Results in Nonlinear Analysis

ISSN 2636-7556

Peer Reviewed Scientific Journal

A modified Laplace transform for certain generalized fractional operators

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Abstract

It is known that Laplace transform converges for functions of exponential order. In order to extend the possibility of working in a large class of functions, we present a modified Laplace transform that we call ρ-Laplace transform, study its properties and prove its own convolution theorem. Then, we apply it to solve some ordinary differential equations in the frame of a certain type generalized fractional derivatives. This modified transform acts as a powerful tool in handling the kernels of these generalized fractional operators.

Keywords: Generalized fractional derivatives, Generalized Caputo, ρ-Laplace transform
2010 MSC: 26A33, 35A22, 44A10.

1. Introduction

The fractional calculus and its applications in various fields of science and engineering is considered now an important subfield of mathematics capable to bring new insides the dynamics of non-local complex systems [1, 2, 3, 4, 5]. Fractional differential equations were subjected to an intense debate from both theoretical and numerical viewpoints but still many open problems exists in this area (see [2, 4, 5] and the references therein). Since the fractional differential operators describe better the memory effect caused by the non-locality of these operators [1, 2, 3, 4, 5], there has been an interest in generalizing these operators in order to better understand the impact of the non-locality [6, 7, 8, 9, 10, 11, 12, 13].

One of the main difficulty is to find some appropriate transformations in order to find analytic solutions to some classes of fractional differential equations. In order to construct explicit solutions to differential equations with constant coefficients and in the frame of Riemann-Liouville, Caputo and Riez fractional derivatives, integral transforms including Laplace, Mellin and Fourier were found to be strong tools [2, 4, 5].

Recently, the author in [14, 15] presented the so called generalized fractional integrals and derivatives. Such integrals unify the Riemann-Liouville and the Hadamard fractional integrals, while the generalized fractional derivatives unify the Riemann-Liouville and the Hadamard fractional derivatives.

From the classical fractional calculus, we recall the following[2, 4, 5]

For $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$ the left Riemann-Liouville fractional integral of order α starting from a has the following form

$$(_aI^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-u)^{\alpha-1} f(u) du, \tag{1.1}$$

while the right Riemann-Liouville fractional integral of order $\alpha > 0$ ending at $b > a$ is defined by

$$(I_b^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (u-t)^{\alpha-1} f(u) du. \tag{1.2}$$

For $\alpha \in \mathbb{C}$, $Re(\alpha) \geq 0$, the left Riemann-Liouville fractional derivative of order α starting at a is given below

$$(_aD^\alpha f)(t) = (\frac{d}{dt})^n (_aI^{n-\alpha} f)(t), \quad n = [\alpha] + 1. \tag{1.3}$$

Meanwhile, the right Riemann-Liouville fractional derivative of order α ending at b becomes

$$(D_b^\alpha f)(t) = (-\frac{d}{dt})^n (I_b^{n-\alpha} f)(t). \tag{1.4}$$

The left Caputo fractional of order $\alpha, Re(\alpha) \geq 0$ starting from a has the following form

$$(_aD^\alpha f)(t) = (_aI^{n-\alpha} f^{(n)})(t), \quad n = [\alpha] + 1, \tag{1.5}$$

while the right Caputo fractional derivative ending at b becomes

$$(D_b^\alpha f)(t) = (I_b^{n-\alpha}(-1)^n f^{(n)})(t). \tag{1.6}$$

Hadamard-type fractional integrals and derivatives were introduced in [16] as:

The left Hadamard fractional integral of order $\alpha \in \mathbb{C}, Re(\alpha) > 0$ starting from a has the following form

$$(_a\mathcal{J}^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (\ln t - \ln u)^{\alpha-1} f(u) du, \tag{1.7}$$

and the right Hadamard fractional integral of order α ending at $b > a$ is defined by

$$(\mathcal{J}_b^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\ln u - \ln t)^{\alpha-1} f(u) du. \tag{1.8}$$

The left Hadamard fractional derivative of order $\alpha \in \mathbb{C}, Re(\alpha) \geq 0$ starting at a is given as

$$(_a\mathcal{D}^\alpha f)(t) = (t\frac{d}{dt})^n (_aI^{n-\alpha} f)(t), \quad n = [\alpha] + 1, \tag{1.9}$$

whereas the right Hadamard fractional derivative of order α ending at b becomes

$$(\mathcal{D}_b^\alpha f)(t) = (-t\frac{d}{dt})^n (I_b^{n-\alpha} f)(t). \tag{1.10}$$

