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Marcinkiewicz functions on product spaces along surfaces of revolution

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Abstract

In this paper, specific L^p bounds for a class of Marcinkiewicz integral operators on product spaces along surfaces of revolution are established whenever the kernel functions are rough in $L^q(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$. By virtue of the obtained bounds and an extrapolation argument, we prove that the aforementioned operators are bounded on $L^p(\mathbb{R}^n\times\mathbb{R}^m)$ under rather weaker conditions on the kernel functions. The results in this work represent essential improvements and extensions of several results on Marcinkiewicz operators.

Mathematics Subject Classification: 42B20, 42B25

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1. Introduction

Let $\tau \geq 2$ ($\tau = n$ or m), $\mathbb{R}^\tau$ be the Euclidean τ-space and $\mathbb{S}^{\tau-1}$ be the unit sphere in $\mathbb{R}^\tau$ equipped with the induced Lebesgue surface measure $d\rho_\tau(\cdot)$.

For $\rho_1 = c_1 + id_1, \rho_2 = c_2 + id_2\,(c_1, c_2, d_1, d_2 \in \mathbb{R}\, with\, c_1, c_2 > 0)$, let

$$K_{\Theta,g}(\xi,\zeta) = \frac{\Theta(\xi,\zeta)g(|\xi|,|\zeta|)}{|\xi|^{n-\rho_1}|\zeta|^{m-\rho_2}},$$

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where g is a measurable function on $\mathbb{R}_+ \times \mathbb{R}_+$ and Θ is a function on $\mathbb{R}^n \times \mathbb{R}^m$, which is measurable, integrable over $\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}$, homogeneous of degree zero, and satysfying

$$\int_{\mathbb{S}^{n-1}} \Theta(\xi,\zeta)d\rho_n(\xi) = \int_{\mathbb{S}^{m-1}} \Theta(\xi,\zeta)d\rho_m(\zeta) = 0. \tag{1}$$

For an appropriate function $\Lambda : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, we let $\mathcal{M}_{\Theta,\Lambda,g}$ be the Marcinkiewicz operator along the surface of revolution $\Gamma_\Lambda(\xi,\zeta) = \left(\xi,\zeta,\Lambda(|\xi|,|\zeta|)\right)$ given by

$$\mathcal{M}_{\Theta,\Lambda,g}(\mathcal{U})(w,v,s) = \left(\iint_{\mathbb{R}_+\times\mathbb{R}_+} \left|A_{l,t}(\mathcal{U})(w,v,s)\right|^2 \frac{dldt}{lt} \right)^{1/2}, \tag{2}$$

where $\mathcal{U} \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R})$ and

$$A_{l,t}(\mathcal{U})(w,v,s) = \frac{1}{l^{\rho_1} t^{\rho_2}} \int_{|\zeta|\le t} \int_{|\xi|\le l} \mathcal{U}(w-\xi, v-\zeta, s-\Lambda(|\xi|,|\zeta|)) K_{\Theta,g}(\xi,\zeta) d\xi d\zeta.$$

We point out that the operator $\mathcal{M}_{\Theta,\Lambda,g}$ is a natural generalization of the operator $\mathcal{M}^{\psi}_{\Theta,g}$ related to the surface $\Gamma_\Lambda(\xi) = \left(\xi,\Lambda(|\xi|)\right)$ in the one parameter setting, which is defined by

$$\mathcal{M}^{\psi}_{\Theta,g}(\mathcal{U})(w,w_{n+1}) = \left(\int_{\mathbb{R}_+} \left| \frac{1}{l^{\rho_1}} \int_{|\xi|\le l} \mathcal{U}(w-\xi, w_{n+1}-\psi(|\xi|)) \frac{\Theta(\xi) g(|\xi|)}{|\xi|^{n-\rho_1}} d\xi \right|^2 \frac{dl}{l} \right)^{1/2}. \tag{3}$$

The operator $\mathcal{M}^{\psi}_{\Theta,g}$ was initiated in [1] whenever $\psi(l) = l$ and $g \equiv 1$. Precisely, the author of [1] established the boundedness of $\mathcal{M}^{\psi}_{\Theta,1}$ on $L^p(\mathbb{R}^{n+1})$ for $p \in (1,2]$ under the condition $\Theta \in Lip_\gamma(\mathbb{S}^{n-1})$ for some $\gamma \in (0,1]$. Thereafter, the operator $\mathcal{M}^{\psi}_{\Theta,g}$ has been considered by many mathematicians, see for example [2-10].

