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Image compression technique based on two new parametrized thresholding operators

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Abstract

In this paper, we introduce a new way to compress images using two innovative thresholding operators that are carefully designed with parameters. These operators are meant to handle comparison tasks more effectively. Our results show that they have clear benefits and perform better than the usual Hard and Soft thresholding methods. We've also included some example images and data to show how accurate and efficient our approach is, especially when looking at PSNR and CR measurements.

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1. Introduction

In recent years, image compression techniques are still an open problem in the field of image processing. These techniques are classified into lossless and lossy compression [8], where in the lossless compression the compressed image is numerically identical to the original image, whereas the compressed image produced by the lossy compression differs from the original image. Several compression standard methods have been developed in the lossy compression, which are widely used in the image compression like JPEG (Joint Photographic Experts Group) and JPEG2000 (Joint Photographic Experts

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Group 2000) [1]. The JPEG2000 uses the discrete wavelet transform (DWT) [8, 13], which is based on thresholding techniques in its compression algorithms such as Hard and Soft Thresholding proposed by Donoho [6]. The thresholding is depended on the threshold value and directly applied to the wavelet coefficients. However, the choice of the threshold value is a fundamental issue, such that a very large threshold cuts too many coefficients that might contain useful image information. Conversely, a too small threshold value produces many coefficients. Then, the major problem with both thresholding techniques is the selection of a suitable threshold value. Hence, the proper thresholding technique is that achieves a reasonable balance between the wavelet coefficients before and after thresholding. For this purpose, one of evident methods is that enhances the previous techniques in terms of the Peak Signal to Noise Ratio (PSNR) [9], which can be expressed according to the following problems:

Let f be a signal of $L^2(\mathbb{R}^2)$ characterized by its thresholding operators $T_\lambda^{Hard}f$ and $T_\lambda^{Soft}f$ at a threshold λ, are there operators $T_\lambda^{New,Hard}f$, $T_\lambda^{New,Soft}f$ of $L^2(\mathbb{R}^2)$, such that:

$$\left\|T_\lambda^{New,Hard}f-f\right\|_{L^2\left(\mathbb{R}^2\right)} \le \left\|T_\lambda^{Hard}f-f\right\|_{L^2\left(\mathbb{R}^2\right)}, \tag{1}$$

$$\left\|T_\lambda^{New,Soft}f-f\right\|_{L^2\left(\mathbb{R}^2\right)} \le \left\|T_\lambda^{Soft}f-f\right\|_{L^2\left(\mathbb{R}^2\right)}. \tag{2}$$

Our main goal of this work is to solve these problems by proposing an approximation method called Comparative Thresholding (CT), which is based on constructing two convergent operators sequences $(T_{J,\lambda}^{New,Hard}f)_{J\ge1}$, $(T_{J,\lambda}^{New,Soft}f)_{J\ge1}$ such as $T_{J,\lambda}^{New,Hard}f \to T_\lambda^{New,Hard}f$ and $T_{J,\lambda}^{New,Soft}f \to T_\lambda^{New,Soft}f$ in $L^2(\mathbb{R}^2)$ norm sense when $J \to +\infty$.

Our work is structured as follows: Section 2. Reminds the orthogonal Procrustes problem with its solution, whose we conclude a main result that will be useful in our proposed method. Section 3. Gives an overview on the multiresolution approximation and its orthonormal wavelet bases. Section 4. Introduces traditional thresholding techniques, such as Hard and Soft thresholding with theirs practical properties. Section 5. Presents our proposed method CT (Comparative Thresholding) and its compression algorithm. Section 6. Provides us with compression results and discussing them. Finally, we close this work by a conclusion and main references.

2. Preliminaries

In this section,we recall the orthogonal Procrustes problem with its solution. After that, we present a main result that will be used in the principal section (Section 5). The simplest algebraic statement of an orthogonal Procrustes problem [7] is to be given two matrices X_1,X_2 of $\mathcal{M}_{N\times M}(\mathbb{R})$ and seeks to determine an orthogonal matrix T of $\mathcal{M}_N(\mathbb{R})$ that minimises the following quantity

$$\left\|TX_1 - X_2\right\|_F^2, \tag{3}$$

Where $\|.\|_F$ denotes the Frobenius norm.

One of solutions of (3) is presented in [15] as follows:

$$T = UV^t, \tag{4}$$

where U and V are the orthogonal matrices that correspond to the singular value decomposition of $X_2X_1^t$ such that $X_2X_1^t = USV^t$ (S is a diagonal matrix). Note that each solution T of problem (3) satisfies the following relation:

$$\left\|TX_1 - X_2\right\|_F^2 \le \left\|AX_1 - X_2\right\|_F^2, \tag{5}$$

for all orthogonal matrices A of $\mathcal{M}_N(\mathbb{R})$.