In [17, 18, 19], the authors defined the left and right Caputo-Hadamard fractional derivatives of order $\alpha \in \mathbb{C}, Re(\alpha) \geq 0$ respectively as

$$(_a^C\mathcal{D}^\alpha f)(t) = {}_a\mathcal{D}^\alpha[f(u) - \sum_{k=0}^{n-1} \frac{\delta^k f(a)}{k!}(log\frac{u}{a})^k](t), \quad \delta = t\frac{d}{dt}, \tag{1.11}$$

and in the space $AC_\delta^n[a, b] = \{g : [a, b] \to \mathbb{C} : \delta^{n-1}[g(x)] \in AC[a, b]\}$ equivalently by

$$({}_a^C\mathcal{D}^\alpha f)(t) = ({}_a\mathcal{J}^{n-\alpha}(t\frac{d}{dt})^n f)(t), \quad n = [\alpha] + 1. \tag{1.12}$$

and

$$({}^C\mathcal{D}_b^\alpha f)(t) = \mathcal{D}_b^\alpha[f(u) - \sum_{k=0}^{n-1} \frac{(-1)^k \delta^k f(b)}{k!}(\log\frac{b}{u})^k](t), \tag{1.13}$$

and in the space $AC_\delta^n[a, b]$ equivalently by

$$(\mathcal{D}_b^\alpha f)(t) = (\mathcal{J}_b^{n-\alpha}(-t\frac{d}{dt})^n f)(t). \tag{1.14}$$

For $a < b$, $c \in \mathbb{R}$ and $1 \le p < \infty$, define the function space

$$X_c^p(a, b) = \left\{f : [a, b] \to \mathbb{R} : \|f\|_{X_c^p} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t}\right)^{1/p} < \infty\right\}.$$

For $p = \infty$, $\|f\|_{X_c^p} = ess\ sup_{a \le t \le b}[t^c|f(t)|]$. The generalized left and right fractional integrals in the sense of they are defined in [14] have the forms

$$({}_aI^{\alpha,\rho}f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - u^\rho}{\rho}\right)^{\alpha-1} f(u)\frac{du}{u^{1-\rho}}, \tag{1.15}$$

and

$$(I_b^{\alpha,\rho}f)(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\frac{u^\rho - t^\rho}{\rho})^{\alpha-1} f(u)\frac{du}{u^{1-\rho}}, \tag{1.16}$$

respectively. It should mentioned that once $\rho = 1$ the integrals in (1.15) and (1.16) become the Riemann-Liouville fractional integrals (1.1) and (1.2). And in case one takes the limit as $\rho \to 0$ (1.15) and (1.16) become the Hadamard fractional integrals (1.7) and (1.8), respectively.

The left and right generalized fractional derivatives of order $\alpha > 0$ are defined by [15]

$$({}_aD^{\alpha,\rho}f)(t) = \gamma^n({}_aI^{n-\alpha,\rho}f)(t) = \frac{\gamma^n}{\Gamma(n-\alpha)} \int_a^t (\frac{t^\rho - u^\rho}{\rho})^{n-\alpha-1} f(u)\frac{du}{u^{1-\rho}} \tag{1.17}$$

and

$$(D_b^{\alpha,\rho}f)(t) = (-\gamma)^n({}_aI^{n-\alpha,\rho}f)(t) = \frac{(-\gamma)^n}{\Gamma(n-\alpha)} \int_t^b \left(\frac{u^\rho - t^\rho}{\rho}\right)^{n-\alpha-1} f(u)\frac{du}{u^{1-\rho}}, \tag{1.18}$$

respectively, where $\rho > 0$ and where $\gamma = t^{1-\rho}\frac{d}{dt}$. It should noted that once $\rho = 1$ the derivatives in (1.17) and (1.18) become the Riemann-Liouville fractional derivatives (1.3) and (1.4). While, taking the limit of (1.17) and (1.18) as $\rho \to 0$ yields the Hadamard fractional derivatives (1.9) and (1.10) respectively.

In [20], the Caputo modification of the left and right generalized fractional derivatives were presented respectively as

$$({}_a^C D^{\alpha,\rho}f)(t) = ({}_aI^{n-\alpha,\rho}\gamma^n f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\frac{t^\rho - u^\rho}{\rho}\right)^{n-\alpha-1} \gamma^n f(u)\frac{du}{u^{1-\rho}} \tag{1.19}$$

and

$$({}^C D_b^{\alpha,\rho}f)(t) = ({}_aI^{n-\alpha,\rho}(-\gamma)^n f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_t^b \left(\frac{u^\rho - t^\rho}{\rho}\right)^{n-\alpha-1} (-\gamma)^n f(u)\frac{du}{u^{1-\rho}}. \tag{1.20}$$

Putting $\rho = 1$ in the derivatives in (1.19) and (1.20), one obtains become the Caputo fractional derivatives (1.5) and (1.6). Meanwhile when ρ approaches 0 in (1.19) and (1.20) yields the Caputo-Hadamard fractional derivatives (1.11) and (1.12), respectively.

To our best of our knowledge, there has been no trials to use integral transforms in order to find explicit solutions to differential equations in the frame of generalized fractional derivatives. In this line of thought we suggest the ρ-Laplace transform introduced in [21] to solve the fractional partial differential equations.

