_1) \Cup (\delta_2, \Lambda_2)\right] = \circledcirc(\delta_1, \Lambda_1) \Cup \circledcirc(\delta_2, \Lambda_2)$;

(ii) $\circledcirc\left[(\delta_1, \Lambda_1) \Cap (\delta_2, \Lambda_2)\right] = \circledcirc(\delta_1, \Lambda_1) \Cap \circledcirc(\delta_2, \Lambda_2)$;

(iii) $\circledcirc\left[\circledcirc(\delta_1, \Lambda_1)\right] = \circledcirc(\delta_1, \Lambda_1)$.

Proof: Let $\lambda \in \Lambda_1 \cup \Lambda_2$ and $u \in \mathcal{U}$ be arbitrary. For $i = 1, 2$, define the PPINSS:

$$\delta_i(\lambda)(u) = (\tau_i, \iota_i, \kappa_i, \phi_i, \upsilon_i).$$

The necessity operator $(\circledcirc)$ is defined component-wise as:

$$\circledcirc(\tau, \iota, \kappa, \phi, \upsilon) = (\tau, \iota, \kappa, 1 - \tau, \upsilon).$$

The union $\Cup$ and intersection $\Cap$ of two PPINSS are defined pointwise for the pentagonal tuples as follows:

$$(\delta_1 \Cup \delta_2)(\lambda)(u) = \left(\max(\tau_1, \tau_2), \max(\iota_1, \iota_2), \max(\kappa_1, \kappa_2), \min(\phi_1, \phi_2), \min(\upsilon_1, \upsilon_2)\right),$$
$$(\delta_1 \Cap \delta_2)(\lambda)(u) = \left(\min(\tau_1, \tau_2), \min(\iota_1, \iota_2), \min(\kappa_1, \kappa_2), \max(\phi_1, \phi_2), \max(\upsilon_1, \upsilon_2)\right).$$

(i) Let $\delta_{\Cup} = \delta_1 \Cup \delta_2$. By definition:

$$\delta_{\Cup}(\lambda)(u) = \left(\max(\tau_1,\tau_2), \max(\iota_1,\iota_2), \max(\kappa_1,\kappa_2), \min(\phi_1,\phi_2), \min(\upsilon_1,\upsilon_2)\right).$$

Applying the necessity operator:

$$\circledcirc\delta_{\Cup}(\lambda)(u) = \left(\max(\tau_1,\tau_2), \max(\iota_1,\iota_2), \max(\kappa_1,\kappa_2), 1-\max(\tau_1,\tau_2), \min(\upsilon_1,\upsilon_2)\right).$$

Now, compute the right-hand side. First, for $i = 1, 2$:

$$\circledcirc\delta_i(\lambda)(u) = (\tau_i, \iota_i, \kappa_i, 1-\tau_i, \upsilon_i).$$

Their union is:

$$\left(\circledcirc\delta_1 \Cup \circledcirc\delta_2\right)(\lambda)(u) = \left(\max(\tau_1,\tau_2), \max(\iota_1,\iota_2), \max(\kappa_1,\kappa_2), \min(1-\tau_1, 1-\tau_2), \min(\upsilon_1,\upsilon_2)\right).$$

Since $\min(1-\tau_1, 1-\tau_2) = 1-\max(\tau_1,\tau_2)$, the two expressions are equal. This proves (i).

(ii) Let $\delta_{\Cap} = \delta_1 \Cap \delta_2$. By definition:

$$\delta_{\Cap}(\lambda)(u) = \left(\min(\tau_1,\tau_2), \min(\iota_1,\iota_2), \min(\kappa_1,\kappa_2), \max(\phi_1,\phi_2), \max(\upsilon_1,\upsilon_2)\right).$$

Applying the necessity operator:

$$\circledcirc\delta_{\Cap}(\lambda)(u) = \left(\min(\tau_1,\tau_2), \min(\iota_1,\iota_2), \min(\kappa_1,\kappa_2), 1-\min(\tau_1,\tau_2), \max(\upsilon_1,\upsilon_2)\right).$$

Now, compute the intersection of the necessities:

$$\left(\circledcirc\delta_1 \Cap \circledcirc\delta_2\right)(\lambda)(u) = \left(\min(\tau_1,\tau_2), \min(\iota_1,\iota_2), \min(\kappa_1,\kappa_2), \max(1-\tau_1, 1-\tau_2), \max(\upsilon_1,\upsilon_2)\right).$$

Since $\min(1-\tau_1, 1-\tau_2) = 1-\max(\tau_1,\tau_2)$, the two expressions are equal. This proves (ii).

(iii) Idempotence follows directly from the definition. For any (δ_1, Λ_1):

$$\circledcirc(\delta_1,\Lambda_1)(\lambda)(u) = (\tau_1, \iota_1, \kappa_1, 1-\tau_1, \upsilon_1).$$

Applying $(\circledcirc)$ again:

$$\begin{aligned} \circledcirc\left[\circledcirc(\delta_1,\Lambda_1)\right](\lambda)(u) &= \circledcirc(\tau_1, \iota_1, \kappa_1, 1-\tau_1, \upsilon_1) \\ &= (\tau_1, \iota_1, \kappa_1, 1-\tau_1, \upsilon_1) \\ &= \circledcirc(\delta_1,\Lambda_1)(\lambda)(u). \end{aligned}$$

This proves (iii). □

Theorem 4.3: *Let (δ_1,Λ_1) and (δ_2,Λ_2) be two PPINSS over a universe $\mathcal{U}$.*

Then:

(*i*) $\circledast\left[(\delta_1,\Lambda_1) \Cup (\delta_2,\Lambda_2)\right] = \circledast(\delta_1,\Lambda_1) \Cup \circledast(\delta_2,\Lambda_2)$;

(*ii*) $\circledast\left[(\delta_1,\Lambda_1) \Cap (\delta_2,\Lambda_2)\right] = \circledast(\delta_1,\Lambda_1) \Cap \circledast(\delta_2,\Lambda_2)$;

(*iii*) $\circledast\left[\circledast(\delta_1,\Lambda_1)\right] = \circledast(\delta_1,\Lambda_1)$.

Proof. We prove (i) in detail; (ii) and (iii) follow similarly.

Let $(\delta_1,\Lambda_1) \Cup (\delta_2,\Lambda_2) = (\delta_{\Cup},\Lambda_{\Cup})$ with $\Lambda_{\Cup} = \Lambda_1 \cup \Lambda_2$.
For any $\lambda \in \Lambda_{\Cup}$ and $u \in \mathcal{U}$, define:

$$\delta_i(\lambda)(u) = (\tau_i, \iota_i, \kappa_i, \phi_i, \upsilon_i), \quad \text{for } i = 1, 2.$$

By the definition of union for PPNSS:

$$\delta_{\Cup}(\lambda)(u) = \left(\max(\tau_1, \tau_2), \max(\iota_1, \iota_2), \max(\kappa_1, \kappa_2), \min(\phi_1, \phi_2), \min(\upsilon_1, \upsilon_2)\right).$$

The possibility operator $(\circledast)$ is defined as:

$$\circledast(\tau, \iota, \kappa, \phi, \upsilon) = (1 - \phi, \iota, \kappa, \phi, \upsilon).$$

Applying $(\circledast)$ to the union:

$$\begin{aligned}\circledast\delta_{\Cup}(\lambda)(u) &= \circledast\left(\max(\tau_1, \tau_2), \max(\iota_1, \iota_2), \max(\kappa_1, \kappa_2), \min(\phi_1, \phi_2), \min(\upsilon_1, \upsilon_2)\right) \\ &= \left(1 - \min(\phi_1, \phi_2), \max(\iota_1, \iota_2), \max(\kappa_1, \kappa_2), \min(\phi_1, \phi_2), \min(\upsilon_1, \upsilon_2)\right).\end{aligned}$$

Now compute the right-hand side. First, apply $(\circledast)$ individually:

$$\circledast\delta_i(\lambda)(u) = (1 - \phi_i, \iota_i, \kappa_i, \phi_i, \upsilon_i), \quad \text{for } i = 1, 2.$$

Their union is:

$$\left(\circledast\delta_1 \Cup \circledast\delta_2\right)(\lambda)(u) = \left(\max(1 - \phi_1, 1 - \phi_2), \max(\iota_1, \iota_2), \max(\kappa_1, \kappa_2), \min(\phi_1, \phi_2), \min(\upsilon_1, \upsilon_2)\right).$$

Since $\max(1 - \phi_1, 1 - \phi_2) = 1 - \min(\phi_1, \phi_2)$, the two expressions are equal:

$$\circledast\delta_{\Cup}(\lambda)(u) = \left(\circledast\delta_1 \Cup \circledast\delta_2\right)(\lambda)(u).$$

This holds for all $\lambda \in \Lambda_{\Cup}$ and $u \in \mathcal{U}$, proving (i).