In this work, we are interested in studying the operator $\mathcal{M}_{\Theta,\Lambda,g}$. When $\Lambda \equiv 0$ and $\rho_1 = 1 = \rho_2$, we denote the operator $\mathcal{M}_{\Theta,\Lambda,g}$ by $\mathcal{M}_{\Theta,g}$. In addition, when $g \equiv 1$, then $\mathcal{M}_{\Theta,g}$ reduces to the classical Marcinkiewicz integral on product domains, which is denoted by $\mathcal{M}_\Theta$. The discussion of the operator $\mathcal{M}_\Theta$ has attracted the attentions of many researchers for along time. Historically, the L^p boundedness of $\mathcal{M}_\Theta$ was begun in [11] in which the author established only the L^2 boundedness of $\mathcal{M}_\Theta$ whenever Θ belongs to the space $L(\log L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$. Thereafter, the authors of [12] proved the L^p boundedness of $\mathcal{M}_\Theta$ for all $p \in (1,\infty)$ under the assumption $\Theta \in L(\log L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$, and also they mentioned that similar argument as that in [13] gives the optimality to the condition $\Theta \in L(\log L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$. On the other side, the author of [14] found that the operator $\mathcal{M}_\Theta$ is of type (p,p) for $p \in (1,\infty)$ provided that $\Theta \in B_q^{(0,0)}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ with $q > 1$, and that the condition $\Theta \in B_q^{(0,0)}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ is optimal. Here, $B_q^{(0,v)}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ refers to the block space introduced in [15].

Later on, Yano's extrapolation argument [16] was employed by the authors of [17] to find the L^p boundedness of $\mathcal{M}_{\Theta,g}$ for $|1/p - 1/2| < \min\{1/\kappa', 1/2\}$ whenever the kernel function Θ lies either in the space $L(\log L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ or in the space $B_q^{(0,0)}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ and the mapping function g belongs to $\nabla_\kappa(\mathbb{R}_+ \times \mathbb{R}_+)$ for some $\kappa > 1$, where $\nabla_\kappa(\mathbb{R}_+ \times \mathbb{R}_+)$ (for $\kappa > 1$) is the class of all measurable functions g satisfying

$$\|g\|_{\nabla_\kappa(\mathbb{R}_+\times\mathbb{R}_+)} = \sup_{j,k\in\mathbb{Z}} \left(\int_{2^j}^{2^{j+1}} \int_{2^k}^{2^{k+1}} |g(l,t)|^\kappa \frac{dldt}{lt} \right)^{1/\kappa} < \infty.$$

The above results have motivated many mathematicians to study the Marcinkiewicz integral on product spaces along surfaces of revolution of the form

$$\mathcal{M}^{\psi,\phi}_{\Theta,g}(\mathcal{U})(w,w_{n+1},v,v_{m+1})=\left(\iint_{\mathbb{R}_+\times\mathbb{R}_+}\left|A^{\psi,\phi}_{l,t}(\mathcal{U})\right|^2\frac{dldt}{lt}\right)^{1/2}, \tag{4}$$

where

$$A^{\psi,\phi}_{l,t}(\mathcal{U})=\frac{1}{t^{\rho_1}l^{\rho_2}}\int_{|\xi|\le t}\int_{|\zeta|\le l}\mathcal{U}(w-\xi,w_{n+1}-\psi(|\xi|),v-\zeta,v_{m+1}-\phi(|\zeta|))K_{\Theta,g}(\xi,\zeta)d\xi d\zeta.$$

Under various assumptions on the mappings ψ, ϕ, Θ, and g, the operator $\mathcal{M}^{\psi,\phi}_{\Theta,g}$ was studied by many authors (see [18-24]).

