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Mathematical inequalities for optimization and decision-making in engineering and physical sciences

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Abstract

The primary objective of this article is to provide an accessible exposition of the dynamic and evolving field of mathematical inequalities, with a particular emphasis on their role in optimization and decision-making within engineering and the physical sciences. We begin with an elementary introduction to fundamental inequalities, followed by a range of illustrative examples that span from classical applications to contemporary research challenges. To demonstrate the breadth and utility of inequalities, we explore examples from diverse areas of mathematics, including the ubiquitous triangle inequality, which arises in contexts ranging from Euclidean geometry to matrix norms. We highlight key results, such as the interlacing of roots of orthogonal polynomials, which are elegantly formulated through inequality frameworks. In number theory, we present select theorems and conjectures—particularly from the active domain of prime gaps—that are naturally expressed using inequalities. The article also examines inequalities in physics, such as the Clausius inequality in thermodynamics, constraints on electron localization in atomic structures, and bounds related to the cardinality of resistor networks. Overall, this paper aims to introduce readers to the techniques and significance of mathematical inequalities, while showcasing their applications in optimization, theoretical analysis, and practical decision-making across multiple scientific and engineering domains.

Mathematics Subject Classification (2020): 26D05, 26D10

Key words and Phrases: Inequalities, Cauchy–Bunyakovsky–Schwarz inequality, Bernoulli's inequality, Integral inequalities, Optimization, Number Theory and Prime Gaps, Bhatia–Davis inequality, Mathematical Inequalities, Inequalities from Physics.

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1. Introduction

Not only the graduate level but even the advanced level mathematics and other science courses invariably cover inequalities and their solutions. The exposure is from elementary to advanced. The coverage to this important topic can be enriched with further examples and applications [1–9]. We shall look at some examples and applications from mathematics and other sciences. Many important theorems and even conjectures are stated in the form of inequalities. In order to keep the article self-contained, definitions and brief descriptions are provided where required.

Any quantity is said to be greater than another quantity b when $a - b$ is positive. This statement is written as $a > b$. Likewise, we have the statement $a < b$ when $a - b$ is negative. These relations are known as strict inequalities. The other two possibilities are $a \geq b$ and $a \leq b$. Essentially, inequalities are a statement of an *order relationship* (*greater than*, *greater than or equal to*, *less than*, or *less than or equal to*) between two numbers or algebraic expressions.

Euclidean geometry provides one of the oldest inequalities, which is known as the *triangle inequality theorem*. It states that the *sum of any two sides of a triangle is greater than or equal to the third side*. Symbolically, $a + b \geq c$. The part, *or equal to* is often excluded as it leads to the degenerate triangles. Essentially in the Euclidean geometry, the theorem states that the shortest distance between two points is a straight line. The triangle inequality appears in other forms depending on the topic. For any two real numbers, the absolute values satisfy $|a| + |b| \geq |a + b|$. For vector lengths (norm), we have $\|A\| + \|B\| \geq \|A + B\|$. For any two complex numbers, the moduli satisfy $|z_1| + |z_2| \geq |z_1 + z_2|$.

Many inequalities can be derived from the fact that the square of any real number is greater than or equal to zero. Symbolically, $R^2 \geq 0$ and the equality occurs only when $R = 0$. As an example

$$\begin{aligned}(a-b)^2 = a^2 - 2ab + b^2 \geq 0 \\ a^2 + b^2 \geq 2ab.\end{aligned} \tag{1.1}$$

If we substitute $a = x$ and $b = \frac{1}{x}$, then

$$x + \frac{1}{x} \geq 2. \tag{1.2}$$

This inequality can also be derived using the minima/maxima technique via derivatives from calculus [10–12].

Let a_1, a_2, a_3, ... a_n be n positive real numbers. The *arithmetic mean* (A_n), the *geometric mean* (G_n) and the *harmonic mean* (H_n) are defined by

$$A_n = \frac{a_1 + a_2 + a_3 + \cdots + a_n}{n}, \tag{1.3}$$

$$G_n = \sqrt[n]{a_1 a_2 a_3 \cdots a_n}, \tag{1.4}$$

$$H_n = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n}}. \tag{1.5}$$

The inequalities between the three means are [13]

$$A_n \geq G_n \geq H_n. \tag{1.6}$$

and equality occurs iff $a_1 = a_2 = a_3 = \cdots = a_n$.