(ii) For the intersection case, let $(\delta_1, \Lambda_1) \Cap (\delta_2, \Lambda_2) = (\delta_{\Cap}, \Lambda_{\Cap})$. Then:

$$\delta_{\Cap}(\lambda)(u) = \left(\min(\tau_1, \tau_2), \min(\iota_1, \iota_2), \min(\kappa_1, \kappa_2), \max(\phi_1, \phi_2), \max(\upsilon_1, \upsilon_2)\right).$$

Applying $(\circledast)$:

$$\circledast\delta_{\Cap}(\lambda)(u) = \left(1 - \max(\phi_1, \phi_2), \min(\iota_1, \iota_2), \min(\kappa_1, \kappa_2), \max(\phi_1, \phi_2), \max(\upsilon_1, \upsilon_2)\right).$$

Now compute the intersection of possibilities:

$$\left(\circledast\delta_1 \Cap \circledast\delta_2\right)(\lambda)(u) = \left(\min(1 - \phi_1, 1 - \phi_2), \min(\iota_1, \iota_2), \min(\kappa_1, \kappa_2), \max(\phi_1, \phi_2), \max(\upsilon_1, \upsilon_2)\right).$$

Since $\max(1 - \phi_1, 1 - \phi_2) = 1 - \min(\phi_1, \phi_2)$, the expressions are equal, proving (ii).

(iii) For idempotence, let $\delta(\lambda)(u) = (\tau, \iota, \kappa, \phi, \upsilon)$. Then:

$$\circledast\delta(\lambda)(u) = (1 - \phi, \iota, \kappa, \phi, \upsilon).$$

Applying $(\circledast)$ again:

$$\begin{aligned}\circledast\left(\circledast\delta(\lambda)(u)\right) &= \circledast(1 - \phi, \iota, \kappa, \phi, \upsilon) \\ &= (1 - \phi, \iota, \kappa, \phi, \upsilon) \\ &= \circledast\delta(\lambda)(u).\end{aligned}$$

This proves (iii). □

Theorem 4.4: *Let (δ, Λ) be a PPNSS. Then*

(*i*) $\circledast \circ \circledcirc (\delta, \Lambda) = \circledcirc(\delta, \Lambda)$;

(*ii*) $\circledcirc \circ \circledast (\delta, \Lambda) = \circledast(\delta, \Lambda)$.

Proof. Let $\lambda \in \Lambda$ and $u \in \mathcal{U}$ be arbitrary and let

$$\delta(\lambda)(u) = (\tau, \iota, \kappa, \phi, \upsilon).$$

(i) First, compute the necessity operator applied to δ:

$$\circledcirc\delta(\lambda)(u) = (\tau, \iota, \kappa, 1-\tau, \upsilon).$$

Now apply the possibility operator to this result:

$$\begin{aligned} \circledast\left(\circledcirc\delta(\lambda)(u)\right) &= \circledast(\tau, \iota, \kappa, 1-\tau, \upsilon) \\ &= \left(1-(1-\tau), \iota, \kappa, 1-\tau, \upsilon\right) \\ &= (\tau, \iota, \kappa, 1-\tau, \upsilon) \\ &= \circledcirc\delta(\lambda)(u). \end{aligned}$$

Since this holds for all $\lambda \in \Lambda$ and $u \in \mathcal{U}$, we have:

$$\circledast \circ \circledcirc (\delta, \Lambda) = \circledcirc(\delta, \Lambda).$$

(ii) Now compute the possibility operator applied to δ:

$$\circledast\delta(\lambda)(u) = (1-\phi, \iota, \kappa, \phi, \upsilon).$$

Apply the necessity operator to this result:

$$\begin{aligned} \circledcirc\left(\circledast\delta(\lambda)(u)\right) &= \circledcirc(1-\phi, \iota, \kappa, \phi, \upsilon) \\ &= \left(1-\phi, \iota, \kappa, 1-(1-\phi), \upsilon\right) \\ &= (1-\phi, \iota, \kappa, \phi, \upsilon) \\ &= \circledast\delta(\lambda)(u). \end{aligned}$$

Since this holds for all $\lambda \in \Lambda$ and $u \in \mathcal{U}$, we have:

$$\circledcirc \circ \circledast (\delta, \Lambda) = \circledast(\delta, \Lambda).$$

This completes the proof of both identities. □

Theorem 4.5: Let (δ_1, Λ_1) *and* (δ_2, Λ_2) be two PPINSS over a common universe $\mathcal{U}$. Then the following identities hold for all parameters and elements of $\mathcal{U}$:

(*i*) $\circledcirc\left[(\delta_1, \Lambda_1) \wedge (\delta_2, \Lambda_2)\right] = \circledcirc(\delta_1, \Lambda_1) \wedge \circledcirc(\delta_2, \Lambda_2)$;

(*ii*) $\circledcirc\left[(\delta_1, \Lambda_1) \vee (\delta_2, \Lambda_2)\right] = \circledcirc(\delta_1, \Lambda_1) \vee \circledcirc(\delta_2, \Lambda_2)$;

(*iii*) $\circledast\left[(\delta_1, \Lambda_1) \wedge (\delta_2, \Lambda_2)\right] = \circledast(\delta_1, \Lambda_1) \wedge \circledast(\delta_2, \Lambda_2)$;

(*iv*) $\circledast\left[(\delta_1, \Lambda_1) \vee (\delta_2, \Lambda_2)\right] = \circledast(\delta_1, \Lambda_1) \vee \circledast(\delta_2, \Lambda_2)$.

Proof. All statements hold componentwise. We demonstrate (i) and (iii); the others follow analogously by min–max duality.

Let $\lambda_1 \in \Lambda_1$, $\lambda_2 \in \Lambda_2$, and $u \in \mathcal{U}$. Denote

$$\delta_i(\lambda_i)(u) = (\tau_i, \iota_i, \kappa_i, \phi_i, \upsilon_i), \quad i = 1,2.$$

(i) Necessity under the $\wedge$-operation. By definition,

$$\delta_\wedge(\lambda_1, \lambda_2)(u) = \left(\min(\tau_1, \tau_2), \min(\iota_1, \iota_2), \min(\kappa_1, \kappa_2), \max(\phi_1, \phi_2), \min(\upsilon_1, \upsilon_2)\right).$$

Applying the necessity operator:

$$\circledcirc\delta_\wedge(\lambda_1, \lambda_2)(u) = \left(\min(\tau_1, \tau_2), \min(\iota_1, \iota_2), \min(\kappa_1, \kappa_2), 1 - \min(\tau_1, \tau_2), \min(\upsilon_1, \upsilon_2)\right).$$

Individually,

$$\circledcirc\delta_i(\lambda_i)(u) = (\tau_i, \iota_i, \kappa_i, 1 - \tau_i, \upsilon_i),$$

whose $\wedge$-operation gives

$$(\circledcirc\delta_1 \wedge \circledcirc\delta_2)(\lambda_1, \lambda_2)(u) = \left(\min(\tau_1, \tau_2), \min(\iota_1, \iota_2), \min(\kappa_1, \kappa_2), \max(1 - \tau_1, 1 - \tau_2), \min(\upsilon_1, \upsilon_2)\right).$$

Since $\max(1 - \tau_1, 1 - \tau_2) = 1 - \min(\tau_1, \tau_2)$, the two results coincide, proving (i).

(iii) Possibility under the $\wedge$-operation. The $\wedge$-combination gives

$$\delta_\wedge(\lambda_1, \lambda_2)(u) = \left(\min(\tau_1, \tau_2), \min(\iota_1, \iota_2), \min(\kappa_1, \kappa_2), \max(\phi_1, \phi_2), \min(\upsilon_1, \upsilon_2)\right).$$

Applying the possibility operator:

$$\circledast\delta_\wedge(\lambda_1, \lambda_2)(u) = \left(1 - \max(\phi_1, \phi_2), \min(\iota_1, \iota_2), \min(\kappa_1, \kappa_2), \max(\phi_1, \phi_2), \min(\upsilon_1, \upsilon_2)\right).$$

Individually,

$$\circledast\delta_i(\lambda_i)(u) = (1 - \phi_i, \iota_i, \kappa_i, \phi_i, \upsilon_i),$$

and their $\wedge$ gives

$$(\circledast\delta_1 \wedge \circledast\delta_2)(\lambda_1, \lambda_2)(u) = \left(\min(1 - \phi_1, 1 - \phi_2), \min(\iota_1, \iota_2), \min(\kappa_1, \kappa_2), \max(\phi_1, \phi_2), \min(\upsilon_1, \upsilon_2)\right).$$