Our manuscript is organized as follows: In section 2, we present the main theory of the ρ-Laplace transform. In section 3, we give the ρ-Laplace transforms of the generalized fractional integrals and derivatives. In section 4, we solve two linear (generalized) fractional differential equations with constant coefficients using ρ-Laplace transforms. And the last section is devoted to the conclusion.

2. The ρ-Laplace transform

The ρ-Laplace transform was stated in [21] in order to solve differential equations in the frame of conformable derivatives. In this section we set the details for the theory of its basic concepts.

Definition 2.1. (see Definition 5.1 in [21]) Let $f : [0, \infty) \to \mathbb{R}$ be a real valued function. The ρ-Laplace transform of f is defined by

$$\mathcal{L}_\rho\{f(t)\}(s) = \int_0^\infty e^{-s\frac{t^\rho}{\rho}} f(t) \frac{dt}{t^{1-\rho}}, \rho > 0, \tag{2.1}$$

for all values of s, the integral is valid.

The following theorem is part of Example 5.3 in [21].

Theorem 2.2. *Let $f : [0, \infty) \to \mathbb{R}$ be a real valued function such that its ρ-Laplace transform exists. Then*

$$\mathcal{L}_\rho\{f(t)\}(s) = \mathcal{L}\{f((\rho t)^{\frac{1}{\rho}}\}(s), \tag{2.2}$$

where $\mathcal{L}\{f\}$ is the usual Laplace transform of f.

Definition 2.3. A function $f : [0, \infty) \to \mathbb{R}$ is said to be of ρ-exponential order $e^{c\frac{t^\rho}{\rho}}$ if there exist non-negative constants M, c, T such that $|f(t)| \leq M e^{c\frac{t^\rho}{\rho}}$ for $t \geq T$.

Now, we present the conditions for the existence of the ρ-Laplace transform of a function.

Theorem 2.4. *If $f : [0, \infty) \to \mathbb{R}$ is a piecewise continuous function and is of ρ-exponential order, then its ρ-Laplace transform exists for $s > c$.*

The proof of Theorem 2.4 straight forward.

The linearity property is provided in the following theorem.

Theorem 2.5. *If the ρ-Laplace transform of $f : [0, \infty) \to \mathbb{R}$ exists for $s > c_1$ and the ρ-Laplace transform of $g : [0, \infty) \to \mathbb{R}$ exists for $s > c_2$. Then, for any constants a and b, the ρ-Laplace transform of $af + bg$ exists and*

$$\mathcal{L}_\rho\{af(t) + bg(t)\}(s) = a\mathcal{L}_\rho\{f(t)\}(s) + \mathcal{L}_\rho\{g(t)\}(s), \text{ for } s > max\{c_1, c_2\}. \tag{2.3}$$

Some of the ρ-Laplace transforms of elementary functions were given in [21].

Lemma 2.6. *[21]*

1) $\mathcal{L}_\rho\{1\}(s) = \dfrac{1}{s}, \quad s > 0.$

2) $\mathcal{L}_\rho\{t^p\}(s) = \rho^{\frac{p}{\rho}} \dfrac{\Gamma(1 + \frac{p}{\rho})}{s^{1+\frac{p}{\rho}}}, p \in \mathbb{R} \quad s > 0.$

3) $\mathcal{L}_\rho\{e^{\lambda\frac{t^\rho}{\rho}}\}(s) = \dfrac{1}{s - \lambda}, \quad s > \lambda.$

Now we present the ρ-Laplace transforms of the derivatives.

Theorem 2.7. *Let the function $f(t)$ be continuous and of exponential order $e^{c\frac{t^\rho}{\rho}}$ such that $\gamma f(t)$ is piecewise continuous over every finite interval $[0, T]$. Then ρ-Laplace transform of $\gamma f(t)$ exists for $s > c$ and*

$$\mathcal{L}_\rho\{\gamma f(t)\}(s) = s\mathcal{L}_\rho\{f(t)\}(s) - f(0). \tag{2.4}$$

Proof. Let $t_1, t_2, ..., t_n$ be the points in the interval $[0, T]$ where γf is discontinuous. Then we have

$$
\begin{aligned}
\int_0^T e^{-s\frac{t^\rho}{\rho}} (\gamma f)(t)\frac{dt}{t^{1-\rho}} &= \int_0^T e^{-s\frac{t^\rho}{\rho}} f'(t)dt \\
&= \int_0^{t_1} e^{-s\frac{t^\rho}{\rho}} f'(t)dt + \int_{t_1}^{t_2} e^{-s\frac{t^\rho}{\rho}} f'(t)dt \\
&\quad +... + \int_{t_n}^T e^{-s\frac{t^\rho}{\rho}} f'(t)dt.
\end{aligned}
$$