From the foregoing,we deduce our main result which is obtained by taking $A = I_N$ in (5) as it is expressed by the following inequality:

$$\|TX_1 - X_2\|_2^2 \leq \|X_1 - X_2\|_2^2, \tag{6}$$

where $\| . \|_2$ denotes the matrix 2-norm.

3. Overview on Multiresolution Approximation (MRA)

Multiresolution Approximation or Multiresolution Analysis (MRA) is a mathematical notion that has emerged in 1985 through the work of S. Mallat and Yves Meyer [12, 13]. This notion is considered a tool to extract the orthonormal wavelet bases, whose the signal can be represented by a limit of its approximations at different levels or at successive scales. In this section, we present the concept of MRA and its main properties.

3.1. Multiresolution Approximation of $L^2(\mathbb{R}^n)$

Definition 3.1 [13] *A multiresolution approximation of $L^2(\mathbb{R}^n)$ is an increasing sequence $(V_j)_{j\in\mathbb{Z}}$ of closed linear subspaces of $L^2(\mathbb{R}^n)$ with the following properties:*

$$(a) \bigcap_{j\in\mathbb{Z}} V_j, \bigcup_{j\in\mathbb{Z}} V_j \text{is dense in } L^2(\mathbb{R}^n).$$

$$(b) \forall f \in L^2(\mathbb{R}^n), \forall j \in \mathbb{Z}: f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}. \tag{7}$$

$$(c) \forall f \in L^2(\mathbb{R}^n), \forall k \in \mathbb{Z}^n: f(x) \in V_0 \Leftrightarrow f(x-k) \in V_0. \tag{8}$$

$$(d) \exists g \in V_0, \text{ such that } \{g(x-k)\}_{k\in\mathbb{Z}^n} \text{ is a Riesz basis of the space } V_0.$$

Definition 3.2 [13] *A multiresolution approximation $(V_j)_{j\in\mathbb{Z}}$ is called r-regular ($r \in \mathbb{N}$), if the function g of (d) can be chosen in such a way that:*

$$\left|\partial^\alpha g(x)\right| \leq C_m \left(1 + |x|\right)^{-m}, \tag{9}$$

for each integer $m \in \mathbb{N}$ and for every multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ satisfying $|\alpha| \leq r$. (Here $\partial^\alpha = (\partial / \partial x_1)^{\alpha_1} \ldots (\partial / \partial x_n)^{\alpha_n}$ and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$).

Definition 3.3 [13]. *A scaling function ϕ associated to $(V_j)_{j\in\mathbb{Z}}$ is an element of V_0, such as the sequence $\{\phi(x-k)\}_{k\in\mathbb{Z}^n}$ constitutes an orthonormal basis of V_0.*

Theorem 3.4 [13]. *Let $(V_j)_{j\in\mathbb{Z}}$ be a multiresolution approximation of $L^2(\mathbb{R}^n)$. Then there exist two constants $c_2 \geq c_1 > 0$, such that for almost all $\zeta \in \mathbb{R}^n$, we have*

$$c_1 \leq \left(\sum_{k\in\mathbb{Z}^n} \left|\hat{g}(\zeta + 2k\pi)\right|^2 \right)^{\frac{1}{2}} \leq c_2. \tag{10}$$

Furthermore, if $\phi \in L^2(\mathbb{R}^n)$

$$\hat{\phi}(\zeta) = \hat{g}(\zeta) \left(\sum_{k\in\mathbb{Z}^n} \left|\hat{g}(\zeta + 2k\pi)\right|^2 \right)^{-\frac{1}{2}}, \tag{11}$$

then $\{\phi(x-k)\}_{k\in\mathbb{Z}^n}$ is an orthonormal basis of V_0.

Let $(V_j)_{j\in\mathbb{Z}}$ be an r-regular multiresolution approximation of $L^2(\mathbb{R}^n)$ and let W_j denote the orthogonal complement of V_j in V_{j+1}. The W_j verifies the condition (b), (c) and (d) of the Definition 3.1, in dimension 1 the function ψ such that $\{\psi(x-k)\}_{k\in\mathbb{Z}^n}$ is an orthonormal basis of W_0, called wavelet function. Furthermore, the orthogonal projection of $f\in L^2(\mathbb{R}^n)$ onto V_{j+1} is expressed by the following decomposition formula:

$$P_{V_{j+1}}f = P_{V_j}f + P_{W_j}f.$$

In the case of $j\geq 1$, we have

$$P_{V_j}f = P_{V_0}f + \sum_{k=0}^{k=j-1} P_{W_k}f. \tag{12}$$

Moreover,

$$L^2\left(\mathbb{R}^n\right) = \bigoplus_{j\in\mathbb{Z}} W_j = V_0 \bigoplus \left(\bigoplus_{j\geq 0} W_j\right) \tag{13}$$