Using $\min(1 - \phi_1, 1 - \phi_2) = 1 - \max(\phi_1, \phi_2)$, the two results coincide, proving (iii). (ii) and (iv) follow analogously for the $\vee$-operation using the dual equalities

$$\max(1 - \tau_1, 1 - \tau_2) = 1 - \min(\tau_1, \tau_2), \qquad \min(1 - \phi_1, 1 - \phi_2) = 1 - \max(\phi_1, \phi_2)$$

Hence, all identities hold componentwise under the intuitionistic constraint $\tau + \phi \le 1$ for every $u \in \mathcal{U}$ and $\lambda \in \Lambda$. □

Definition 4.6: *Let (δ_1, Λ_1) and (δ_2, Λ_2) be two PPINSS defined over a common universe $\mathcal{U}$. Define*

$$\Lambda_\oplus = \Lambda_\ominus = \Lambda_1 \cup \Lambda_2.$$

(i) The $\oplus$ operator is defined as

$$(\delta_1, \Lambda_1) \oplus (\delta_2, \Lambda_2) = (\delta_\oplus, \Lambda_\oplus),$$

where, for every $\lambda \in \Lambda_\oplus$ and $u \in \mathcal{U}$,

$$\delta_{\oplus}(\lambda)(u)=\begin{cases}\delta_1(\lambda)(u), & \text{if } \lambda\in\Lambda_1\setminus\Lambda_2,\\ \delta_2(\lambda)(u), & \text{if } \lambda\in\Lambda_2\setminus\Lambda_1,\\ \left(\frac{\tau_1+\tau_2}{2},\frac{\iota_1+\iota_2}{2},\frac{\kappa_1+\kappa_2}{2},\frac{\phi_1+\phi_2}{2},\frac{\upsilon_1+\upsilon_2}{2}\right), & \text{if } \lambda\in\Lambda_1\cap\Lambda_2,\end{cases}$$

where $(\tau_i,\iota_i,\kappa_i,\phi_i,\upsilon_i)=\delta_i(\lambda)(u)$ *for* $i=1,2$.

(ii) The $\ominus$ *operator is defined as*

$$(\delta_1,\Lambda_1)\ominus(\delta_2,\Lambda_2)=(\delta_{\ominus},\Lambda_{\ominus}),$$

where, for every $\lambda\in\Lambda_{\oplus}$ *and* $u\in\mathcal{U}$,

$$\delta_{\ominus}(\lambda)(u)=\begin{cases}\delta_1(\lambda)(u), & \text{if } \lambda\in\Lambda_1\setminus\Lambda_2,\\ \delta_2(\lambda)(u), & \text{if } \lambda\in\Lambda_2\setminus\Lambda_1,\\ \left(\frac{2\tau_1\tau_2}{\tau_1+\tau_2},\frac{\iota_1+\iota_2}{2},\frac{\kappa_1+\kappa_2}{2},\frac{2\phi_1\phi_2}{\phi_1+\phi_2},\frac{\upsilon_1+\upsilon_2}{2}\right), & \text{if } \lambda\in\Lambda_1\cap\Lambda_2,\end{cases}$$

Proposition 4.7: *Let* (δ_1,Λ_1) *and* (δ_2,Λ_2) *be non-empty PPINSS over the universe* $\mathcal{U}$. *Then*

(i) $(\delta_1,\Lambda_1)\oplus(\delta_2,\Lambda_2)=(\delta_2,\Lambda_2)\oplus(\delta_1,\Lambda_1)$;

(ii) $\left[(\delta_1,\Lambda_1)^c\oplus(\delta_2,\Lambda_2)^c\right]^c=(\delta_1,\Lambda_1)\oplus(\delta_2,\Lambda_2)$.

Proof. We prove each item componentwise.

(i) For $\lambda\in\Lambda_1\cap\Lambda_2$ and $u\in\mathcal{U}$. Write

$$\delta_i(\lambda)(u)=(\tau_i,\iota_i,\kappa_i,\phi_i,\upsilon_i),\qquad i=1,2.$$

By definition of $\oplus$ (componentwise arithmetic mean) we have

$$\left(\delta_1\oplus\delta_2\right)(\lambda)(u)=\left(\frac{\tau_1+\tau_2}{2},\frac{\iota_1+\iota_2}{2},\frac{\kappa_1+\kappa_2}{2},\frac{\phi_1+\phi_2}{2},\frac{\upsilon_1+\upsilon_2}{2}\right)=\left(\frac{\tau_2+\tau_1}{2},\frac{\iota_2+\iota_1}{2},\frac{\kappa_2+\kappa_1}{2},\frac{\phi_2+\phi_1}{2},\frac{\upsilon_2+\upsilon_1}{2}\right)=\left(\delta_2\oplus\delta_1\right)(\lambda)(u).$$

(ii) For $\lambda\in\Lambda_1\cap\Lambda_2$ and $u\in\mathcal{U}$; use the same notation. First compute complements (using the PPINSS complement rule with $1-\upsilon$):

$$\delta_i^c(\lambda)(u)=(\phi_i,1-\iota_i,1-\kappa_i,\tau_i,1-\upsilon_i),\qquad i=1,2.$$

Apply $\oplus$ to these complements:

$$\left(\delta_1^c\oplus\delta_2^c\right)(\lambda)(u)=\left(\frac{\phi_1+\phi_2}{2},\frac{(1-\iota_1)+(1-\iota_2)}{2},\frac{(1-\kappa_1)+(1-\kappa_2)}{2},\frac{\tau_1+\tau_2}{2},\frac{(1-\upsilon_1)+(1-\upsilon_2)}{2}\right).$$

Now take complement of this result (apply complement componentwise):

$$\left(\left(\delta_1^c\oplus\delta_2^c\right)(\lambda)(u)\right)^c=\left(\frac{\tau_1+\tau_2}{2},1-\frac{(1-\iota_1)+(1-\iota_2)}{2},1-\frac{(1-\kappa_1)+(1-\kappa_2)}{2},\frac{\phi_1+\phi_2}{2},1-\frac{(1-\upsilon_1)+(1-\upsilon_2)}{2}\right).$$

But elementary algebra yields

$$1-\frac{(1-\iota_1)+(1-\iota_2)}{2}=\frac{\iota_1+\iota_2}{2},\qquad 1-\frac{(1-\kappa_1)+(1-\kappa_2)}{2}=\frac{\kappa_1+\kappa_2}{2},$$

and

$$1-\frac{(1-\upsilon_1)+(1-\upsilon_2)}{2}=\frac{\upsilon_1+\upsilon_2}{2}.$$

Hence the complemented tuple simplifies to

$$\left(\frac{\tau_1+\tau_2}{2}, \frac{\iota_1+\iota_2}{2}, \frac{\kappa_1+\kappa_2}{2}, \frac{\phi_1+\phi_2}{2}, \frac{\upsilon_1+\upsilon_2}{2}\right),$$

which is exactly $(\delta_1 \oplus \delta_2)(\lambda)(u)$. Therefore,

$$\left[(\delta_1, \Lambda_1)^c \oplus (\delta_2, \Lambda_2)^c\right]^c = (\delta_1, \Lambda_1) \oplus (\delta_2, \Lambda_2).$$

This completes the proof. □

Proposition 4.8: *Let (δ_1, Λ_1) and (δ_2, Λ_2) be non-empty PPINSS over the universe $\mathcal{U}$. Then*

(*i*) $(\delta_1, \Lambda_1) \ominus (\delta_2, \Lambda_2) = (\delta_2, \Lambda_2) \ominus (\delta_1, \Lambda_1)$;

(*ii*) $\left[(\delta_1, \Lambda_1)^c \ominus (\delta_2, \Lambda_2)^c\right]^c = (\delta_1, \Lambda_1) \ominus (\delta_2, \Lambda_2)$.

Definition 4.9: *Let (δ, Λ) be a PPINSS over a universe $\mathcal{U}$. For a fixed parameter $\lambda \in \Lambda$ and element $u \in \mathcal{U}$ write*

$$\delta(\lambda)(u) = (\tau_\lambda(u), \iota_\lambda(u), \kappa_\lambda(u), \phi_\lambda(u), \upsilon_\lambda(u)),$$

with the intuitionistic margin

$$\pi_\lambda(u) := 1 - \tau_\lambda(u) - \phi_\lambda(u) \geq 0.$$

Let $(\mu \in [0,1])$. The operator (Δ_μ) produces the PPINSS

$$\Delta_\mu(\delta, \Lambda) = (\delta^{(\mu)}, \Lambda),$$

where for every $\lambda \in \Lambda$ and $\in$

$$\delta^{(\mu)}(\lambda)(u) = \left(\tau_\lambda(u) + \mu\pi_\lambda(u), \iota_\lambda(u), \kappa_\lambda(u), \phi_\lambda(u) + (1-\mu)\pi_\lambda(u), \upsilon_\lambda(u)\right).$$

Proposition 4.10: Let $\mu, \nu \in [0,1]$ *with* $\mu \leq \nu$. Then for every PPINSS (δ, Λ) over $\mathcal{U}$, the following properties hold:

(*i*) $\Delta_\mu(\delta, \Lambda) \Subset \Delta_\nu(\delta, \Lambda)$;

(*ii*) $\Delta_0(\delta, \Lambda) = \circledcirc(\delta, \Lambda)$;

(*iii*) $\Delta_1(\delta, \Lambda) = \circledast(\delta, \Lambda)$.