Now integrating by parts gives

$$
\begin{aligned}
\int_0^T e^{-s\frac{t^\rho}{\rho}} (\gamma f)(t)\frac{dt}{t^{1-\rho}} &= e^{-s\frac{t^\rho}{\rho}} f(t)\Big|_0^{t_1} + e^{-s\frac{t^\rho}{\rho}} f(t)\Big|_{t_1}^{t_2} + ... + e^{-s\frac{t^\rho}{\rho}} f(t)\Big|_{t_n}^T \\
&\quad + s\Big[\int_0^{t_1} e^{-s\frac{t^\rho}{\rho}} f(t)\frac{dt}{t^{1-\rho}} + \int_{t_1}^{t_2} e^{-s\frac{t^\rho}{\rho}} f(t)\frac{dt}{t^{1-\rho}} \\
&\quad +... + \int_{t_n}^T e^{-s\frac{t^\rho}{\rho}} f(t)\frac{dt}{t^{1-\rho}} \Big].
\end{aligned}
$$

Thus, we have

$$\int_0^T e^{-s\frac{t^\rho}{\rho}} (\gamma f)(t)\frac{dt}{t^{1-\rho}} = e^{-s\frac{T^\rho}{\rho}} f(T) - f(0) + s \int_0^T e^{-s\frac{t^\rho}{\rho}} f(t)\frac{dt}{t^{1-\rho}}. \tag{2.5}$$

The result is then obtained by taking the limit as $T \to \infty$ of both sides of equation (2.5). $\qquad\square$

The above theorem is the piecewise continuous version of Theorem 5.1 in [21]. In Theorem 5.1 there, $\gamma f(t)$ was expressed by means of the conformable derivative as $\gamma f(t) = T_\rho(f)(t)$ and more generally $\gamma^n f(t) = \underbrace{T_\rho T_\rho ... T_\rho}_{\texttt{n times}} f(t) = {}^{(n)}T_\rho f(t)$ (see eqn. (29) in [21] about the sequential conformable derivative of order n).
Hence, the result in Theorem 2.7 can be generalized as follows.

Corollary 2.8. *Let $f \in C_\gamma^{n-1}[0, \infty)$ such that $\gamma^i f, i = 0, 1, 2..., n-1$ are of ρ-exponential order $e^{c\frac{t^\rho}{\rho}}$. Let $\gamma^n f$ be a piecewise continuous function on the interval $[0, T]$. Then, the ρ-Laplace transform of $\gamma^n f(t)$ exists for $s > c$ and*

$$\mathcal{L}_\rho\{(\gamma^n f)(t)\}(s) = s^n \mathcal{L}_\rho\{f(t)\}(s) - \sum_{k=0}^{n-1} s^{n-k-1}(\gamma^k f)(0). \tag{2.6}$$

Proof. The proof can be done by mathematical induction. $\qquad\square$

In order to find the ρ-Laplace transforms of the generalized fractional integrals and derivatives, we need to define the ρ-convolution integral.

Definition 2.9. Let f and g be two functions which are piecewise continuous at each interval $[0, T]$ and of exponential order. We define the ρ-convolution of f and g by

$$(f *_\rho g)(t) = \int_0^t f\Big((t^\rho - \tau^\rho)^{\frac{1}{\rho}} \Big) g(\tau)\frac{d\tau}{\tau^{1-\rho}}. \tag{2.7}$$

The commutativity of the ρ-convolution of two functions is given in the following lemma.

Lemma 2.10. *Let f and g be two functions which are piecewise continuous at each interval $[0,T]$ and of exponential order. Then*

$$f *_\rho g = g *_\rho f. \tag{2.8}$$

Proof.

$$
\begin{aligned}
(f *_\rho g)(t) &= \int_0^t f\left((t^\rho - \tau^\rho)^{\frac{1}{\rho}}\right) g(\tau) \frac{d\tau}{\tau^{1-\rho}} \\
&= -\int_t^0 f(u) g\left((t^\rho - u^\rho)^{\frac{1}{\rho}}\right) \frac{du}{u^{1-\rho}} \quad \text{after substituting } u^\rho = t^\rho - \tau^\rho \\
&= \int_0^t f(u) g\left((t^\rho - u^\rho)^{\frac{1}{\rho}}\right) \frac{du}{u^{1-\rho}} \\
&= (g *_\rho f)(t).
\end{aligned}
$$

$\square$

Below we present the ρ-Laplace transform of the ρ-convolution integral.

Theorem 2.11. *Let f and g be two functions which are piecewise continuous at each interval $[0,T]$ and of exponential order $e^{c\frac{t^\rho}{\rho}}$. Then*

$$\mathcal{L}_\rho\{f *_\rho g\} = \mathcal{L}_\rho\{f\}\mathcal{L}_\rho\{g\} \quad s > c. \tag{2.9}$$

Proof.