Very recently, the authors of [25] discussed the operator $\mathcal{M}_{\Theta,\Lambda,g}$ for several classes of Λ. In fact, they proved its boundedness on $L^p(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})$ for all $|1/2-1/p|<\min\{1/\kappa',1/2\}$ provided that $\Theta\in L(\log L)(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})\cup B^{(0,0)}_q(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$ and $g\in\nabla_\kappa(\mathbb{R}_+\times\mathbb{R}_+)$ with $\kappa>1$. For more information as well as a sample of past studies regarding the development and applications of the operator $\mathcal{M}_{\Theta,\Lambda,g}$, we refer the readers to see [26-30] and their references.

In this paper, we study the operator $\mathcal{M}_{\Theta,\Lambda,g}$ whenever the mapping Λ belongs to a new class differs from those in [25]. In fact, we assume that $\Lambda(l,t)=f(lt)$, where $f\in C^1(\mathbb{R}_+)$, f' is convex and increasing function with $f'(0)=0$.

The main results of this work are the following:

Theorem 1.1 *Let $\Lambda(l,t)=f(t\ell)$, where f in $C^1(\mathbb{R}_+)$ and f' is increasing and convex function with $f'(0)=0$. Suppose that $g\in\nabla_\kappa(\mathbb{R}_+\times\mathbb{R}_+)$ for some $\kappa>1$ and $\Theta\in L^q\left(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}\right)$ for some $q\in(1,2]$. Then, there is a bounded real number $C_p>0$ such that*

$$\left\|\mathcal{M}_{\Theta,\Lambda,g}(\mathcal{U})\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})}\le C_p\frac{\kappa}{(\kappa-1)(q-1)}\|\mathcal{U}\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})}\|\Theta\|_{L^q(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})}\|g\|_{\nabla_\kappa(\mathbb{R}_+\times\mathbb{R}_+)} \tag{5}$$

for all $|1/2-1/p|<\min\{1/\kappa',1/2\}$.

The estimate (5) along with Yano's extrapolation approach (see [16, 31]) lead to the following result:

Theorem 1.2 *Let Θ satisfy the condition* (1). *Assume that g and Λ are given as in Theorem 1.1.*

1. If $\Theta\in B^{(0,0)}_q(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$ with $q>1$, then the estimate

$$\left\|\mathcal{M}_{\Theta,\Lambda,g}(\mathcal{U})\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})}\le C_p\|g\|_{\nabla_\kappa(\mathbb{R}_+\times\mathbb{R}_+)}\|\mathcal{U}\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})}\left(1+\|\Theta\|_{B^{(0,0)}_q(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})}\right)$$

   holds for $|1/2-1/p|<\min\{1/\kappa'1/2\}$;

1. If $\Theta\in L(\log L)(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$, then the estimate

$$\left\|\mathcal{M}_{\Theta,\Lambda,g}(\mathcal{U})\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})}\le C_p\|g\|_{\nabla_\kappa(\mathbb{R}_+\times\mathbb{R}_+)}\|\mathcal{U}\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})}\left(1+\|\Theta\|_{L(logL)(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})}\right)$$

   holds for all $|1/p-1/2|<\min\{1/2,1/\kappa'\}$.

Remark

1. The assumptions on Θ in Theorem 1.2 are the weakest assumptions in their particular classes. In fact, they are optimal (see [12, 14]).

2. The authors of [30] proved the L^p $(1<p<\infty)$ boundedness of $\mathcal{M}_{\Theta,0,1}$ whenever $\Theta\in L^q\left(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}\right)$ with $q>1$. Hence, since $L^q\left(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}\right)\subset B_q^{(0,0)}(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})\cup L(\log L)(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$, then Theorem 1.2 extends and improves the results in [30].
3. When we consider the case $g\in\nabla_\kappa(\mathbb{R}_+\times\mathbb{R}_+)$ with $\kappa>2$, we obtain the boundedness of $\mathcal{M}_{\Theta,\Lambda,g}$ for the full range of $p\in(1,\infty)$.
4. For the special case $\Lambda\equiv 0$, Theorem 1.2 proves that $\mathcal{M}_{\Theta,\Lambda,g}$ is bounded on $L^p(\mathbb{R}^n\times\mathbb{R}^m)$ for $|1/2-1/p|<\min\{1/\kappa',1/2\}$, which is the main finding in [17]. Hence, our results fundamentally improve the main results in [17].
5. The surfaces of revolutions $\Gamma_\Lambda(\xi,\zeta)=\left(\xi,\zeta,\Lambda(|\xi|,|\zeta|)\right)$ considered in Theorems 1.1 and 1.2 cover various substantial natural classical surfaces as $\Lambda(l,t)=(lt)^m$ with $m>0$, $\Lambda(l,t)=(lt)^2\ln(1+lt)$ and $\Lambda(l,t)=e^{lt}-lt-1$.