Proof. For each $\lambda \in \Lambda$ and $u \in \mathcal{U}$, write

$$\delta(\lambda)(u) = (\tau, \iota, \kappa, \phi, \upsilon), \qquad \pi_{\delta(\lambda)}(u) = 1 - \tau - \phi.$$

Then, by definition of the operator Δ_α,

$$\Delta_\alpha(\delta, \Lambda) \ni (\tau + \alpha\pi, \iota, \kappa, \phi + (1-\alpha)\pi, \upsilon),$$

where $\pi = \pi_{\delta(\lambda)}(u) \geq 0$.

(i) Let $0 \leq \mu \leq \nu \leq 1$. Then for the truth and falsity components we have

$$\tau + \mu\pi \leq \tau + \nu\pi, \qquad \phi + (1-\nu)\pi \leq \phi + (1-\mu)\pi.$$

The indeterminacy ι, contradiction κ, and unawareness υ remain unchanged. Hence, under the PPINSS subset relation (where higher truth corresponds to "larger" and higher uncertainty corresponds to "smaller" information states),

$$\Delta_\mu(\delta, \Lambda) \Subset \Delta_\nu(\delta, \Lambda).$$

Thus Δ_μ is monotone increasing with respect to μ.

(ii) *Endpoint $\mu = 0$ corresponds to the necessity operator*. Setting $\mu = 0$ gives

$$\tau^{(0)} = \tau, \qquad \phi^{(0)} = \phi + \pi = \phi + (1 - \tau - \phi) = 1 - \tau.$$

Hence

$$\Delta_0(\delta, \Lambda) = (\tau, \iota, \kappa, 1 - \tau, \upsilon) = \circledcirc(\delta, \Lambda).$$

(iii) *Endpoint $\mu = 1$ corresponds to the possibility operator. For $\mu = 1$ we have*

$$\tau^{(1)} = \tau + \pi = \tau + (1 - \tau - \phi) = 1 - \phi, \qquad \phi^{(1)} = \phi.$$

Therefore

$$\Delta_1(\delta, \Lambda) = (1 - \phi, \iota, \kappa, \phi, \upsilon) = \circledast(\delta, \Lambda).$$

□

Definition 4.11: *Let (δ, Λ) be a PPINSS defined over a universe $\mathcal{U}$, where for each parameter $\lambda \in \Lambda$ and object $u \in \mathcal{U}$,*

$$\delta(\lambda)(u) = \left(\tau_\lambda(u), \iota_\lambda(u), \kappa_\lambda(u), \phi_\lambda(u), \upsilon_\lambda(u)\right),$$

and the intuitionistic margin *is given by*

$$\pi_\lambda(u) = 1 - \tau_\lambda(u) - \phi_\lambda(u).$$

Let $\mu, \nu \in [0,1]$ be parameters satisfying $\mu + \nu \le 1$. Then, the bi-parametric redistribution operator $\Delta_{\mu,\nu}$ is defined as

$$\Delta_{\mu,\nu}(\delta, \Lambda) = (\delta^{(\mu,\nu)}, \Lambda),$$

where for every $\lambda \in \Lambda$ and $u \in \mathcal{U}$,

$$\delta^{(\mu,\nu)}(\lambda)(u) = \left(\tau_\lambda(u) + \mu\pi_\lambda(u), \iota_\lambda(u), \kappa_\lambda(u), \phi_\lambda(u) + \nu\pi_\lambda(u), \upsilon_\lambda(u)\right).$$

Here:

$$\tau_\lambda^{(\mu,\nu)}(u) + \phi_\lambda^{(\mu,\nu)}(u) = 1 - (1 - \mu - \nu)\pi_\lambda(u) \le 1.$$

When $\mu + \nu = 1$, the operator fully redistributes the indeterminate margin $\pi_\lambda(u)$, eliminating uncertainty. When $\mu + \nu \le 1$, a residual margin $(1 - \mu - \nu)\pi_\lambda(u)$ is preserved, retaining partial indeterminacy.

Theorem 4.12: *Let $\mu, \nu \in [0,1]$ with $\mu + \nu \le 1$. Then for every PPINSS (δ, Λ) the following hold:*

(i) *$\Delta_{\mu,\nu}(\delta, \Lambda)$ is a PPINSS*;

(ii) *If $0 \le \chi \le \mu$ then $\Delta_{\chi,\nu}(\delta, \Lambda) \Subset \Delta_{\mu,\nu}(\delta, \Lambda)$*;

(*iii*) *If* $0 \le \chi \le \nu$ *then* $\Delta_{\mu,\nu}(\delta,\Lambda) \Subset \Delta_{\mu,\chi}(\delta,\Lambda)$;

(*iv*) $\Delta_{\mu}(\delta,\Lambda) = \Delta_{\mu,1-\mu}(\delta,\Lambda)$;

(*v*) $\circledcirc(\delta,\Lambda) = \Delta_{0,1}(\delta,\Lambda)$;

(*vi*) $\circledast(\delta,\Lambda) = \Delta_{1,0}(\delta,\Lambda)$;

(*vii*) $\left(\left(\Delta_{\mu,\nu}(\delta,\Lambda)\right)^{c} = \Delta_{\nu,\mu}\left(\delta^{c},\Lambda\right)\right)$, *and hence* $\left(\left(\Delta_{\mu,\nu}(\delta,\Lambda)^{c}\right)^{c} = \Delta_{\mu,\nu}(\delta,\Lambda)\right)$.

Proof. Fix $\lambda \in \Lambda$ and $u \in \mathcal{U}$ and write

$$\delta(\lambda)(u) = (\tau,\iota,\kappa,\phi,\upsilon), \qquad \pi := 1-\tau-\phi\ (\ge 0).$$

(i) For $\Delta_{\mu,\nu}$ we have

$$\tau^{(\mu,\nu)} = \tau + \mu\pi, \qquad \phi^{(\mu,\nu)} = \phi + \nu\pi.$$

Clearly $0 \le \tau^{(\mu,\nu)} \le \tau + \pi = 1-\phi \le 1$ and $0 \le \phi^{(\mu,\nu)} \le \phi + \pi = 1-\tau \le 1$. Moreover

$$\tau^{(\mu,\nu)} + \phi^{(\mu,\nu)} = \tau + \phi + (\mu+\nu)\pi = 1-(1-\mu-\nu)\pi \le 1,$$

since $\mu+\nu \le 1$ and $\pi \ge 0$. The other components remain in [0, 1] unchanged. Thus $\Delta_{\mu,\nu}(\delta,\Lambda)$ satisfies the PPINSS bounds.

(ii) If $0 \le \chi \le \mu$ then

$\tau + \chi\pi \le \tau + \mu\pi,$

while ϕ and the uncertainty components are unchanged. By the PPINSS-subset convention, we obtain

$$\Delta_{\chi,\nu}(\delta,\Lambda) \Subset \Delta_{\mu,\nu}(\delta,\Lambda).$$

(iii) If $0 \le \chi \le \nu$ then

$\phi + \nu\pi \ge \phi + \chi\pi,$

so the PPINSS-subset ordering gives

$$\Delta_{\mu,\nu}(\delta,\Lambda) \Subset \Delta_{\mu,\chi}(\delta,\Lambda).$$

(iv)–(vi) are immediate from the definitions by setting $\nu = 1-\mu$, $(\mu,\nu) = (0,1)$ and (1, 0) respectively.
(vii) Using the complement mapping

$$(\tau,\iota,\kappa,\phi,\upsilon)^{c} = (\phi, 1-\iota, 1-\kappa, \tau, 1-\upsilon),$$

compute the complement of $\Delta_{\mu,\nu}(\delta,\Lambda)$. Starting from

$$\delta^{(\mu,\nu)}(\lambda)(u) = (\tau+\mu\pi,\ \iota,\ \kappa,\ \phi+\nu\pi,\ \upsilon),$$

its complement is

$$\left(\Delta_{\mu,\nu}(\delta,\Lambda)\right)^{c} \ni (\phi+\nu\pi, 1-\iota, 1-\kappa, \tau+\mu\pi, 1-\upsilon).$$