$$
\begin{aligned}
\mathcal{L}_\rho\{f\}\mathcal{L}_\rho\{g\} &= \int_0^\infty e^{-s\frac{t^\rho}{\rho}} f(t) \frac{dt}{t^{1-\rho}} \int_0^\infty e^{-s\frac{u^\rho}{\rho}} g(u) \frac{du}{u^{1-\rho}} \\
&= \int_0^\infty \int_0^\infty e^{-s\frac{t^\rho + u^\rho}{\rho}} f(t) g(u) \frac{dt}{t^{1-\rho}} \frac{du}{u^{1-\rho}} \quad \tau^\rho = t^\rho + u^\rho \\
&= \int_0^\infty \int_u^\infty e^{-s\frac{\tau^\rho}{\rho}} f\left((\tau^\rho - u^\rho)^{\frac{1}{\rho}}\right) g(u) \frac{d\tau}{\tau^{1-\rho}} \frac{du}{u^{1-\rho}}, \text{ changing the order of integration} \\
&= \int_0^\infty e^{-s\frac{\tau^\rho}{\rho}} \left[\int_0^\tau f\left((\tau^\rho - u^\rho)^{\frac{1}{\rho}}\right) g(u) \frac{du}{u^{1-\rho}} \right] \frac{d\tau}{\tau^{1-\rho}} \\
&= \mathcal{L}_\rho\{f *_\rho g\}.
\end{aligned}
$$

$\square$

3. The ρ-Laplace transforms of the generalized fractional integrals and derivatives

In the following theorem, we present the ρ-Laplace transform of the left generalized fractional integral starting at 0.

Theorem 3.1. *Let $\alpha > 0$ and f be a piecewise continuous function on each interval $[0,t]$ and of ρ-exponential order $e^{c\frac{t^\rho}{\rho}}$. Then*

$$\mathcal{L}_\rho\{(\,_0I^{\alpha,\rho}f)(t)\} = s^{-\alpha}\mathcal{L}_\rho\{f(t)\}, \quad s > c. \tag{3.1}$$

Proof.

$$
\begin{aligned}
\mathcal{L}_\rho\{(\,_0I^{\alpha,\rho}f)(t)\} &= \int_0^\infty e^{-s\frac{t^\rho}{\rho}}\Big[\frac{1}{\Gamma(\alpha)}\int_0^t \Big(\frac{t^\rho-\tau^\rho}{\rho}\Big)^{\alpha-1}f(\tau)\frac{d\tau}{\tau^{1-\rho}}\Big]\frac{dt}{t^{1-\rho}} \\
&= \int_0^\infty e^{-s\frac{t^\rho}{\rho}}\Big[\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t \Big((t^\rho-\tau^\rho)^{\frac{1}{\rho}}\Big)^{\rho(\alpha-1)}f(\tau)\frac{d\tau}{\tau^{1-\rho}}\Big]\frac{dt}{t^{1-\rho}} \\
&= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^\infty e^{-s\frac{t^\rho}{\rho}}\Big(t^{\rho(\alpha-1)}*_\rho f(t)\Big)\frac{dt}{t^{1-\rho}} \\
&= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\mathcal{L}_\rho\{t^{\rho(\alpha-1)}*_\rho f(t)\} \\
&= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\mathcal{L}_\rho\{t^{\rho(\alpha-1)}\}\mathcal{L}_\rho\{f(t)\} \\
&= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\rho^{\frac{\rho(\alpha-1)}{\rho}}\frac{\Gamma(1+\frac{\rho(\alpha-1)}{\rho})}{s^{1+\frac{\rho(\alpha-1)}{\rho}}}\mathcal{L}_\rho\{f(t)\} \\
&= s^{-\alpha}\mathcal{L}_\rho\{f(t)\}.
\end{aligned}
$$

□

Now we can present the ρ-Laplace transform of the left generalized fractional derivative starting at 0.

Corollary 3.2. *Let $\alpha > 0$ and $f \in AC_\gamma^n[0,a]$ for any $a > 0$ and $\,_aI^{n-k-\alpha,\rho}f, k = 0,1,...,n-1$ be of ρ-exponential order $e^{c\frac{t^\rho}{\rho}}$. Then*

$$
\mathcal{L}_\rho\{(\,_0D^{\alpha,\rho}f)(t)\}(s) = s^\alpha\mathcal{L}_\rho\{f(t)\} - \sum_{k=0}^{n-1}s^{n-k-1}\Big(\,_aI^{n-k-\alpha}f\Big)(0), \quad s > c. \tag{3.2}
$$

Proof.

$$
\begin{aligned}
\mathcal{L}_\rho\{(\,_0D^{\alpha,\rho}f)(t)\}(s) &= \mathcal{L}_\rho\{\gamma^n(\,_0I^{n-\alpha,\rho}f)(t)\} \quad \text{using equation (17)} \\
&= s^n\mathcal{L}_\rho\{\,_0I^{n-\alpha,\rho}f)(t)\} - \sum_{k=0}^{n-1}s^{n-k-1}\Big(\gamma^k\,_0I^{n-\alpha,\rho}f\Big)(0) \text{ by Corollary 2.8} \\
&= s^n\frac{\mathcal{L}_\rho\{f(t)\}}{s^{n-\alpha}} - \sum_{k=0}^{n-1}s^{n-k-1}\Big(\gamma^k\,_0I^{n-\alpha,\rho}f\Big)(0) \text{ by Theorem 3.1} \\
&= s^\alpha\mathcal{L}_\rho\{f(t)\} - \sum_{k=0}^{n-1}s^{n-k-1}\Big(\gamma^k\,_0I^{n-\alpha,\rho}f\Big)(0).
\end{aligned}
$$

The result then comes out after using Theorem 2.5 in [20].   □

In the following corollary we present a formula for the ρ-Laplace transform of the left generalized Caputo fractional derivative starting at 0.