Theorem 3.5 *Let $(V_j)_{j\in\mathbb{Z}}$ be an r-regular multiresolution approximation of $L^2\left(\mathbb{R}^n\right)$ characterized by its scaling function ϕ, then the sequence $\{\phi_{j,k}\}_{k\in\mathbb{Z}^n}$ such that*

$$\phi_{j,k}\left(x\right) = 2^{n\frac{j}{2}}\phi\left(2^j x - k\right), \tag{14}$$

is an orthonormal basis of V_j.

In dimension n = 1, we can construct an r-regular multiresolution approximation $(V_j)_{j\in\mathbb{Z}}$ from an r-regular multiresolution approximation $(V_j^1)_{j\in\mathbb{Z}}$ of $L^2\left(\mathbb{R}\right)$, the following theorem affirms that.

Theorem 3.6 *Let $(V_j^1)_{j\in\mathbb{Z}}$ be an r-regular multiresolution approximation of $L^2\left(\mathbb{R}\right)$ and φ is its scaling function, we define $(V_j)_{j\in\mathbb{Z}}$ by the following way:*

For each $j\in\mathbb{Z}$,

$$V_j = V_j^1 \bigotimes V_j^1 \bigotimes \ldots \bigotimes V_j^1 \text{(n times)}. \tag{15}$$

Then, $(V_j)_{j\in\mathbb{Z}}$ is an r-regular multiresolution approximation of $L^2\left(\mathbb{R}^n\right)$ characterized by its scaling function ϕ, such that:

$$\forall x = \left(x_1, x_2, \ldots, x_n\right) \in \mathbb{R}^n, \phi\left(x\right) = \varphi\left(x_1\right)\varphi\left(x_2\right)\ldots.\varphi\left(x_n\right). \tag{16}$$

Remark 3.7. *The multiresolution approximation of $L^2(\mathbb{R}^n)$ which is defined by (15), called the separable multiresolution approximation.*

Theorem 3.8. *Let $(V_j^1)_{j\in\mathbb{Z}}$ be an r-regular multiresolution approximation of $L^2\left(\mathbb{R}\right)$ characterized by its scaling and wavelet function, respectively φ and ψ. Let also ϕ be a scaling function associated to a multiresolution approximation $(V_j)_{j\in\mathbb{Z}}$ of $L^2\left(\mathbb{R}^2\right)$ such as:*

$$\phi\left(x,y\right) = \varphi\left(x\right)\varphi\left(y\right), \tag{17}$$

Then the following three wavelet functions of W_0:

$$\psi^1\left(x,y\right) = \varphi\left(x\right)\psi\left(y\right), \tag{18}$$

$$\psi^2\left(x,y\right) = \psi\left(x\right)\varphi\left(y\right), \tag{19}$$

$$\psi^3\left(x,y\right) = \psi\left(x\right)\psi\left(y\right), \tag{20}$$

are satify, for all $j \in \mathbb{Z}$, the sequence

$$\left\{\psi^1_{j,(n,m)},\psi^2_{j,(n,m)},\psi^3_{j,(n,m)}\right\}_{(n,m)\in\mathbb{Z}^2}, \tag{21}$$

constitute an orthonormal basis of W_j. Hence, the elements

$$\left\{\psi^1_{j,(n,m)},\psi^2_{j,(n,m)},\psi^3_{j,(n,m)}\right\}_{(j,n,m)\in\mathbb{Z}^3}, \tag{22}$$

is an orthonormal basis of $L^2\left(\mathbb{R}^2\right)$, where

$$\psi^1_{j,(n,m)}(x,y) = 2^j\psi^1\left(2^jx-n,2^jy-m\right), \tag{23}$$

$$\psi^2_{j,(n,m)}(x,y) = 2^j\psi^2\left(2^jx-n,2^jy-m\right), \tag{24}$$

$$\psi^3_{j,(n,m)}(x,y) = 2^j\psi^3\left(2^jx-n,2^jy-m\right). \tag{25}$$

Proof. For each $j \in \mathbb{Z}$, we have

$$\begin{aligned} V_{j+1} &= V^1_{j+1}\otimes V^1_{j+1} \\ &= \left(V^1_j \oplus W^1_j\right)\otimes\left(V^1_j\oplus W^1_j\right) \\ &= \left(V^1_j\otimes V^1_j\right)\oplus\left(V^1_j\otimes W^1_j\right)\oplus\left(W^1_j\otimes V^1_j\right)\oplus\left(W^1_j\otimes W^1_j\right). \end{aligned}$$

Hence, $W_j = \left(V^1_j\otimes W^1_j\right)\oplus\left(W^1_j\otimes V^1_j\right)\oplus\left(W^1_j\otimes W^1_j\right)$, showing that the sequence $\left\{\psi^1_{j,(n,m)},\psi^2_{j,(n,m)},\psi^3_{j,(n,m)}\right\}_{(n,m)\in\mathbb{Z}^2}$ constitute an orthonormal basis of W_j.