On the other hand, the complement set δ^{c} has

$$\delta^{c}(\lambda)(u) = (\phi, 1-\iota, 1-\kappa, \tau, 1-\upsilon),$$

whose margin equals the original π. Applying $\Delta_{\nu,\mu}$ to δ^c yields

$$\Delta_{\nu,\mu}(\delta^c,\Lambda) \ni (\phi+\nu\pi, 1-\iota, 1-\kappa, \tau+\mu\pi, 1-\upsilon),$$

which coincides with the previous complement. Hence

$$\left(\Delta_{\mu,\nu}(\delta,\Lambda)\right)^c = \Delta_{\nu,\mu}(\delta^c,\Lambda).$$

Taking complements again gives the stated proof. □

Definition 4.13: *Let (δ,Λ) be a PPINSS over universe $\mathcal{U}$. For $\mu,\nu\in[0,1]$ define the operator*

$$\Phi_{\mu,\nu}(\delta,\Lambda) = (\delta^{(\mu,\nu)},\Lambda)$$

by, for every $\lambda\in\Lambda$ and $u\in\mathcal{U}$,

$$\delta^{(\mu,\nu)}(\lambda)(u) = \left(\mu\tau_{\delta(\lambda)}(u), \iota_{\delta(\lambda)}(u), \kappa_{\delta(\lambda)}(u), \nu\phi_{\delta(\lambda)}(u), \upsilon_{\delta(\lambda)}(u)\right).$$

Theorem 4.14: Let $\mu,\nu,\chi\in[0,1]$. Then for every PPINSS (δ,Λ) the following properties hold:

- (*i*) $\Phi_{\mu,\nu}(\delta,\Lambda)$ *is a PPINSS*;
- (*ii*) *If* $\mu\le\chi$ *then* $\Phi_{\mu,\nu}(\delta,\Lambda)\Subset\Phi_{\chi,\nu}(\delta,\Lambda)$;
- (*iii*) *If* $\nu\le\chi$ *then* $\Phi_{\mu,\nu}(\delta,\Lambda)\Supset\Phi_{\mu,\chi}(\delta,\Lambda)$;
- (*iv*) *For all* $\mu,\nu,\chi,\delta\in[0,1]$,
  $\Phi_{\mu,\nu}\left(\Phi_{\chi,\delta}(\delta,\Lambda)\right) = \Phi_{\mu\chi,\nu\delta}(\delta,\Lambda) = \Phi_{\chi,\delta}\left(\Phi_{\mu,\nu}(\delta,\Lambda)\right)$.
- (*v*) $\left(\Phi_{\mu,\nu}(\delta,\Lambda)\right)^c = \Phi_{\nu,\mu}(\delta^c,\Lambda)$, *and hence* $\left(\Phi_{\mu,\nu}(\delta,\Lambda)^c\right)^c = \Phi_{\mu,\nu}(\delta,\Lambda)$.

Proof. Fix $\lambda\in\Lambda$ and $u\in\mathcal{U}$ and write

$$\delta(\lambda)(u) = (\tau,\iota,\kappa,\phi,\upsilon), \qquad \tau,\iota,\kappa,\phi,\upsilon\in[0,1], \quad \tau+\phi\le 1.$$

(i) The new truth and falsity are

$$\tau' = \mu\tau, \qquad \phi' = \nu\phi.$$

Clearly $0\le\tau'\le\mu\le 1$ and $0\le\phi'\le\nu\le 1$. Moreover

$$\tau'+\phi' = \mu\tau+\nu\phi \le \max\{\mu,\nu\}(\tau+\phi) \le \max\{\mu,\nu\} \le 1,$$

since $\tau+\phi\le 1$ and $\max\{\mu,\nu\}\le 1$. The remaining components are unchanged and lie in [0,1]. Thus $\Delta_{\mu,\nu}(\delta,\Lambda)$ satisfies the PPINSS constraints.

(ii) If $\mu\le\chi$ then $(\mu\tau\le\chi\tau)$ for every $\tau\in[0,1]$. The uncertainty components are identical; hence by the PPINSS-subset ordering (truth nondecreasing, uncertainty components nonincreasing) we have

$$\Phi_{\mu,\nu}(\delta,\Lambda)\Subset\Phi_{\chi,\nu}(\delta,\Lambda).$$

(iii) If $\nu\le\chi$ then $(\nu\phi\le\chi\phi)$. Since larger falsity corresponds to a "smaller" information state under $\Subset$ (see subset definition), we obtain

$$\Phi_{\mu,\nu}(\delta,\Lambda)\Supset\Phi_{\mu,\chi}(\delta,\Lambda).$$

(iv) Apply $\Phi_{\chi,\delta}$ first:

$$\Phi_{\chi,\delta}(\delta,\Lambda) \ni (\chi\tau,\iota,\kappa,\delta\phi,\upsilon).$$

Applying $\Phi_{\mu,\nu}$ to this yields

$$(\mu(\chi\tau),\iota,\kappa,\nu(\delta\phi),\upsilon) = (\mu\chi\tau,\iota,\kappa,\nu\delta\phi,\upsilon).$$

Symmetry of multiplication gives the other order; hence the compositions coincide and equal $\Phi_{\mu\chi,\nu\delta}$.

(v) Using the PPINSS complement mapping

$$(\alpha,\beta,\gamma,\delta,\epsilon)^c = (\delta,1-\beta,1-\gamma,\alpha,1-\epsilon),$$

compute the complement of $\Phi_{\mu,\nu}(\delta,\Lambda)$. Starting from

$$\Phi_{\mu,\nu}(\delta,\Lambda) \ni (\mu\tau,\iota,\kappa,\nu\phi,\upsilon),$$

its complement equals

$$(\nu\phi, 1-\iota, 1-\kappa, \mu\tau, 1-\upsilon).$$

On the other hand, the complement δ^c equals $(\phi,1-\iota,1-\kappa,\tau,1-\upsilon)$, and applying $\Phi_{\mu,\nu}$ to δ^c gives

$$(\nu\phi, 1-\iota, 1-\kappa, \mu\tau, 1-\upsilon),$$

which coincides with the previous expression. Therefore

$$\left(\Phi_{\mu,\nu}(\delta,\Lambda)\right)^c = \Phi_{\nu,\mu}(\delta^c,\Lambda).$$

Applying complement again yields the double-complement equality

$$\left(\Phi_{\mu,\nu}(\delta,\Lambda)^c\right)^c = \Phi_{\mu,\nu}(\delta,\Lambda).$$

This completes the proof. □

5. Similarity Measures and Pattern Recognition Applications

Building upon the nonlinear algebraic behaviour of the ⊖ operator, we construct a family of similarity measures for PPINSS. These measures are designed to quantify both direct and interaction-based relationships among PPINSS elements, providing a unified framework for pattern recognition and decision analysis.

Definition 5.1: *Let $X = (\delta_1, \Lambda_1)$ and $Y = (\delta_2, \Lambda_2)$ be two PPINSS over the same universe $\mathcal{U}$. We define three types of similarity measures:*

(1) Direct ⊖-Similarity:

$$[\mathfrak{S}_D(X,Y) = \frac{1}{|\Lambda_1 \cup \Lambda_2| \cdot |\mathcal{U}|} \sum_{\lambda \in \Lambda_1 \cup \Lambda_2} \sum_{u \in \mathcal{U}} \left[\sum_{k=1}^{5} w_k \cdot sim_k(\delta_1(\lambda)(u), \delta_2(\lambda)(u)) \right],$$

where the componentwise similarities are given by:

$$sim_1(\tau_1,\tau_2) = 1-|\tau_1-\tau_2|, \qquad sim_2(\iota_1,\iota_2) = 1-|\iota_1-\iota_2|,$$
$$sim_3(\kappa_1,\kappa_2) = 1-|\kappa_1-\kappa_2|, \qquad sim_4(\phi_1,\phi_2) = 1-|\phi_1-\phi_2|,$$
$$sim_5(\upsilon_1,\upsilon_2) = 1-|\upsilon_1-\upsilon_2|.$$

Here, $w_k \in [0,1]$ *are weight coefficients satisfying* $\sum_{k=1}^{5} w_k = 1$.