Cauchy–Bunyakovsky–Schwarz Inequality:

If $(a_1, a_2, a_3, \ldots, a_n)$ and $(b_1, b_2, b_3, \ldots, b_n)$ are two sequences of real numbers, then

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geq \left(\sum_{i=1}^{n} a_i b_i\right). \tag{1.7}$$

The equality occurs if the sequences $(a_1, a_2, a_3, \dots, a_n)$ and $(b_1, b_2, b_3, \dots, b_n)$ are proportional, i.e., there is a constant λ such that $a_k = \lambda b_k$ for each $k \in \{1, 2, 3, \dots, n\}$. This inequality occurs in many areas of mathematics and sciences.

Exponential Inequality:

The exponential and power inequalities include

$$e^x \geq 1 + x, \text{for any real } x, \tag{1.8}$$

$$\frac{x^p - 1}{p} \geq \ln x \geq \frac{x^p - 1}{p\, x^p}, \text{ if } x > 0 \text{ and } p > 0. \tag{1.9}$$

Bernoulli's Inequality:

The Bernoulli inequality states

$$\begin{aligned} (1+x)^r &\geq 1 + rx, \text{ for } r \geq 1 \text{ and } x \geq -1, \\ (1+x)^r &\leq 1 + rx, \text{ for } 0 \leq r \leq 1 \text{ and } x \geq -1. \end{aligned} \tag{1.10}$$

A Solved Example:

As an application, let us consider the following example [14]. Find the integer part of the series for the first one million terms,

$$S(n) = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}}.$$

Solution: Let us first obtain inequalities for $\frac{1}{\sqrt{n}}$.

$$\begin{aligned} 2\sqrt{n+1} - 2\sqrt{n} &= \frac{2\left(\sqrt{n+1} - \sqrt{n}\right)\left(\sqrt{n+1} + \sqrt{n}\right)}{\left(\sqrt{n+1} + \sqrt{n}\right)} \\ &= \frac{2}{\left(\sqrt{n+1} + \sqrt{n}\right)} < \frac{1}{\sqrt{n}}. \end{aligned} \tag{1.11}$$

Note, that $\sqrt{n+1} > \sqrt{n}$. The upper bound is obtained by starting with $2\sqrt{n} - 2\sqrt{n-1}$. Combining the two bounds, we obtain

$$2\sqrt{n+1} - 2\sqrt{n} < \frac{1}{\sqrt{n}} < 2\sqrt{n} - 2\sqrt{n-1}. \tag{1.12}$$

To get the bounds of $S(n)$, we add the inequalities

$$\begin{aligned} 2\sqrt{3} - 2\sqrt{2} &< \frac{1}{\sqrt{2}} < 2\sqrt{2} - 2\sqrt{1} \\ 2\sqrt{4} - 2\sqrt{3} &< \frac{1}{\sqrt{3}} < 2\sqrt{3} - 2\sqrt{2} \\ &\vdots \\ 2\sqrt{n+1} - 2\sqrt{n} &< \frac{1}{\sqrt{n}} < 2\sqrt{n} - 2\sqrt{n-1}. \end{aligned} \tag{1.13}$$

The first and last part of (1.13) are telescoping and we obtain

$$2\sqrt{n+1}-2\sqrt{2}<S(n)-1<2\sqrt{n}-2,$$
$$2\sqrt{n+1}-2\sqrt{2}+1<S(n)<2\sqrt{n}-1. \tag{1.14}$$

The difference of the upper bound and lower bound is $2\left(\sqrt{2}-1\right)-2\left(\sqrt{n+1}-\sqrt{n}\right)$, which is less than 1 for all n. When $n = 10^6$, $2\sqrt{10^6+1}-2\sqrt{2}+1=1998.17$ and $2\sqrt{10^6}-1=1999$. With $2\sqrt{10^6}-1=1999$, the integer part of $S(10^6)$, written symbolically, we have $\lfloor S(10^6)\rfloor=1998$. If we had started with $n = 1$ in (1.13), we would have used $1 < 2$ and obtained $S(n)<2\sqrt{n}$ and not the better bound of $S(n)<2\sqrt{n}-1$. Hence, utmost care is to be exercised while using inequalities in order to get better bounds.