Since $L^2\left(\mathbb{R}^2\right) = \bigoplus_{j\in\mathbb{Z}} W_j$, then $\left\{\psi^1_{j,(n,m)},\psi^2_{j,(n,m)},\psi^3_{j,(n,m)}\right\}_{(j,n,m)\in\mathbb{Z}^3}$ is an orthonormal basis of $L^2\left(\mathbb{R}^2\right)$.

3.2. Image Approximation Using Wavelet 2-D

In this subsection, we describe the approximation method of an image $f \in L^2\left(\mathbb{R}^2\right)$ using 2-Dimension wavelet bases.

We consider an image $f \in L^2\left(\mathbb{R}^2\right)$; let P_j (respectively Q_j) be the orthogonal projection of f onto V_j (respectively W_j), then

$$P_jf = \sum_{(n,m)\in\mathbb{Z}^2}\left(f,\phi_{j,(n,m)}\right)\phi_{j,(n,m)}. \tag{26}$$

And the projection of f onto W_j is decomposed as follows:

– Projection onto $V^1_j\otimes W^1_j$

$$Q^{(1)}_jf = \sum_{(n,m)\in\mathbb{Z}^2}\left(f,\psi^{(1)}_{j,(n,m)}\right)\psi^{(1)}_{j,(n,m)}. \tag{27}$$

– Projection onto $W^1_j\otimes V^1_j$

$$Q^{(2)}_jf = \sum_{(n,m)\in\mathbb{Z}^2}\left(f,\psi^{(2)}_{j,(n,m)}\right)\psi^{(2)}_{j,(n,m)}. \tag{28}$$

– Projection onto $W^1_j\otimes W^1_j$

$$Q^{(3)}_jf = \sum_{(n,m)\in\mathbb{Z}^2}\left(f,\psi^{(3)}_{j,(n,m)}\right)\psi^{(3)}_{j,(n,m)}. \tag{29}$$

The three detail spaces $V_j^1 \bigotimes W_j^1$, $W_j^1 \bigotimes V_j^1$ and $W_j^1 \bigotimes W_j^1$ are called respectively the horizontal, vertical and diagonal spaces. Furthermore, the image approximation at level j is obtained as the sum of its orthogonal projections onto V_{j-1}, $V_{j-1}^1 \bigotimes W_{j-1}^1$, $W_{j-1}^1 \bigotimes V_{j-1}^1$ and $W_{j-1}^1 \bigotimes W_{j-1}^1$.

4. Traditional Thresholding Techniques

Wavelet thresholding is a technique based on an estimation operator used for image compression. The principal work on this technique is done by Donoho [6], where he has classified the thresholding into two categories namely: Hard and Soft thresholding. In this section, we present the two thresholding techniques with their practical properties.

The Hard thresholding function chooses all wavelet coefficients that are greater than the given threshold λ and sets the others to zero in absolute value, which is described as:

$$\eta_{Hard}(x,\lambda) = \begin{cases} x, \text{ if } |x| > \lambda, \\ 0, \text{ if } |x| \le \lambda. \end{cases} \tag{30}$$

Thus, the Hard thresholding operator corresponding to $f \in L^2(\mathbb{R}^2)$ at a threshold λ is defined as follows:

$$T_\lambda^{Hard} f = P_0 f + \sum_{j=0}^{+\infty} \sum_{i=1}^{3} \sum_{(n,m)\in\mathbb{Z}^2} \eta_{Hard}\left(\left\langle f, \psi_{j,(n,m)}^{(i)} \right\rangle, \lambda\right) \psi_{j,(n,m)}^{(i)}. \tag{31}$$

The Soft thresholding function shrinks the wavelet coefficients that are above the threshold λ and sets the others to zero in absolute value, which is described as:

$$\eta_{Soft}(x,\lambda) = \begin{cases} x - \lambda, \text{ if } x \ge \lambda, \\ 0, \text{ if } |x| \le \lambda, \\ x + \lambda \text{ if } x \le -\lambda. \end{cases} \tag{32}$$

Thus, the Soft thresholding operator for $f \in L^2(\mathbb{R}^2)$ at a threshold λ is defined as follows:

$$T_\lambda^{Soft} f = P_0 f + \sum_{j=0}^{+\infty} \sum_{i=1}^{3} \sum_{(n,m)\in\mathbb{Z}^2} \eta_{Soft}\left(\left\langle f, \psi_{j,(n,m)}^{(i)} \right\rangle, \lambda\right) \psi_{j,(n,m)}^{(i)}. \tag{33}$$

Note that each of $T_\lambda^{Hard} f$, $T_\lambda^{Soft} f \to f$ in $L^2(\mathbb{R}^2)$ norm sense when $\lambda \to 0$.