2. Preliminary Lemmas

In this section, we establish some auxiliary lemmas which will be needed to prove the main results. For $\nu\geq 2$ and a suitable mapping Λ on $\mathbb{R}_+\times\mathbb{R}_+$, we define the family of measures $\{\lambda_{\Theta,\Lambda,g,l,t}:=\lambda_{l,t}:l,t\in\mathbb{R}_+\}$ and its corresponding maximal operators λ_g^* and $\mathrm{M}_{g,\nu}$ on $\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}$ by

$$\iiint_{\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}}\mathcal{U}d\lambda_{l,t}=\frac{1}{l^{\rho_1}t^{\rho_2}}\int_{1/2t\leq|\zeta|\leq t}\int_{1/2l\leq|\xi|\leq l}\mathcal{U}(\xi,\zeta,\Lambda(|\xi|,|\zeta|))K_{\Theta,g}(\xi,\zeta)d\xi d\zeta,$$

$$\lambda_g^*(\mathcal{U})=\sup_{l,t\in\mathbb{R}_+}||\lambda_{l,t}|*\mathcal{U}|,$$

and

$$\mathrm{M}_{g,\nu}(\mathcal{U})=\sup_{j,k\in\mathbb{Z}}\int_{\nu^j}^{\nu^{j+1}}\int_{\nu^k}^{\nu^{k+1}}||\lambda_{l,t}|*\mathcal{U}|\frac{dldt}{lt},$$

where $|\lambda_{l,t}|$ is defined similar to $\lambda_{l,t}$ but with replacing Θ by $|\Theta|$ and g by $|g|$.

Let us start this section with the following result which is due to the authors of [25].

Lemma 2.1 *Let $\Theta\in L^q\left(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}\right)$ with $q>1$ be a homogeneous function of degree zero and satisfy (1). Suppose that $\nu\geq 2$, $\Lambda\in C^1(\mathbb{R}_+\times\mathbb{R}_+)$ and $g\in\nabla_\kappa\left(\mathbb{R}_+\times\mathbb{R}_+\right)$ with $\kappa>1$. Then for all $j,k\in\mathbb{Z}$, a positive constant C exists such that*

$$\left\|\lambda_{l,t}\right\|\leq C_{g,\Theta},\tag{6}$$

$$\int_{\nu^j}^{\nu^{j+1}}\int_{\nu^k}^{\nu^{k+1}}\left|\hat{\lambda}_{l,t}(x,y,z)\right|^2\frac{dldt}{lt}\leq C_{g,\Theta}(\ln\nu)^2\left|x\nu^k\right|^{\pm\frac{2\eta}{q'\varepsilon'}}\left|y\nu^j\right|^{\pm\frac{2\eta}{q'\varepsilon'}},\tag{7}$$

where $C_{g,\Theta}=C\|g\|_{\nabla_\kappa(\mathbb{R}_+\times\mathbb{R}_+)}\|\Theta\|_{L^q(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})}$, $0<\eta<\frac{1}{2q'}$, $\varepsilon=\max\{2,\kappa'\}$, and $\left\|\lambda_{l,t}\right\|$ is the total variation of $\lambda_{l,t}$.

The next lemma plays a key role in proving our main results.

Lemma 2.2 *Let g, Λ and Θ be given as in Theorem 1.1. Then, there exists $C_p>0$ such that*

$$\|\lambda_h^*(\mathcal{U})\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})}\leq\|\mathcal{U}\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})}C_{p,g,\Theta}\tag{8}$$

and

$$\| \mathrm{M}_{g,\nu}(\mathcal{U}) \|_{L^p(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})} \le C_{p,g,\Theta}(\ln\nu)^2 \| \mathcal{U} \|_{L^p(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})} \tag{9}$$

for $p \in (\kappa', \infty)$.