2. Inequalities from Trigonometry

We note some inequalities from trigonometry [15–23]. For an angle, θ (measured in radians) in the first quadrant, we have

$$\frac{2}{\pi}\theta\le\sin\theta\le\theta, \qquad \text{(Jordan's Inequality)} \tag{2.1}$$

$$1-\frac{2}{\pi}\theta\le\cos\theta, \qquad \text{(Kober's Inequality)} \tag{2.2}$$

$$1-\cos\theta\le\theta, \tag{2.3}$$

$$\sin\theta<\theta<\tan\theta, \tag{2.4}$$

$$\cos\theta<\frac{\sin\theta}{\theta}<1. \tag{2.5}$$

Aristarchus's Inequality:
If α and β are two angles in the first quadrant and $\alpha>\beta$ then

$$\frac{\sin\alpha}{\sin\beta}<\frac{\alpha}{\beta}<\frac{\tan\alpha}{\tan\beta}. \tag{2.6}$$

3. Inequalities from Calculus and algebra to optimization

We now list important inequalities from calculus and inequalities that bridge algebra with optimization theory. These are foundational in linear and nonlinear programming, especially in the context of convex of convex analysis, and optimality conditions. There are a number of inequalities for Riemann-integrable functions [10–12]. We list some of them below:

1. A very basic inequality for integrals is given below

$$\int_a^b f^2(x)\,dx\ge 0. \tag{3.1}$$

2. If $f(x)$ is an integrable function on $[a, b]$ and $m\le f(x)\le M$ for all x in $[a, b]$, then

$$m(b-a)\le\int_a^b f(x)\,dx\le M(b-a). \tag{3.2}$$

3. If $f(x)\le g(x)$ for each x in $[a, b]$, then

$$\int_a^b f(x)\,dx\le\int_a^b g(x)\,dx. \tag{3.3}$$

4. The case of absolute value

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx. \tag{3.4}$$

5. The Cauchy–Schwarz inequality for integrals

$$\left(\int_a^b f(x)\,g(x)\,dx\right)^2 \le \left(\int_a^b (f(x))^2\,dx\right)\left(\int_a^b (g(x))^2\,dx\right). \tag{3.5}$$

6. If

$$\frac{1}{p}+\frac{1}{q}=1$$

with $p,q > 1$. Then the Hölder inequality for integrals states

$$\int_a^b |f(x)\,g(x)|\,dx \le \left(\int_a^b |f(x)|^p\,dx\right)^{1/p}\left(\int_a^b |g(x))|^q\,dx\right)^{1/q}. \tag{3.6}$$

For the real numbers $x, y \in R$, and $x, y \ge 0, p$ the following hold:

$$x + y \ge 2\sqrt{xy}. \quad (AM - GM\ Inequality) \tag{3.7}$$

Equality holds if and only if $x = y$. This is used to establish convexity and lower bounds.

$$|x + y| \le |x||y|. \quad (Triangle\ Inequality) \tag{3.8}$$

This is used in normal spaces and in convergence analysis of optimization algorithms.

$$(x + y)^2 \le 2(x^2 + y^2). \quad (Cauchy's\ Inequality) \tag{3.9}$$

Helps in bounding expressions involving squares of variables.

$$\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right) \ge \left(\sum_{i=1}^n a_i b_i\right)^2 \quad (Cauchy - Schwarz's\ Inequality) \tag{3.10}$$

Used in proving dual feasibility and bounding inner products in vector spaces.