Corollary 3.3. *Let $\alpha > 0$ and $f \in AC_\gamma^n[0,a]$ for any $a > 0$ and $\gamma^k f, k = 0,1,...,n$ be of ρ-exponential order $e^{c\frac{t^\rho}{\rho}}$. Then*

$$
\mathcal{L}_\rho\{(\,_0^CD^{\alpha,\rho}f)(t)\}(s) = s^\alpha\Big[\mathcal{L}_\rho\{f(t)\} - \sum_{k=0}^{n-1}s^{-k-1}\Big(\gamma^k f\Big)(0)\Big], \quad s > c. \tag{3.3}
$$

Proof.

$$\begin{aligned}
\mathcal{L}_\rho\{(\, _0^C D^{\alpha,\rho} f)(t)\}(s) &= \mathcal{L}_\rho\{(\, _0 I^{n-\alpha,\rho} \gamma^n f)(t)\} \text{ using equation (19)} \\
&= s^{\alpha-n} \mathcal{L}_\rho\{\gamma^n f)(t)\} \text{ by Theorem 3.1} \\
&= s^{\alpha-n}\Big[s^n \mathcal{L}_\rho\{f(t)\}(s) - \sum_{k=0}^{n-1} s^{n-k-1}(\gamma^k f)(0)\Big] \text{by Corollary 2.8} \\
&= s^{\alpha}\Big[\mathcal{L}_\rho\{f(t)\} - \sum_{k=0}^{n-1} s^{-k-1}\Big(\gamma^k f\Big)(0)\Big].
\end{aligned}$$

$\square$

The Mittag-Leffler functions play an important role in the theory of the fractional calculus [2, 4, 5]. Since we expect the solutions of the Cauchy problems in the frame generalized fractional derivatives, we have to set the relation between these functions and ρ-Laplace transform. The Mittag-Leffler function is given by [2, 4, 5]

$$E_\alpha(z) = \sum_{k=o}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)}, \qquad z \in \mathbb{C}, \operatorname{Re}(\alpha) > 0. \tag{3.4}$$

A more generalized Mittag-Leffler function is given by [2, 4, 5]

$$E_{\alpha,\beta}(z) = \sum_{k=o}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \qquad z \in \mathbb{C}, \operatorname{Re}(\alpha) > 0. \tag{3.5}$$

It can be observed clearly from equations (3.4) and (3.5) that

$$E_{\alpha,1}(z) = E_\alpha(z). \tag{3.6}$$

In the lemma below we present the ρ-Laplace transforms of some Mittag-Leffler functions.

Lemma 3.4. *Let $Re(\alpha) > 0$ and $\left|\frac{\lambda}{s^\alpha}\right| < 1$. Then*

$$\mathcal{L}_\rho\Big\{E_\alpha\Big(\lambda\Big(\frac{t^\rho}{\rho}\Big)^\alpha\Big)\Big\} = \frac{s^{\alpha-1}}{s^\alpha - \lambda}, \tag{3.7}$$

and

$$\mathcal{L}_\rho\Big\{\Big(\frac{t^\rho}{\rho}\Big)^{\alpha-1} E_{\alpha,\alpha}\Big(\lambda\Big(\frac{t^\rho}{\rho}\Big)^\alpha\Big)\Big\} = \frac{1}{s^\alpha - \lambda}. \tag{3.8}$$

Proof.

$$\begin{aligned}
\mathcal{L}_\rho\Big\{E_\alpha\Big(\lambda\Big(\frac{t^\rho}{\rho}\Big)^\alpha\Big)\Big\} &= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(k\alpha + 1)\rho^{k\alpha}} \mathcal{L}_\rho\{t^{k\rho\alpha}\} \text{ from the definition (3.4)} \\
&= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(k\alpha + 1)\rho^{k\alpha}} \rho^{k\alpha} \frac{\Gamma(k\alpha + 1)}{s^{1+k\alpha}} \text{ by Lemma 2.6} \\
&= \frac{1}{s} \sum_{k=0}^{\infty} \Big(\frac{\lambda}{s^\alpha}\Big)^k \\
&= \frac{s^{\alpha-1}}{s^\alpha - \lambda}.
\end{aligned}$$