(2) ***Combined*** $\ominus$ ***-Similarity:***

$$[\mathfrak{S}_C(X,Y) = \mathfrak{S}_D(X \ominus Y, X \cap Y),$$

which evaluates how the nonlinear $\ominus$ *-interaction between X and Y aligns with their shared intersection region.*

(3) Weighted Comprehensive Similarity:

$$\mathfrak{S}_W(X,Y) = \alpha\mathfrak{S}_D(X,Y) + \beta\mathfrak{S}_C(X,Y) + \gamma\mathfrak{S}_D(X^c,Y^c), \text{ where } \alpha+\beta+\gamma = 1.$$

The parameters $\alpha,\beta,\gamma \in [0,1]$ *allow the analyst to emphasize direct, interaction-based, or complement-invariant similarities according to the decision context.*

Theorem 5.2: Let X, Y, Z be PPINSS over the same universe $\mathcal{U}$. With component weights $w_k \in [0,1]$ satisfying $\sum_{k=1}^{5} w_k = 1$ and convex parameters $\alpha,\beta,\gamma \in [0,1]$ with $\alpha+\beta+\gamma = 1$, the similarity measures $\mathfrak{S}_D$, $\mathfrak{S}_C$, *and* $\mathfrak{S}_W$ satisfy:

- (i) $0 \le \mathfrak{S}_*(X,Y) \le 1$ *for* $* \in \{D,C,W\}$;
- (ii) $\mathfrak{S}_*(X,Y) = 1$ *if and only if* $X = Y$;
- (iii) $\mathfrak{S}_*(X,Y) = \mathfrak{S}_*(Y,X)$;
- (iv) $\mathfrak{S}_*(X^c,Y^c) = \mathfrak{S}_*(X,Y)$, *provided* $w_1 = w_4$ (*i.e., equal weights for truth and falsity components*);
- (v) *For the direct similarity measure*,

$$\mathfrak{S}_D(X,Y) \ge 1 - \frac{1}{|\Lambda_1 \cup \Lambda_2| \cdot |\mathcal{U}|} \sum_{\lambda \in \Lambda_1 \cup \Lambda_2} \sum_{u \in \mathcal{U}} \sum_{k=1}^{5} w_k \left(d_k(X,Z;\lambda,u) + d_k(Z,Y;\lambda,u) \right),$$

where $d_k(A,B;\lambda,u)$ *denotes the absolute difference of the k-th component between sets A and B at parameter λ and element u.*

Proof. **(i)** For each component, $\text{sim}_k(a,b) = 1 - |a-b|$ where $a,b \in [0,1]$, so $0 \le \text{sim}_k(a,b) \le 1$. Since the weights w_k are non-negative and sum to one, their convex combination also lies in [0,1]. Averaging over λ and u preserves this range. The combined measure $\mathfrak{S}_W$ is a convex combination of bounded terms $\mathfrak{S}_D$, $\mathfrak{S}_C$, and $\mathfrak{S}_D(X^c,Y^c)$, so $0 \le \mathfrak{S}_*(X,Y) \le 1$ for all cases.

(ii) If $\mathfrak{S}_D(X,Y) = 1$, then for every λ, u, the inner weighted average of sim_k equals 1. Because each $\text{sim}_k \le 1$ and weights are positive, we must have $\text{sim}_k(a,b) = 1$ for all k, which implies $|a-b| = 0$ and hence $a = b$ for all components. Therefore $\delta_1(\lambda)(u) = \delta_2(\lambda)(u)$ for all λ, u, so $X = Y$. Conversely, if $X = Y$, then every $|a-b| = 0$, so $\mathfrak{S}_D(X,Y) = 1$. The same reasoning holds for $\mathfrak{S}_C$ and $\mathfrak{S}_W$ since they are derived from $\mathfrak{S}_D$.

(iii) For each component, $\text{sim}_k(a,b) = 1 - |a-b| = 1 - |b-a| = \text{sim}_k(b,a)$, hence $\mathfrak{S}_D(X,Y) = \mathfrak{S}_D(Y,X)$. The operators $\ominus$ and $\cap$ are symmetric, so $\mathfrak{S}_C(X,Y) = \mathfrak{S}_C(Y,X)$. Consequently, $\mathfrak{S}_W$ is also symmetric.

(iv) Let the complement of each element be $(\tau,\iota,\kappa,\phi,\upsilon)^c = (\phi, 1-\iota, 1-\kappa, \tau, \upsilon)$. Then componentwise:

$$\text{sim}_1^c = 1 - |\phi_1 - \phi_2|, \qquad \text{sim}_2^c = 1 - |\iota_1 - \iota_2|,$$

$$\text{sim}_3^c = 1 - |\kappa_1 - \kappa_2|, \qquad \text{sim}_4^c = 1 - |\tau_1 - \tau_2|,$$
$$\text{sim}_5^c = 1 - |\upsilon_1 - \upsilon_2|.$$

Thus, the set of similarity values for (X^c, Y^c) is identical to that for (X, Y) except that the truth and falsity components are interchanged. If the weights satisfy $w_1 = w_4$, the inner weighted sum remains unchanged, hence $\mathfrak{S}_D(X^c, Y^c) = \mathfrak{S}_D(X, Y)$. Since $\ominus$ and $\cap$ are complement-consistent, the same invariance holds for $\mathfrak{S}_C$ and $\mathfrak{S}_W$.

(v) For any real $a, b, c \in [0,1]$, the triangle inequality gives $|a-b| \le |a-c| + |c-b|$. Therefore,

$$1 - |a-b| \ge 1 - (|a-c| + |c-b|).$$

Applying this to each component k of $\delta(\lambda)(u)$, multiplying by w_k, and summing over k, we obtain

$$\sum_k w_k (1 - |a-b|) \ge \sum_k w_k \left[1 - (|a-c| + |c-b|)\right].$$

Averaging over all parameters and universe elements preserves the inequality, yielding the stated additive triangle bound for $\mathfrak{S}_D$. □

Algorithmic 1: PPINSS Pattern Recognition using Similarity Measures (improved)

Require: Database of reference patterns $\{P_1, \ldots, P_n\}$, query pattern Q, weight parameters (α, β, γ)
Ensure: Best matching pattern, confidence score, and normalized rankings

1: **Ensure weights sum to one:** if $\alpha + \beta + \gamma \ne 1$, set

$$(\alpha, \beta, \gamma) \leftarrow \frac{1}{\alpha + \beta + \gamma}(\alpha, \beta, \gamma)$$

2: Initialize $S_i \leftarrow 0$ forall $i = 1, \ldots, n$
3: **for** each pattern P_i in database **do**
4: **Preprocessing:** compute nonlinear signature

$$T(P_i) \leftarrow P_i \ominus P_i^c$$

5: Compute direct similarity:

$$s_{i1} \leftarrow \mathfrak{S}_D(Q, P_i)$$

6: Compute combined similarity (uses $\ominus$ and intersection):

$$s_{i2} \leftarrow \mathfrak{S}_C(Q, P_i) = \mathfrak{S}_D\big(Q \ominus P_i, Q \cap P_i\big)$$

7: Compute complement similarity:

$$s_{i3} \leftarrow \mathfrak{S}_D(Q^c, P_i^c)$$

8: Optional (recommended) operator-based similarity using nonlinear signatures:

$$s_{i4} \leftarrow \mathfrak{S}_D\left(\mathcal{O}_{\ominus}(Q, P_i), \mathcal{O}_{\ominus}(P_i, T(P_i))\right)$$

9: Aggregate weighted score (if s_{i4} used, give it a small weight δ, or fold it into β):

$$S_i \leftarrow \alpha s_{i1} + \beta s_{i2} + \gamma s_{i3}$$
$$(\textit{If using } s_{i4}, \textit{you can use e.g. } S_i \leftarrow (1-\delta)S_i + \delta s_{i4}.)$$

end for
Identify best match(s): $I^* \leftarrow \arg\max_i S_i$ if multiple, apply tie-break below
If $|I^{\wedge *}|>1$ thene
choose $i^* \in I^*$ by (in order): highest s_{i2}, then highest s_{i4}, then smallest index
else
    $i^* \leftarrow$ the (unique) element of I^*
end if
Compute confidence score:

$$\text{confidence} \leftarrow \frac{S_{i^*}}{\sum_{i=1}^{n} S_i}$$

Normalize scores:

$$\text{norm\_scores} \leftarrow \left(\frac{S_1}{\max_i S_i}, \ldots, \frac{S_n}{\max_i S_i} \right)$$

return (P_i* norm_scores)}

5.1. Pattern Recognition Algorithm. We now present a comprehensive pattern recognition framework that utilizes the proposed PPINSS similarity measures.

The operational flow of the proposed PPINSS recognition model is depicted in Fig. 1. It illustrates how the nonlinear $\ominus$ operator functions as a preprocessing stage before similarity computation, providing a more robust basis for pattern differentiation.

5.2. Medical Diagnosis Application. To illustrate the applicability of the proposed PPINSS similarity framework, consider a medical diagnosis system comprising three diseases and a patient profile. Symptoms are represented as PPINSS over the universe $\mathcal{U}$ = {fever, cough, fatigue} with parameters Λ = {severity, duration}.

Disease Patterns.