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y),\ 0 < \lambda < 1. \quad (Cauchy\ Function\ Inequality) \tag{3.11}$$

$$\left(\sum_{i=1}^n |a_i b_i|\right)\left(\sum_{i=1}^n |a_i|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^n |b_i|^q\right)^{\frac{1}{q}} \quad (Cauchy - Schwarz's\ Inequality) \tag{3.12}$$

A generalization of Cauchy-Schwarz used in norm bounds.

4. Orthogonal Polynomials

Any sequence of polynomials $\{P_n(x)\}$ in which the degree of $P_n(x)$ is n for all n is said to be orthogonal with respect to the measure $d\alpha(x)$ if

$$\begin{aligned}&\int_a^b P_n(x)P_m(x)\,d\alpha(x) = 0, \quad n \ne m,\\ &\int_a^b P_n(x)P_n(x)\,d\alpha(x) \ne 0, \quad \forall n \ge 0.\end{aligned} \tag{4.1}$$

The limits of integration can be finite (as in the case of Laguerre polynomials and Jacobi polynomials) or infinite (as in the case of Hermite polynomials). When this measure is absolutely continuous, i.e., $d\alpha(x) = w(x)dx$, where $w(x)$ is called the *weight function*, then the relations in (4.1) become

$$\int_a^b P_n(x)P_m(x)w(x)\,dx = 0, \quad n \neq m,$$
$$\int_a^b P_n(x)P_n(x)w(x)\,dx \neq 0, \quad \forall n \geq 0. \tag{4.2}$$

The *classical* orthogonal polynomials are Hermite polynomials, Laguerre polynomials and Jacobi polynomials [24–26]. The orthogonal polynomials occur across mathematics and sciences. They have many properties and we note the following two. The roots of the orthogonal polynomials are all real, distinct and lie in the interval of orthogonality. This property acquires an extra significance as no formula is known for the roots of any of the orthogonal polynomials with the exception of the Chebyshev polynomials. The Chebyshev polynomials are one of the special cases of the Jacobi polynomials. Chebyshev polynomials of the *first kind* occur in the expansion of $\cos(n\theta) = T_n(\cos\theta)$. Chebyshev polynomials of the *second kind* occur in the expansion of $\sin(n\theta) = \sin\theta U_{n-1}(\cos\theta)$.

Chebyshev polynomials of both the kinds are unique as each of the degree n polynomials have n different simple roots, (known as Chebyshev nodes or Chebyshev roots), in the interval of orthogonality [–1, 1]. The n Chebyshev roots of $T_n(x)$ are

$$x_k = \cos\left(\frac{\left(k+\frac{1}{2}\right)}{n}\pi\right), \quad k = 0,\ldots,n-1. \tag{4.3}$$

The n Chebyshev roots of $U_n(x)$ are

$$x_k = \cos\left(\frac{k}{n+1}\pi\right), \quad k = 1,\ldots,n. \tag{4.4}$$

As no such formulae are known for any of the other orthogonal polynomials, the topic is of active research interest. The second property is the interlacing of zeros (roots) of the orthogonal polynomials. If $\{x_{n,k}\}_{k=1}^{n}$ and $\{x_{n+1,k}\}_{k=1}^{n+1}$ denote the consecutive zeros of $P_n(x)$ and $P_{n+1}(x)$ respectively, then we have

$$\begin{aligned} a < x_{n+1,1} < x_{n,1} < x_{n+1,2} < x_{n,2} \ldots \\ < x_{n+1,n} < x_{n,n} \\ < x_{n+1,n+1} < b. \end{aligned} \tag{4.5}$$

In passing, we note that all these polynomials have been generalized in various ways. One possibility of the generalizations is through the *quantum algebras* [27–30].