This was the proof of (3.7). The proof of (3.8) is as follows

$$
\begin{aligned}
\mathcal{L}_\rho\Big\{\Big(\frac{t^\rho}{\rho}\Big)^{\alpha-1}E_{\alpha,\alpha}\Big(\lambda\Big(\frac{t^\rho}{\rho}\Big)^\alpha\Big)\Big\} &= \sum_{k=0}^\infty \frac{\lambda^k}{\Gamma(k\alpha+\alpha)\rho^{k\alpha+\alpha-1}}\mathcal{L}_\rho\{t^{k\rho\alpha+\rho\alpha-\rho}\} \\
&= \sum_{k=0}^\infty \frac{\lambda^k}{\Gamma(k\alpha+\alpha)\rho^{k\alpha+\alpha-1}}\rho^{k\alpha+\alpha-1}\frac{\Gamma(k\alpha+\alpha)}{s^{\alpha+k\alpha}} \\
&= \frac{1}{s^\alpha}\sum_{k=0}^\infty \Big(\frac{\lambda}{s^\alpha}\Big)^k \\
&= \frac{1}{s^\alpha-\lambda}.
\end{aligned}
$$

$\square$

4. Solution of some generalized fractional differential equations by ρ-Laplace transforms

In this section we are consider the following two linear differential equations in the frame of the generalized fractional derivatives and the generalized Caputo fractional derivatives.

Theorem 4.1. *The Cauchy problem*

$$
\begin{aligned}
{}_0D^{\alpha,\rho}y(t) - \lambda y(t) &= f(t), \quad t>0, \ 0<\alpha\le 1, \ \lambda\in\mathbb{R}, \\
({}_0I^{1-\alpha,\rho}y)(0) &= b, \quad b\in\mathbb{R},
\end{aligned}
\tag{4.1}
$$

has the solution

$$
y(t) = b\Big(\frac{t^\rho}{\rho}\Big)^{\alpha-1}E_{\alpha,\alpha}\Big(\lambda\Big(\frac{t^\rho}{\rho}\Big)^\alpha\Big) + \int_0^t \Big(\frac{t^\rho-\tau^\rho}{\rho}\Big)^{\alpha-1}E_{\alpha,\alpha}\Big(\lambda\Big(\frac{t^\rho-\tau^\rho}{\rho}\Big)^\alpha\Big)f(\tau)\frac{d\tau}{\tau^{1-\rho}}.
\tag{4.2}
$$

Proof. Applying the ρ-Laplace transform to both sides of the equation (4.1) and then using Corollary 3.2 with $n=1$, one gets

$$
\begin{aligned}
\mathcal{L}_\rho\{y(t)\} &= b\,\frac{1}{s^\alpha-\lambda} + \frac{1}{s^\alpha-\lambda}\mathcal{L}_\rho\{f(t)\} \\
&= b\mathcal{L}_\rho\Big\{\Big(\frac{t^\rho}{\rho}\Big)^{\alpha-1}E_{\alpha,\alpha}\Big(\lambda\Big(\frac{t^\rho}{\rho}\Big)^\alpha\Big)\Big\} + \mathcal{L}_\rho\Big\{\Big(\frac{t^\rho}{\rho}\Big)^{\alpha-1}E_{\alpha,\alpha}\Big(\lambda\Big(\frac{t^\rho}{\rho}\Big)^\alpha\Big)\Big\}\mathcal{L}_\rho\{f(t)\} \\
&= \mathcal{L}_\rho\Big\{b\Big(\frac{t^\rho}{\rho}\Big)^{\alpha-1}E_{\alpha,\alpha}\Big(\lambda\Big(\frac{t^\rho}{\rho}\Big)^\alpha\Big) + \Big(\frac{t^\rho}{\rho}\Big)^{\alpha-1}E_{\alpha,\alpha}\Big(\lambda\Big(\frac{t^\rho}{\rho}\Big)^\alpha\Big) *_\rho f(t)\Big\}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
y(t) &= b\Big(\frac{t^\rho}{\rho}\Big)^{\alpha-1}E_{\alpha,\alpha}\Big(\lambda\Big(\frac{t^\rho}{\rho}\Big)^\alpha\Big) + \Big(\frac{t^\rho}{\rho}\Big)^{\alpha-1}E_{\alpha,\alpha}\Big(\lambda\Big(\frac{t^\rho}{\rho}\Big)^\alpha\Big) *_\rho f(t) \\
&= b\Big(\frac{t^\rho}{\rho}\Big)^{\alpha-1}E_{\alpha,\alpha}\Big(\lambda\Big(\frac{t^\rho}{\rho}\Big)^\alpha\Big) + \int_0^t \Big(\frac{t^\rho-\tau^\rho}{\rho}\Big)^{\alpha-1}E_{\alpha,\alpha}\Big(\lambda\Big(\frac{t^\rho-\tau^\rho}{\rho}\Big)^\alpha\Big)f(\tau)\frac{d\tau}{\tau^{1-\rho}}.
\end{aligned}
$$

$\square$

Next, we consider a differential equation in the frame of generalized Caputo fractional derivatives.