- **Influenza** (D_1) — high fever, moderate cough, high fatigue:

$$\delta_1(\text{severity})(\text{fever}) = (0.8, 0.1, 0.1, 0.1, 0.1),$$
$$\delta_1(\text{severity})(\text{cough}) = (0.7, 0.2, 0.2, 0.2, 0.1),$$
$$\delta_1(\text{duration})(\text{fatigue}) = (0.6, 0.3, 0.2, 0.3, 0.2).$$

- **COVID-19** (D_2) — very high fever, mild cough, very high fatigue:

$$\delta_1(\text{severity})(\text{fever}) = (0.8, 0.1, 0.1, 0.1, 0.1),$$
$$\delta_1(\text{severity})(\text{cough}) = (0.7, 0.2, 0.2, 0.2, 0.1),$$
$$\delta_1(\text{duration})(\text{fatigue}) = (0.6, 0.3, 0.2, 0.3, 0.2).$$

- **Common Cold** (D_3) — low fever, high cough, moderate fatigue:

$$\delta_3(\text{severity})(\text{fever}) = (0.4, 0.3, 0.4, 0.5, 0.3),$$
$$\delta_3(\text{severity})(\text{cough}) = (0.3, 0.4, 0.4, 0.6, 0.3),$$
$$\delta_3(\text{duration})(\text{fatigue}) = (0.5, 0.3, 0.3, 0.4, 0.2).$$

Patient Profile (P).

$$\delta_P(\text{severity})(\text{fever}) = (0.8, 0.2, 0.1, 0.1, 0.1),$$
$$\delta_P(\text{severity})(\text{cough}) = (0.7, 0.2, 0.2, 0.2, 0.1),$$
$$\delta_P(\text{duration})(\text{fatigue}) = (0.7, 0.2, 0.2, 0.2, 0.1).$$

Parameter Consistency. Fever and cough are evaluated using the "severity" parameter, while fatigue is assessed via the "duration" parameter. This aligns with clinical practice, where duration is diagnostically more relevant for fatigue, and severity is emphasized for fever and cough.

Similarity Framework. Using the proposed PPINSS similarity measures:

- $\mathfrak{S}_D$: Direct similarity between patient and disease patterns.
- $\mathfrak{S}_C$: Complement-based similarity.
- $\mathfrak{S}_D(P^c, D^c)$: Similarity between complement patterns.

The aggregated similarity is computed as:

$$S_i = \alpha\mathfrak{S}_D(P, D_i) + \beta\mathfrak{S}_C(P, D_i) + \gamma\mathfrak{S}_D(P^c, D_i^c),$$

with weights $\alpha = 0.4$, $\beta = 0.4$, and $\gamma = 0.2$.

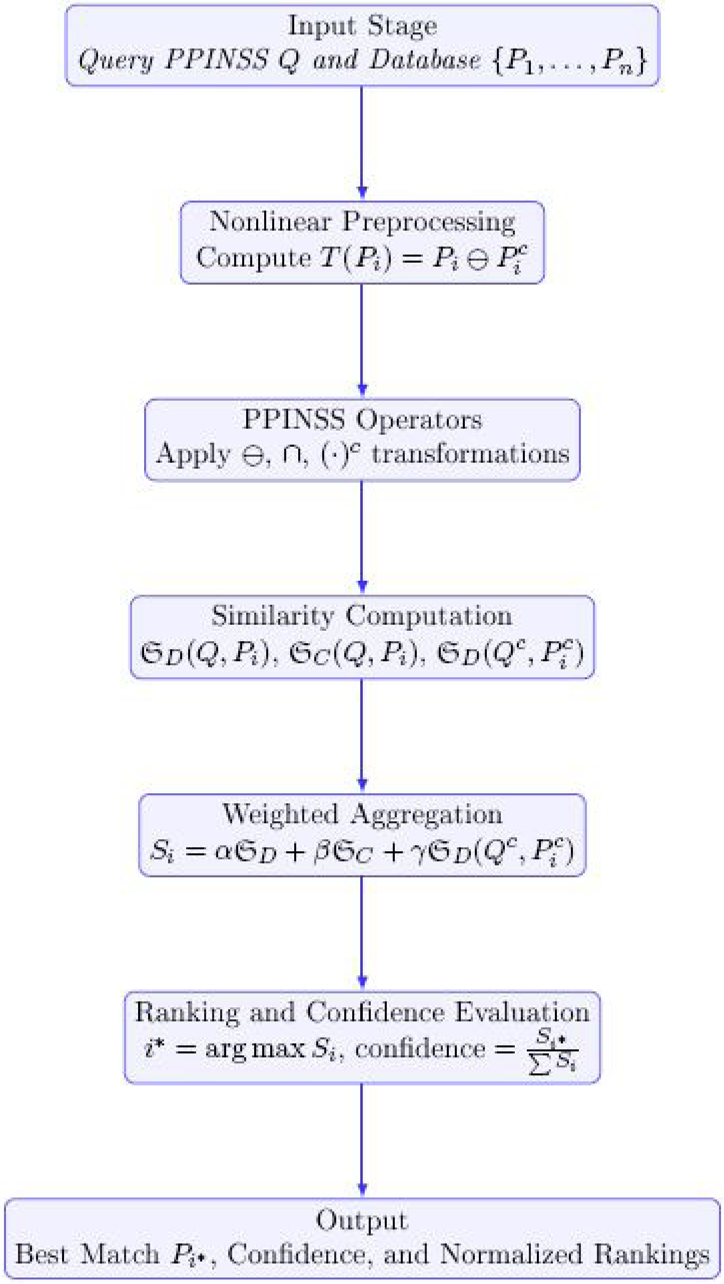

Figure 1: Workow of the PPINSS similarity-based pattern recognition system.

Similarity Computation.

$$\mathfrak{S}_D(P,D_1)=0.85,\quad \mathfrak{S}_C(P,D_1)=0.78,\quad \mathfrak{S}_D(P^c,D_1^c)=0.83,$$
$$\mathfrak{S}_D(P,D_2)=0.92,\quad \mathfrak{S}_C(P,D_2)=0.85,\quad \mathfrak{S}_D(P^c,D_2^c)=0.90,$$
$$\mathfrak{S}_D(P,D_3)=0.45,\quad \mathfrak{S}_C(P,D_3)=0.38,\quad \mathfrak{S}_D(P^c,D_3^c)=0.42.$$

Weighted Aggregation.

$$S_1=0.4(0.85)+0.4(0.78)+0.2(0.83)=0.818,$$
$$S_2=0.4(0.92)+0.4(0.85)+0.2(0.90)=0.888,$$
$$S_3=0.4(0.45)+0.4(0.38)+0.2(0.42)=0.416.$$

Diagnosis Results.

- **Primary Diagnosis:** COVID-19 (D_2), with confidence:

$$\text{Confidence}=\frac{S_2}{\sum_{i=1}^{3}S_i}=\frac{0.888}{2.122}\approx 0.4186\,(41.86\%).$$

- **Secondary Diagnosis:** Influenza (D_1).

$$\frac{S_1}{S_2}=0.9212\,(92.12\%),\qquad \frac{S_1}{\sum_i S_i}=0.3856\,(38.56\%).$$

- **Tertiary Diagnosis:** Common Cold (D_3).

$$\frac{S_3}{S_2}=0.4685\,(46.85\%),\qquad \frac{S_3}{\sum_i S_i}=0.1960\,(19.60\%).$$

Clinical Interpretation. The close similarity between the patient profile and D_2 (COVID-19), followed by D_1 (Influenza), reflects the symptomatic overlap among respiratory infections. However, the model effectively distinguishes these from D_3 (Common Cold), demonstrating its diagnostic sensitivity and robustness. The nonlinear $\ominus$ operator and complement-based similarity enhance the

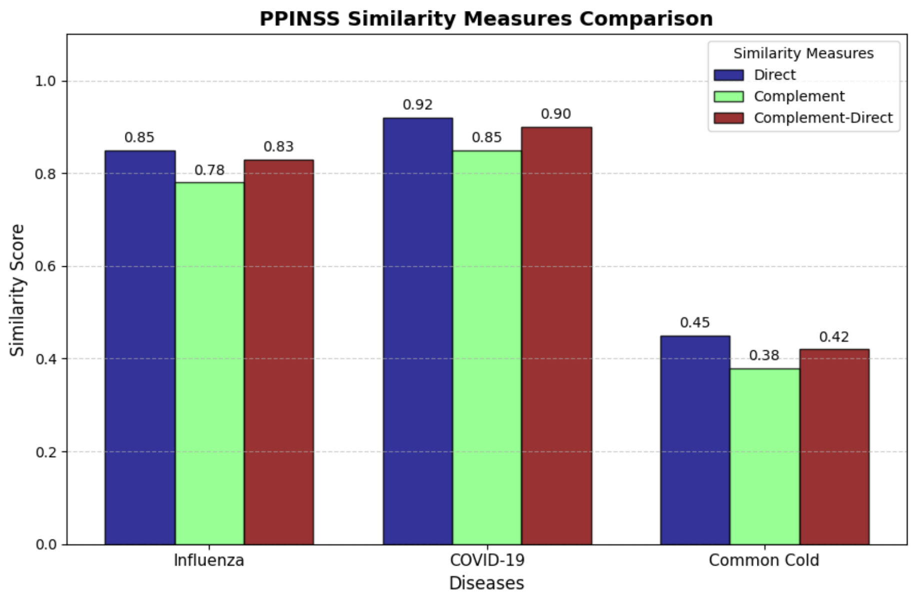

Figure 2: Comparison of PPINSS similarity measures ($\mathfrak{S}_D, \mathfrak{S}_C$, and $\mathfrak{S}_D(Q^c,P_i^c)$) across disease profiles. The weighted similarity $\mathfrak{S}_W$ exhibits enhanced discriminative power and complement invariance.

framework's ability to capture partial contradictions and latent symptom uncertainty, resulting in a more reliable and context-aware diagnosis.