5. Number Theory

Number theory is known for its rich and growing stock of conjectures often expressed using inequalities! Here, we shall focus on a few of them [31–35]. The product of the first prime numbers is known as the *primorial* and denoted by $P_n\#$. For example, $p_5\# = 2\times3\times5\times7\times11 = 2310$. The lower bound is given by Bonse's inequality and the upper bound is due to Erdős

$$p_{n+1}^2 < p_n\# = p_1p_2p_3\cdots p_n < 4^{p_n}. \tag{5.1}$$

Prime Gaps: A prime gap is the difference between two successive prime numbers. The difference between the $(n + 1)$-th and the n-th prime numbers is denoted by g_n.
Symbolically, we have

$$g_n = p_{n+1} - p_n. \tag{5.2}$$

Theorem 1. ***Bertrand–Chebyshev Theorem:*** *For any integer $n > 3$, there is always at least one prime number p such that*

$$n < p < 2n - 2. \tag{5.3}$$

A less restrictive statement:

$$n < p < 2n, \quad \text{for } n > 1.$$

Equivalent Formulation:

$$P_{n+1} < 2P_n.$$

It is also called Bertrand's postulate! In 1845, Joseph Bertrand stated this statement as a conjecture with numerical support of n = 3,000,000. In 1852, Chebyshev provided a proof using certain non-elementary methods. Simpler proofs were later provided by many mathematicians including Srinivasa Ramanujan (1919) and Paul Erdős (1932). Bertrand–Chebyshev theorem leads to

$$g_n < P_n. \tag{5.4}$$

The prime gaps is an active area of research employing very diverse techniques. However, many questions remain unanswered. Many conjectures remain open to be proved or disproved. We note the following conjectures

1. *Andrica's Conjecture:*

$$\begin{aligned} &\sqrt{p_{n+1}} - \sqrt{p_n} < 1. \\ \textit{Equivalently}: \quad & g_n < 2\sqrt{p_n} + 1. \end{aligned} \tag{5.5}$$

   This is due to Dorin Andrica (1986). It has a numerical support of $n = 4 \times 10^{18}$.

2. *Firoozbakht's Conjecture:*

$$\begin{aligned} &\sqrt[n+1]{p_{n+1}} < \sqrt[n]{p_n}\,. \\ \textit{Equivalently}: \quad & p_{n+1} < p_n^{1+\frac{1}{n}}\,. \end{aligned} \tag{5.6}$$

   This is due to Farideh Firoozbakht (1982). It is true for all primes less than $2^{64} \sim 1.844 \times 10^{19}$.

3. *Legendre's Conjecture:* For every positive integer n, there is a prime p such that

$$n^2 < p < (n+1)^2. \tag{5.7}$$

   This is due to Adrien-Marie Legendre (1752–1833). It has a numerical support of $n = 2 \times 10^9$.

4. *Brocard's Conjecture:* For every $n \geq 2$, there are at least four primes between p_n^2 and p_{n+1}^2. This is due to Pierre René Jean Baptiste Henri Brocard (1845–1922).

5. *Oppermann's Conjecture:* For every positive integer n, there is a prime p and another prime p' such that

$$\begin{aligned} &n(n-1) < p < n^2 \\ &\quad n^2 < p' < n(n+1) \end{aligned} \tag{5.8}$$

   This is due to Ludvig Henrik Ferdinand Oppermann (1877).

All these five conjectures are strongly interrelated and proof of any one of them implies the proofs of most others. The size of the n-th prime, $p_n \sim n\ln(n) < n^2$. The average gap between the primes

$$\begin{aligned}\bar{g}_n &\equiv \frac{1}{n}\sum_{k=1}^{n} g_k \\ &= \frac{p_{n+1}-2}{n} \\ &\sim \ln(n).\end{aligned} \tag{5.9}$$

Likewise, the average of Andrica's function $A_n = \sqrt{p_{n+1}} - \sqrt{p_n}$ is

$$\begin{aligned}\bar{A}_n &\equiv \frac{1}{n}\sum_{k=1}^{n} A_k \\ &= \frac{\sqrt{p_{n+1}}-\sqrt{2}}{n} \\ &\sim \frac{\sqrt{(n+1)\ln(n+1)}-\sqrt{2}}{n} < 1.\end{aligned} \tag{5.10}$$

On an average, $A_n < 1$. This is another supportive statement (though weak) for Andrica's conjecture.