Proof. It is clear that Hölder's inequality gives

$$\begin{aligned} | \, | \lambda_{l,t} | * \mathcal{U}(w,v,s) |^{\kappa'} &\le C \|\Theta\|_{L^1(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})} \| g \|^{\kappa'}_{\nabla_\kappa(\mathbb{R}_+\times\mathbb{R}_+)} \\ &\times \frac{1}{lt} \int_{t/2}^{t} \int_{l/2}^{l} \int_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}} \left| \mathcal{U}(w-l\xi, v-t\zeta, s-\Lambda(l,t)) \right|^{\kappa'} | \Theta(\xi,\zeta) | \, d\rho_n(\xi) d\rho_m(\zeta) dl dt, \end{aligned}$$

which leads, by Minkowski's inequality for integrals, to

$$\begin{aligned} \| \lambda_g^*(\mathcal{U}) \|_{L^p(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})} &\le C \|\Theta\|^{1/\kappa'}_{L^1(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})} \| g \|_{\nabla_\kappa(\mathbb{R}_+\times\mathbb{R}_+)} \\ &\times \left(\| \sigma_\Lambda^*(|\mathcal{U}|^{\kappa'}) \|_{L^{(p/\kappa')}(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})} \right)^{1/\kappa'}, \end{aligned}$$

where

$$\iiint_{\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}} \mathcal{U} d\sigma_{l,t} = \frac{1}{l^{\rho_1} t^{\rho_2}} \int_{1/2t\le|\zeta|\le t} \int_{1/2l\le|\xi|\le l} \mathcal{U}(\xi,\zeta,\Lambda(|\xi|,|\zeta|)) \frac{\Theta(\xi,\zeta)}{|\xi|^{n-\rho_1} |\zeta|^{m-\rho_2}} d\xi d\zeta$$

and

$$\sigma_\Lambda^*(\mathcal{U}) = \sup_{l,t\in\mathbb{R}_+} | \, | \sigma_{l,t} | * \mathcal{U} |.$$

Hence, to prove this lemma, it is enough to show that for any $p > 1$,

$$\| \sigma_\Lambda^*(\mathcal{U}) \|_{L^p(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})} \le C \|\Theta\|_{L^q(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})} \| \mathcal{U} \|_{L^p(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})}. \tag{10}$$

By the arguments employed in the proof of [Lemma 1, [25]], we get for $(x,y,z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$,

$$\left| \hat{\sigma}_{l,t}(x,y,z) \right| \le C \|\Theta\|_{L^q(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})} \left| 0.05inxl \right|^{-\frac{\eta}{q'\varepsilon'}} \left| 0.05inyt \right|^{-\frac{\eta}{q'\varepsilon'}};$$

$$\left| \hat{\sigma}_{l,t}(x,y,z) - \hat{\sigma}_{l,t}(0,y,z) \right| \le C \|\Theta\|_{L^q(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})} \left| 0.05inxl \right|^{\frac{\eta}{q'\varepsilon'}} \left| 0.05inyt \right|^{-\frac{\eta}{q'\varepsilon'}};$$

$$\left| \hat{\sigma}_{l,t}(x,y,z) - \hat{\sigma}_{l,t}(x,0,z) \right| \le C \|\Theta\|_{L^q(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})} \left| 0.05inxl \right|^{-\frac{\eta}{q'\varepsilon'}} \left| 0.05inyt \right|^{\frac{\eta}{q'\varepsilon'}};$$

and

$$\begin{aligned} &\left| \hat{\sigma}_{l,t}(x,y,z) - \hat{\sigma}_{l,t}(0,y,z) - \hat{\sigma}_{l,t}(x,0,z) + \hat{\sigma}_{l,t}(0,0,z) \right| \\ &\le C \|\Theta\|_{L^q(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})} \left| 0.05inxt \right|^{\frac{\eta}{q'\varepsilon'}} \left| 0.05inyt \right|^{\frac{\eta}{q'\varepsilon'}}. \end{aligned}$$