Theorem 4.2. *The Cauchy problem*

$$
\begin{aligned}
{}_0^C D^{\alpha,\rho}y(t) - \lambda y(t) &= f(t), \quad t>0, \ 0<\alpha\le 1, \ \lambda\in\mathbb{R}, \\
y(0) &= b, \quad b\in\mathbb{R},
\end{aligned}
\tag{4.3}
$$

has the solution

$$
y(t) = bE_\alpha\Big(\lambda\Big(\frac{t^\rho}{\rho}\Big)^\alpha\Big) + \int_0^t \Big(\frac{t^\rho-\tau^\rho}{\rho}\Big)^{\alpha-1}E_{\alpha,\alpha}\Big(\lambda\Big(\frac{t^\rho-\tau^\rho}{\rho}\Big)^\alpha\Big)f(\tau)\frac{d\tau}{\tau^{1-\rho}}.
\tag{4.4}
$$

Proof. Applying the ρ-Laplace transform to both sides of the equation (4.3) and then using Corollary 3.3 with $n = 1$, one obtains

$$
\begin{aligned}
\mathcal{L}_\rho\{y(t)\} &= b\,\frac{s^{\alpha-1}}{s^\alpha - \lambda} + \frac{1}{s^\alpha - \lambda}\mathcal{L}_\rho\{f(t)\} \\
&= b\mathcal{L}_\rho\Big\{E_\alpha\Big(\lambda\Big(\frac{t^\rho}{\rho}\Big)^\alpha\Big)\Big\} + \mathcal{L}_\rho\Big\{\Big(\frac{t^\rho}{\rho}\Big)^{\alpha-1}E_{\alpha,\alpha}\Big(\lambda\Big(\frac{t^\rho}{\rho}\Big)^\alpha\Big)\Big\}\mathcal{L}_\rho\{f(t)\} \\
&= \mathcal{L}_\rho\Big\{bE_\alpha\Big(\lambda\Big(\frac{t^\rho}{\rho}\Big)^\alpha\Big) + \Big(\frac{t^\rho}{\rho}\Big)^{\alpha-1}E_{\alpha,\alpha}\Big(\lambda\Big(\frac{t^\rho}{\rho}\Big)^\alpha\Big) *_\rho f(t)\Big\}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
y(t) &= bE_\alpha\Big(\lambda\Big(\frac{t^\rho}{\rho}\Big)^\alpha\Big) + \Big(\frac{t^\rho}{\rho}\Big)^{\alpha-1}E_{\alpha,\alpha}\Big(\lambda\Big(\frac{t^\rho}{\rho}\Big)^\alpha\Big) *_\rho f(t) \\
&= bE_\alpha\Big(\lambda\Big(\frac{t^\rho}{\rho}\Big)^\alpha\Big) + \int_0^t \Big(\frac{t^\rho - \tau^\rho}{\rho}\Big)^{\alpha-1}E_{\alpha,\alpha}\Big(\lambda\Big(\frac{t^\rho - \tau^\rho}{\rho}\Big)^\alpha\Big)f(\tau)\frac{d\tau}{\tau^{1-\rho}}.
\end{aligned}
$$

$\square$

5. Conclusions

1. The generalized fractional derivatives starting at $a = 0$ are the fractional versions of the conformable derivative starting at 0 (see the open problem in item 6 of Section 6 in [21]). As the usual Laplace transform (1-Laplace) is a tool to solve classical fractional Riemann-Liouville and Caputo derivatives, we employed the $\rho-$Laplace transform (or the conformable Laplace transform) to solve dynamical systems in the frame of Riemann-Liouville and Caputo type fractional generalized operators introduced by Katugampola in [14, 15] and studied by Jarad etal in [20] and Almeida etal. in [22]. This confirms that conformable derivatives are local derivatives of arbitrary order that can be used to produce more general types of fractional derivatives with memory effect (see the open problem in [21]). Therefore, it is always of interest to introduce new local derivatives of arbitrary order and use them by a fractionalization process (either by iterating the local derivatives or their correspondent integrals) to produce new types of fractional derivatives of different kernels.

2. We studied the basic theory for the $\rho-$Laplace transform and proved its convolution theorem to find the $\rho-$Laplace transform of the generalized fractional integrals and derivatives through the evaluation of the $\rho-$Laplace for certain $\rho-$weighted Mittag-Leffler functions (see Lemma 3.4). Then, finally we used it in Section 5 to solve nonhomogeneous linear fractional dynamic equations in the frame of generalized fractional Riemann and Caputo type operators (see Theorem 4.1 and Theorem 4.2). We have shown that the solution representations are expressible by means of the $\rho-$weighted Mittag-Leffler functions.

3. The limiting case $\rho \to 1$ converts into classical Riemann-Liouville and Caputo fractional operators and their linear fractional dynamical systems.

4. The $\rho-$Laplace transform which we have used as an effective tool to solve dynamical systems depending on generalized fractional operators, whose kernel is singular, open the door for the possibility of developing new integral transforms that fit the kernels of some recently introduced and studied fractional operators with nonsingular kernels [6]-[13].

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