6. Matrix Inequalities

Matrices are used in most scientific fields [36–41]. Even classical mechanics has been formulated using matrices. Maxwell theory of electromagnetism is the correct theory of light. In many situations, it is not straightforward to deal with the Maxwell equations directly. One possible way to circumvent this difficulty is to use a matrix representation of Maxwell equations [39–41]. The trace of a square matrix A, denoted by $\mathrm{tr}(A)$, is defined to be the sum of elements on the main diagonal. The sum of the eigenvalues of a matrix equals the trace of the matrix. The determinant is equal to the product of its eigenvalues. The trace occurs in many contexts and we note the

$$\text{Jacobi's Formula: } \det\left(e^{A}\right) = e^{\mathrm{tr}(A)}. \tag{6.1}$$

In general, $e^A e^B \neq e^{A+B}$ (equality occurs if $AB = BA$) and the

$$\begin{aligned}&\text{Golden–Thompson Inequality:} \\ &\mathrm{tr}\, e^{A+B} \leq \mathrm{tr}\left(e^{A} e^{B}\right).\end{aligned} \tag{6.2}$$

The notion of a matrix norm has some similarities to the magnitude of vectors. There are several types of matrix norms. The Frobenius norm (also known as the Hilbert–Schmidt norm or Schur norm) of a matrix is defined as the square root of the sum of the absolute squares of its elements

$$\|A\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \left|a_{ij}\right|^2}. \tag{6.3}$$

In terms of trace,

$$\|A\|_F = \sqrt{\mathrm{tr}\left(A\, A^{H}\right)}, \tag{6.4}$$

where A^H is the conjugate transpose (also denoted by $A^{\dagger}$). The Frobenius norm obeys the triangular inequality

$$\|A+B\| \leq \|A\| + \|B\| \tag{6.5}$$

and the products have the following inequality

$$\|A\,B\| \le \|A\|\,\|B\|. \tag{6.6}$$

The matrix norms are widely used in programming and optimization problems.

There are many types of matrices. A matrix A is said to be Hermitian, if $A = A^H$ (also denoted by $A^\dagger$). Hermitian matrices have real eigenvalues. Hermitian matrices are fundamental to quantum mechanics because they describe operators, which necessarily have real eigenvalues. For any $n \times n$ Hermitian matrix, let the eigenvalues be $\lambda_{\min} = \lambda_1 \le \lambda_2 \le \lambda_3 \cdots \le \lambda_n = \lambda_{\max}$, then

$$\lambda_{\min} \le \frac{1}{n}\mathrm{tr}(A) \le \lambda_{\max}. \tag{6.7}$$

Let us recall that, trace is the sum of the eigenvalues, $\mathrm{tr}(A) = \sum_{i=1}^{n}\lambda_i$. There are many inequalities specific to the type of the matrix.

7. Inequalities from Physics

There are several inequalities restricting certain physical processes. We shall cover just a few of them.

Clausius Theorem: For any thermodynamic system (such as a heat engine or a heat pump) exchanging heat with an external reservoirs and undergoing a thermodynamic cycle, then

$$\oint \frac{\delta Q}{T_{\text{surr}}} \le 0, \tag{7.1}$$

where δQ is the infinitesimal amount of heat absorbed by the system from the reservoir and T_{surr} is the temperature of the external reservoir (or surroundings) at a particular instant of time. The equality holds in the reversible case. As a consequence, the entropy always increases [42].