In practice, the Hard thresholding technique is instinctively appealing, but sometimes it introduces artifacts in the compressed image. The Soft thresholding technique is more smooth, such that it reduces the false structures provided by the Hard thresholding. However, the Soft thresholding technique does not perform well in certain cases when the vast majority of wavelet coefficients are approximately zero.

5. Proposed Method CT (Comparative Thresholding)

Our proposed Comparative Thresholding method (CT), is a new image compression technique based on constructing approximation operators such that theirs limits solve the problems (1) and (2). This new method is developed to overcome the drawbacks of the Hard and the Soft thresholding technique. In this section, we introduce the principle and the compression algorithm of our proposed method.

5.1. Principle of Proposed Method

In this subsection, we illustrate and prove the CT method through the following theorem.

Theorem 5.1 *Let $(V_j^1)_{j\in\mathbb{Z}}$ be an r-regular multiresolution approximation of $L^2(\mathbb{R})$ characterized by its scaling and wavelet functions φ and ψ, respectively. Let also $(V_j)_{j\in\mathbb{Z}}$ be the separable multiresolution* *approximation of $L^2\left(\mathbb{R}^2\right)$ associated to $(V_j^1)_{j\in\mathbb{Z}}$. For $\lambda > 0$, for all $f \in L^2\left(\mathbb{R}^2\right)$ and each $J \geq 1$, we define the following operators:*

$$T_{J,\lambda}^{Hard} f = P_0 f + \sum_{j=0}^{J-1}\sum_{i=1}^{3}\sum_{(n,m)\in\mathbb{Z}^2} \eta_{Hard}\left(\left\langle f, \psi_{j,(n,m)}^{(i)}\right\rangle, \lambda\right)\psi_{j,(n,m)}^{(i)}, \tag{34}$$

$$T_{J,\lambda}^{Soft} f = P_0 f + \sum_{j=0}^{J-1}\sum_{i=1}^{3}\sum_{(n,m)\in\mathbb{Z}^2} \eta_{Soft}\left(\left\langle f, \psi_{j,(n,m)}^{(i)}\right\rangle, \lambda\right)\psi_{j,(n,m)}^{(i)}, \tag{35}$$

If both f and ψ have compact support, then there exist two operators $T_{J,\lambda}^{New,Hard} f$ and $T_{J,\lambda}^{New,Soft} f$ of V_J such that:

$$\left\|T_{J,\lambda}^{New,Hard} f - P_J f\right\|_{L^2\left(\mathbb{R}^2\right)} \leq \left\|T_{J,\lambda}^{Hard} f - P_J f\right\|_{L^2\left(\mathbb{R}^2\right)}, \tag{36}$$

$$\left\|T_{J,\lambda}^{New,Soft} f - P_J f\right\|_{L^2\left(\mathbb{R}^2\right)} \leq \left\|T_{J,\lambda}^{Soft} f - P_J f\right\|_{L^2\left(\mathbb{R}^2\right)}. \tag{37}$$

Moreover, $T_{J,\lambda}^{New,Hard} f$ and $T_{J,\lambda}^{New,Soft} f$ are converge to solution operators, corresponding to problems (1) *and* (2), respectively *when $J \to +\infty$.*

Proof. Since both f and ψ have compact support, there exist two integers $N, M \in \mathbb{N}$ such that the following truncated forms hold:

$$T_{J,\lambda}^{Hard} f = P_0 f + \sum_{j=0}^{J-1}\sum_{i=1}^{3}\sum_{n=1}^{N}\sum_{m=1}^{M} \eta_{Hard}\left(\langle f, \psi_{j,(n,m)}^{(i)}\rangle, \lambda\right)\psi_{j,(n,m)}^{(i)}, \tag{38}$$

$$T_{J,\lambda}^{Soft} f = P_0 f + \sum_{j=0}^{J-1}\sum_{i=1}^{3}\sum_{n=1}^{N}\sum_{m=1}^{M} \eta_{Soft}\left(\langle f, \psi_{j,(n,m)}^{(i)}\rangle, \lambda\right)\psi_{j,(n,m)}^{(i)}, \tag{39}$$

$$P_J f = P_0 f + \sum_{j=0}^{J-1}\sum_{i=1}^{3}\sum_{n=1}^{N}\sum_{m=1}^{M} \langle f, \psi_{j,(n,m)}^{(i)}\rangle \psi_{j,(n,m)}^{(i)}. \tag{40}$$