5.3. Graphical Analysis of PPINSS Similarity Measures. To visualize the performance of the proposed similarity model, Fig. 2 presents the comparative similarity scores obtained for the three diseases across the direct, complement, and complement–direct measures $(\mathfrak{S}_D, \mathfrak{S}_C, \mathfrak{S}_D(Q^c, P_i^c))$. The results clearly demonstrate that the weighted similarity measure $\mathfrak{S}_W$ achieves higher diagnostic discrimination, with COVID-19 showing the highest confidence (41.8%), followed by Influenza (38.6%) and Common Cold (19.6%). This confirms the model's capacity to handle uncertainty, contradiction, and contextual ignorance in real-world medical decision systems.

6. Conclusion

In this work, we established the algebraic structure of the PPINSS framework. Through an extension of the traditional neutrosophic as well as quadri-partitioned soft set models, PPINSS provides a five-valued logical structure that includes *truth, indeterminacy, contradiction, falsity* and *contextual ignorance*. At the center of this definition is the intuitionistic interdependence of truth and falsity, articulated by the balance relation $\pi = 1 - \tau - \phi$ that promotes boundedness and semantic consistency within the multi-valued universe. The fifth aspect—contextual ignorance—facilitates direct representation of hidden or delayed knowledge and, therefore, real-world uncertainty that is incomplete, dynamic, or context-dependent. A comprehensive set of algebraic operators was introduced and studied, encompassing the necessity and possibility transformations $(\circledcirc, \circledast)$, the aggregation operators $(\oplus, \ominus)$ and the parametric transformation families $(\Delta_{\mu,u}, \Phi_{\mu,u})$. Each operator was systematically tested for algebraic soundness, meeting properties including commutativity, associativity, idempotency, monotonicityand duality. The findings verify that the PPINSS structure is both logically coherent and algebraically closed, with conventional neutrosophic and quadri-partitioned models presenting themselves as special cases—hence ensuring continuity throughout the neutrosophic hierarchy. Addition of the fifth component is not just an addition to the structure but a *functional requirement* in order to express multi-context, time-varyingand conflict-ridden informational systems. PPINSS thus presents an expressive and computationally feasible basis for managing multi-layered uncertainty and as such is a useful instrument for *multi-criteria decision analysis (MCDA), distributed intelligence, expert evaluation systems*, and *cognitive reasoning environments*. Directions for future work encompass the construction of PPINSS-based aggregation and entropy measures, similarity and distance measure formulationand optimization-based decision algorithm integration. Additionally, practical application of PPINSS to domains like healthcare diagnostics, engineering design optimizationand real-time uncertainty modeling will prove robustness, interpretabilityand computational efficiency. The algebraic structures shown here thus form a strong theoretical foundation for advancing neutrosophic soft set theory to richer, intuitionistically inspiredand application-focused models of uncertainty representation.

Bibliography

[1] S. Ali, A. Ali, A. B. Azim, A. Aloqailyand N. Mlaiki, *Utilizing aggregation operators based on q-rung orthopair neutrosophic soft sets and their applications in multiattribute decision-making problems*, Heliyon, 10(15) (2024), Article No. e35059.

[2] K. T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, 20 (1986), 87–96.

[3] Q. T. Bui, M. P. Ngo, V. Snášel, W. Pedryczand B. Vo, *The sequence of neutrosophic soft sets and a decision-making problem in medical diagnosis*, International Journal of Fuzzy Systems, 24 (2022), 2036–2053.

[4] R. Chatterjee, P. Majumdarand S. K. Samanta, *On some similarity measures and entropy on quadripartitioned single-valued neutrosophic sets, Journal of Intelligent & Fuzzy Systems*, 30 (2016), 2475–2485.

[5] V. Chinnadurai and A. Bobin, *Simplified intuitionistic neutrosophic soft set and its application on diagnosing psychological disorder by using similarity measure*, Applications and Applied Mathematics: An International Journal, 16(1) (2021), Article 34.

[6] V. Chinnadurai and A. Bobin, *Interval-valued intuitionistic neutrosophic soft set and its application on diagnosing psychiatric disorder by using similarity measure*, Neutrosophic Sets and Systems, 41(1) (2021), 215–245.
[7] M. Datta, *Interval-valued pentapartitioned neutrosophic sets*, Neutrosophic Sets and Systems, 60 (2023), 147–158.
[8] M. Dhar, *Some basic concepts of neutrosophic soft block matrices*, Neutrosophic Sets and Systems, 51 (2022), 46–59.
[9] M. Jamil, F. Afzal, A. Maqbooland others, *Multiple attribute group decision-making approach for selection of robot under induced bipolar neutrosophic aggregation operators*, Complex Intelligent Systems, 10 (2024), 2765–2779.
[10] F. Karaaslan, *Possibility neutrosophic soft sets and PNS-decision making approach*, Applied Soft Computing, 54 (2014), 403–414.
[11] E. Karatas, A. Yolcuand T. Y. Ozturk, *Effective neutrosophic soft set theory and its application to decision-making*, African Mathematician (Afr. Mat.), 34 (2023), Article No. 62.
[12] S. R. Kumar and A. V. Senthil Kumar, *Quadri-partitioned neutrosophic soft sets and their applications in topology*, Journal of Intelligent & Fuzzy Systems, 40(2) (2021), 2345–2356.
[13] P. K. Maji, A. R. Royand R. Biswas, *An application of soft sets in a decision making problem*, Computers & Mathematics with Applications, 44(8–9) (2002), 1077–1083.
[14] R. Mallick and S. Pramanik, *Pentapartitioned neutrosophic sets and their properties*, Neutrosophic Sets and Systems, 36 (2020), 184–192.
[15] D. Molodtsov, *Soft set theory — first results*, Computers & Mathematics with Applications, 37(4–5) (1999), 19–31.
[16] S. Pramanik, R. Mallickand M. Biswas, *Interval-valued pentapartitioned neutrosophic sets and decision-making applications*, Soft Computing, 27(7) (2023), 15123–15140.
[17] S. G. Quek, G. Selvachandran, D. Ajay, P. Chellamani, D. Taniar, H. Fujita, P. Duong, L. H. Sonand N. L. Giang, *New concepts of pentapartitioned neutrosophic graphs and applications for determining safest paths and towns in response to COVID-19*, Computational and Applied Mathematics, 41 (2022), Article No. 319.
[18] J. J. M. Rabuni, N. Balamaniand F. Smarandache, *Applications of neutrosophic soft open sets in decision making via operation approach*, Journal of Mathematics and Computer Science, 31(1) (2023), 1–14.
[19] D. Sarkar and S. Ghosh, *Hausdor similarity measure on neutrosophic soft set with its applicability in decision making*, Fuzzy Information and Engineering, 16(3) (2024), 194–206.
[20] F. Smarandache, *A unifying eld in logics: Neutrosophic logic*, American Research Press, 2005.
[21] J. Vimala, S. Rajareega, D. Preethi, M. Al-Tahan, P. Mahalakshmi, K. Jeevithaand K. Ashma Banu, *Lattice ordered neutrosophic soft set and its usage for a large-scale company in adopting best NGO to deliver funds during COVID-19*, Journal of Computational and Cognitive Engineering, 2(3) (2022), 210–219.
[22] L. A. Zadeh, *Fuzzy sets*, Information and Control, 8(3) (1965), 338–353.
[23] R. M. Zulqarnain, A. Iampan, I. Siddique, H. Abdand E.-W. Khalifa, *Some fundamental operations for multi-polar interval-valued neutrosophic soft set and a decision-making approach to solve MCDM problem*, Neutrosophic Sets and Systems, 51 (2022), 205–220.