Heisenberg's Uncertainty Principle: In regards to a moving particle, there are fundamental limits to the accuracy with which we can measure its properties. It is impossible to know both the precise position and the precise momentum of a particle at the same instant. Mathematically, the product of the uncertainty Δx in the position of a particle and the uncertainty Δp in its momentum in the x-direction at the same instant is always greater than one-half of the reduced Planck's constant $\hbar = h/2\pi$

$$\Delta x\,\Delta p \ge \frac{\hbar}{2} \tag{7.2}$$

and the Planck's constant, $h = 6.62607015 \times 10^{-34}$ Joule.Second. The position and momentum form one *conjugate pair*. The other conjugate pair is energy and time and the corresponding uncertainty relation is

$$\Delta t\,\Delta E \ge \frac{\hbar}{2}. \tag{7.3}$$

A consequence of the uncertainty principle is that electrons cannot reside inside the atomic nucleus. The classically assigned size of the atomic nucleus is 5×10^{-15} m. If we assign this value to Δx, the corresponding value of Δp is 1.1×10^{-20} $kg.m/s$. The rest mass energy of an electron is $m_0c^2 = 0.51\,MeV$. The corresponding kinetic energy (KE) calculated using the relativistic formula yields $KE = \sqrt{p^2c^2 + m_0^2\,c^4} \approx pc \ge (1.1\times10^{-20})(3\times10^{8}) \ge 3.3\times10^{-12} Joules = 20 MeV$. It has been established by numerous experiments that the electrons emitted by the nuclei through the β-decay (or the rarer double beta decay) have an energy of less than 5 MeV. Thus, starting with an inequality, it is possible to rule out the idea that electrons reside inside the atomic nucleus [43].

Scherzer's Theorem: There is a limit of resolution for electron lenses because of unavoidable aberrations. One of the aberrations known as the spherical aberration is always present and it has a nonzero value, which is given by this theorem. This affects the performance of electron microscopes and any device based on electron lenses [44].

Resistor Networks: The properties of sets of equivalent resistances formed by connecting identical resistors in series and parallel have been studied using diverse techniques including number theory [45–47]. The inequalities derived using the number theoretic methods are supported by an independent investigation that makes use of graph theory [48]. Let us consider the set obtained from three identical resistors of value R_0. The equivalent resistances are $3R_0$ (all three in series); $(1/3)R_0$ (all three in parallel; $(2/3)R_0$ (block of two in series with one in parallel) and $(1/3)R_0$ (block of two in parallel with one in series) respectively. The corresponding numerical set (multiple of R_0) is $A(3) = \left\{\frac{1}{3}, \frac{2}{3}, \frac{3}{2}, 3\right\}$ and the order is $|A(3)| = 4$. There is no known formula for the order of an arbitrary set $|A(n)|$ and the following inequalities have been established

$$\frac{1}{4}\left(1+\sqrt{2}\right)^n < \left|A(n)\right| < \left(1-\frac{1}{n}\right)(0.318)(2.618)^n. \tag{7.4}$$

The lower bound is obtained using combinatorial arguments. The upper bound is obtained using number theoretic tools of Fibonacci numbers and the Farey sequences. A numerical fit consumes a lot of computer memory and a study using up to 27 resistors leads to

$$\left|A(n)\right|_{n\le 27} \sim 2.53^n. \tag{7.5}$$

8. Concluding Remarks

We noted some basic inequalities from, geometry, trigonometry and calculus. There are thousands of inequalities and many are being discovered, which is evident by the fact there are even journals containing the word inequalities in their titles. We included the inequalities required in the introductory mathematics and science courses. We end this article with the following inequality from statistics. *Bhatia–Davis Inequality:* If any bounded probability distribution has minimum m, maximum M, and expected value μ, then the variance σ^2 obeys

$$\sigma^2 \le (M-\mu)(\mu-m). \tag{8.1}$$

Equality holds precisely if all of the probability is concentrated at the endpoints m and M.

Through the medium of inequalities, some frontline areas of research were described. We saw the interlacing of roots of orthogonal polynomials. We pointed to the absence of a formula for the roots of orthogonal polynomials. We noted the easy to express conjectures on the prime gaps, an area of active research. We also saw some inequalities from physics and their influence on physical processes. In particular, we used Heisenberg's uncertainty principle to rule out the idea that electrons can reside inside the atomic nucleus. Results from sets of resistor networks were noted. We hope that this article will generate interest in the use of inequalities (and their refinements, where applicable) and in the set of open problems listed from different disciplines. Lastly, the *social inequalities* or inequalities in the society are a crucial topic and of course, way beyond the scope of this article.

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