Step 1: Matrix formulation

For each $j = 0, 1, \ldots, J-1$ and $i \in \{1,2,3\}$, let us define the following real matrices of order $N \times M$:

$$D_j^{(i)}(n,m) = \langle f, \psi_{j,(n,m)}^{(i)}\rangle, \tag{41}$$

$$D_{j,\lambda}^{Hard,(i)}(n,m) = \eta_{Hard}(\langle f, \psi_{j,(n,m)}^{(i)}\rangle, \lambda), \tag{42}$$

$$D_{j,\lambda}^{Soft,(i)}(n,m) = \eta_{Soft}(\langle f, \psi_{j,(n,m)}^{(i)}\rangle, \lambda). \tag{43}$$

Hence, all the classical thresholding operators can be rewritten in terms of these matrices.

Step 2: Orthogonal correction via Procrustes problem

To construct improved thresholding operators, we associate with each matrix $D_{j,\lambda}^{Hard,(i)}$ and $D_{j,\lambda}^{Soft,(i)}$ an orthogonal correction obtained as the solution of two orthogonal Procrustes problems:

$$\min_{T_1 \in O(N)} \| T_1 D_{j,\lambda}^{Hard,(i)} - D_j^{(i)} \|_F^2, \quad \min_{T_2 \in O(N)} \| T_2 D_{j,\lambda}^{Soft,(i)} - D_j^{(i)} \|_F^2 . \tag{44}$$

According to (6), any optimal orthogonal matrix T satisfies:

$$\| TX_1 - X_2 \|_2 \leq \| X_1 - X_2 \|_2, \tag{45}$$

which means that the orthogonal transformation minimizes the distance between the modified and the original coefficients.

Step 3: Definition of the new coefficient matrices

We now define the new matrices by applying the orthogonal corrections:

$$D_{j,\lambda}^{New,Hard,(i)} = T_1 D_{j,\lambda}^{Hard,(i)}, \tag{46}$$

$$D_{j,\lambda}^{New,Soft,(i)} = T_2 D_{j,\lambda}^{Soft,(i)}. \tag{47}$$

These matrices preserve the energy structure of the original coefficients while reducing the overall reconstruction error.

Step 4: Construction of the new operators

The new thresholding operators are then defined as:

$$T_{J,\lambda}^{New,Hard} f = P_0 f + \sum_{j=0}^{J-1}\sum_{i=1}^{3}\sum_{n=1}^{N}\sum_{m=1}^{M} D_{j,\lambda}^{New,Hard,(i)}(n,m)\psi_{j,(n,m)}^{(i)}, \tag{48}$$

$$T_{J,\lambda}^{New,Soft} f = P_0 f + \sum_{j=0}^{J-1}\sum_{i=1}^{3}\sum_{n=1}^{N}\sum_{m=1}^{M} D_{j,\lambda}^{New,Soft,(i)}(n,m)\psi_{j,(n,m)}^{(i)}. \tag{49}$$

From (12), it follows that $T_{J,\lambda}^{New,Hard} f, T_{J,\lambda}^{New,Soft} f \in V_J$.

Step 5: Energy inequality

From (40) and (48), we have

$$\begin{aligned}\left\|T_{J,\lambda}^{New,Hard} f - P_J f\right\|_{L^2(\mathbb{R}^2)}^2 &= \sum_{j=0}^{J-1}\sum_{i=1}^{3}\sum_{n=1,m=1}^{N,M}\left|D_{j,\lambda}^{New,Hard,(i)}(n,m) - D_j^{(i)}(n,m)\right|^2 \\ &= \sum_{j=0}^{J-1}\sum_{i=1}^{3}\left\|D_{j,\lambda}^{New,Hard,(i)} - D_j^{(i)}\right\|_2^2 \\ &= \sum_{j=0}^{J-1}\sum_{i=1}^{3}\left\|T_1 D_{j,\lambda}^{Hard,(i)} - D_j^{(i)}\right\|_2^2,\end{aligned}$$

by using our result (6), we find that

$$\left\|T_{J,\lambda}^{New,Hard} f - P_J f\right\|_{L^2(\mathbb{R}^2)}^2 \le \sum_{j=0}^{J-1}\sum_{i=1}^{3}\left\|D_{j,\lambda}^{Hard,(i)} - D_j^{(i)}\right\|_2^2 = \left\|T_{J,\lambda}^{Hard} f - P_J f\right\|_{L^2(\mathbb{R}^2)}^2.$$

Consequently,

$$\left\|T_{J,\lambda}^{New,Hard} f - P_J f\right\|_{L^2(\mathbb{R}^2)}^2 \le \left\|T_{J,\lambda}^{Hard} f - P_J f\right\|_{L^2(\mathbb{R}^2)}^2.$$

Similarly, from (36) and (49), we get

$$\begin{aligned}\left\|T_{J,\lambda}^{New,Soft} f - P_J f\right\|_{L^2(\mathbb{R}^2)}^2 &= \sum_{j=0}^{J-1}\sum_{i=1}^{3}\sum_{n=1,m=1}^{N,M}\left|D_{j,\lambda}^{New,Soft,(i)}(n,m) - D_j^{(i)}(n,m)\right|^2 \\ &= \sum_{j=0}^{J-1}\sum_{i=1}^{3}\left\|D_{j,\lambda}^{New,Soft,(i)} - D_j^{(i)}\right\|_2^2 \\ &= \sum_{j=0}^{J-1}\sum_{i=1}^{3}\left\|T_2 D_{j,\lambda}^{Soft,(i)} - D_j^{(i)}\right\|_2^2,\end{aligned}$$

by using our result (6), we find that

$$\left\|T_{J,\lambda}^{New,Soft} f - P_J f\right\|_{L^2\left(\mathbb{R}^2\right)}^2 \le \sum_{j=0}^{J-1}\sum_{i=1}^{3}\left\|D_{j,\lambda}^{Soft,(i)} - D_j^{(i)}\right\|_2^2 = \left\|T_{J,\lambda}^{Soft} f - P_J f\right\|_{L^2\left(\mathbb{R}^2\right)}^2 .$$

Hence,

$$\left\|T_{J,\lambda}^{New,Soft} f - P_J f\right\|_{L^2\left(\mathbb{R}^2\right)}^2 \le \left\|T_{J,\lambda}^{Soft} f - P_J f\right\|_{L^2\left(\mathbb{R}^2\right)}^2 .$$

This proves inequalities (36) and (37).

Step 6: Convergence of the new operators

Since $P_J f \to f$ in $L^2(\mathbb{R}^2)$ as $J \to +\infty$, by taking limits in the inequalities (36) and (37), we obtain:

$$\| T_{\lambda}^{New,Hard} f - f \|_{L^2\left(\mathbb{R}^2\right)}^2 \le \| T_{\lambda}^{Hard} f - f \|_{L^2\left(\mathbb{R}^2\right)}^2, \tag{50}$$

$$\| T_{\lambda}^{New,Soft} f - f \|_{L^2\left(\mathbb{R}^2\right)}^2 \le \| T_{\lambda}^{Soft} f - f \|_{L^2\left(\mathbb{R}^2\right)}^2 . \tag{51}$$

Therefore, the new parametrized thresholding operators provide a better approximation of f in the L^2-sense, which theoretically implies higher PSNR and improved compression performance compared to traditional Hard and Soft thresholding.

5.2. Compression Algorithm Based on Proposed Method

This subsection describes the steps and the flowchart corresponding to comparative thresholding compression algorithm.

Step 1. Input the original image.
Step 2. Decompose the original image using DWT.
Step 3. Apply Hard or Soft thresholding to wavelet coefficients at the threshold value λ.
Step 4. Apply the Comparative Thresholding technique to resultant wavelet coefficients.
Step 5. Reconstruct the image using the new thresholding operator.
Step 6. Use an entropy encoder such as Huffman encoder.
Step 7. We get the compressed image for transmission or storage.

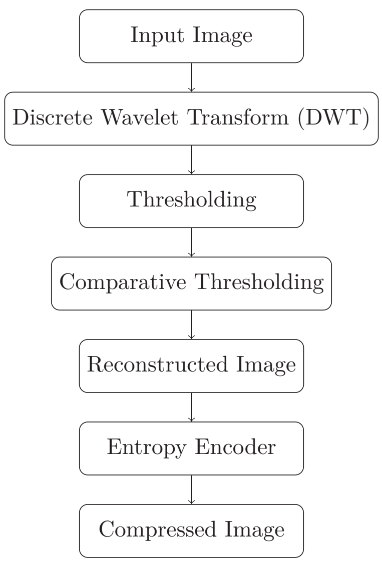

Figure 1. Flowchart of the proposed compression algorithm.
Source: Authors.

6. Experimental Results and Discussion

In order to show that our proposed compression method (CT) performs better than DWT using the Hard and the Soft thresholding, we have used a grayscale test image that is ('barbara.png') of 512 × 512 pixels,whose size is 184.73 KB, as presented in Figure 2.

This image was processed using traditional thresholding techniques and the Comparative Thresholding technique with various threshold levels. The algorithm was implemented in MATLAB, where the Haar wavelet was employed and the decomposition was performed at level $J = 1$. So, the computed PSNR and CR values for DWT and CT using Hard and Soft thresholding are given in the following table:

Table 1. Comparison of PSNR (dB) and CR results between the classical and proposed compression methods.

Criterion	λ	Hard	CT Hard	Soft	CT Soft
PSNR (dB)	5	32.3038	**33.9434**	26.2039	**26.4442**
	10	26.6280	**27.1453**	22.4675	**22.5905**
	15	22.7030	**23.2312**	21.0073	**21.0941**
	20	21.0624	**21.3755**	20.7035	**20.7858**
	30	20.6608	**20.7342**	20.6461	**20.6567**
CR (ratio)	5	12.4800:1	12.4800:1	12.4800:1	12.4800:1
	10	12.4000:1	12.4000:1	12.4000:1	12.4000:1
	15	12.1500:1	**12.3200:1**	12.4000:1	12.4000:1
	20	11.8400:1	**12.1500:1**	12.3200:1	**12.4000:1**
	30	11.5500:1	**11.7700:1**	12.0000:1	**12.1500:1**

Source: Authors.

This table presents two image compression criteria, namely PSNR and CR, the PSNR is used to evaluate the quality of the reconstructed image; generally, a higher PSNR indicates better image quality. The CR (Compression Ratio) is used to measure the ability of data compression and to compare different methods by relating the size of the compressed image to that of the original image, where the greater (CR) means the less size of compressed image, indicating a better compression method.

Obviously, the experimental results show that for each threshold value λ from 5 to 30, the PSNR and CR obtained with the proposed method (CT) are higher than those obtained with DWT using both traditional thresholding techniques. Furthermore, the effectiveness of the proposed method is

Figure 2. Original image.
Source: Authors.

Figure 3. Compressed images using Hard Thresholding.
Source: Authors.

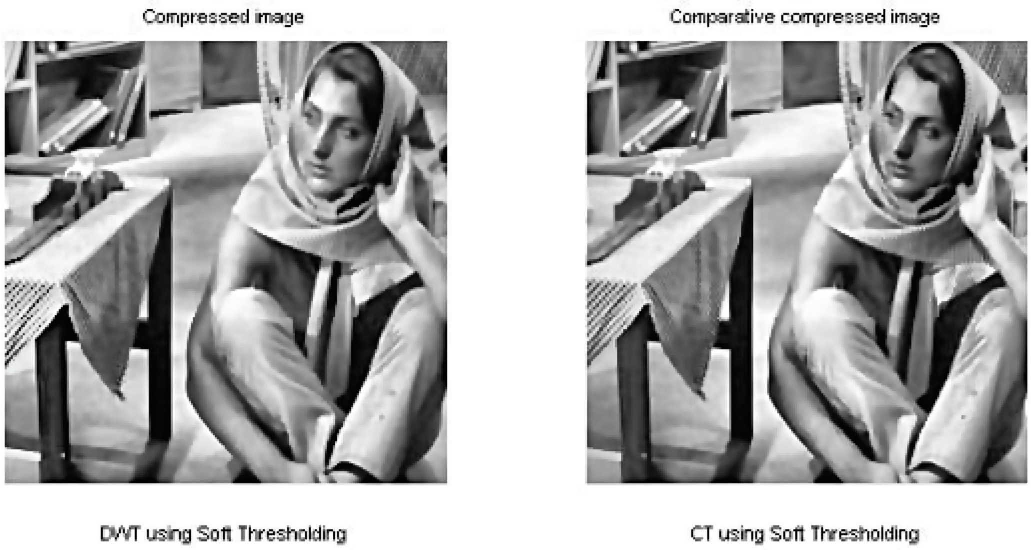

Figure 4. Compressed images using Soft Thresholding.
Source: Authors.

observed at $\lambda = 30$, as shown in Figures 3 and 4, where the CR values of 11.77:1 and 12.15:1 indicate that the compressed image produced by the proposed method occupies only 8.5% and 8.23% of the storage space, respectively of the storage space of the original image. Hence, the Comparative Thresholding method performs better than classical thresholding methods.

7. Conclusion

In this paper, we've introduced a new approach called Comparative Thresholding for image compression. It offers some clear advantages over the traditional Hard and Soft thresholding methods. Our technique is designed to improve both the PSNR and CR, making the compressed images look better without losing quality. From the tests we ran, it's clear that our method outperforms the usual thresholding techniques, especially when using a lambda value of 30. The results show sharper images with higher quality and better compression ratios. Because of these benefits, we believe this new method could be really useful for everyday life and practical applications.

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