Let $\Psi^{(1)} \in \mathcal{S}\left(\mathbb{R}^n\right)$ and $\Psi^{(2)} \in \mathcal{S}\left(\mathbb{R}^m\right)$ be two Schwartz functions such that $\widehat{\Psi^{(1)}}\left(x\right)=1$ for $\left|x\right| \le \frac{1}{2}$, $(\widehat{\Psi^{(1)}})\left(x\right)=0$ for $\left|x\right| \ge 1$, $\widehat{\Psi^{(2)}}\left(y\right)=1$ for $\left|y\right| \le \frac{1}{2}$, and $(\widehat{\Psi^{(2)}})\left(y\right)=0$ for $\left|y\right| \ge 1$. For $l,t \in \mathbb{R}^+$, let $\widehat{\Psi_{1,l}}(x)=\widehat{\Psi^{(1)}}\left(lx\right)$ and $\widehat{\Psi_{2,t}}(y)=\widehat{\Psi^{(2)}}\left(ty\right)$. Define the sequence of measures $\left\{\vartheta_{l,t}\right\}$ by

$$\hat{\vartheta}_{l,t}(x,y,z)=\hat{\sigma}_{l,t}(x,y,z)-\widehat{\Psi_{1,l}}\left(x\right)\hat{\sigma}_{l,t}(0,y,z)-\widehat{\Psi_{2,t}}\left(y\right)\hat{\sigma}_{l,t}(x,0,z)+\widehat{\Psi_{1,l}}\left(x\right)\widehat{\Psi_{2,t}}\left(y\right)\hat{\sigma}_{l,t}\left(0,0,z\right). \tag{11}$$

Hence, by a standard argument we obtain that

$$\left|\hat{\vartheta}_{l,t}(x,y,z)\right| \le C\left\|\Theta\right\|_{L^q(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})}\left|0.05inxl\right|^{\pm\frac{\eta}{q'\varepsilon'}}\left|0.05inyt\right|^{\pm\frac{\eta}{q'\varepsilon'}}. \tag{12}$$

Set

$$V(\mathcal{U})(w,v,s)=\left(\int_{\mathbb{R}}\left|\vartheta_{l,t}*\mathcal{U}\,(w,v,s)\right|^2\right)^{\frac{1}{2}},\ \vartheta^*\left(\mathcal{U}\right)=\sup_{l,t\in\mathbb{R}}\left\|\vartheta_{l,t}\right|*\mathcal{U}\right|,0.05in$$

$$\sigma^{(1)}_{l,t}\mathcal{U}(w,v,s)=\sup_{l,t\in\mathbb{R}_+}\int_{t/2\le|\zeta|<0.05int}\left(\int_{l/2}^{l}\left|\mathcal{U}(w,v-\zeta,s-f(lt))\right|\Theta_2\left(\zeta\right)\right)\frac{dl}{l}d\zeta,$$

$$\sigma^{(2)}_{l,t}\mathcal{U}(w,v,s)=\sup_{l,t\in\mathbb{R}_+}\int_{l/2\le|\xi|<0.05inl}\left(\int_{t/2}^{t}\left|\mathcal{U}(w-\xi,v,s-f(lt))\right|\Theta_1\left(\xi\right)\right)\frac{dt}{t}d\xi,$$

$$\sigma^{(3)}_{l,t}\mathcal{U}(w,v,s)=\left\|\Theta\right\|_{L^q(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})}\sup_{l,t\in\mathbb{R}_+}\int_{t/2}^{t}\int_{l/2}^{l}\left|\mathcal{U}(w,v,s-f(lt))\right|\frac{dldt}{lt},$$

where

$$\Theta_1\left(\xi\right)=0.05in\int_{\mathbb{S}^{m-1}}\left|\Theta\left(\xi,\zeta\right)\right|d\sigma_m\left(\zeta\right)\ and\ \Theta_2\left(\zeta\right)=0.05in\int_{\mathbb{S}^{n-1}}\left|\Theta\left(\xi,\zeta\right)\right|d\sigma_n\left(\xi\right).$$

We notice that $\Theta_1 \in L^q(\mathbb{S}^{n-1})$ and $\Theta_2 \in L^q(\mathbb{S}^{m-1})$. Hence,

$$\begin{aligned}\vartheta^*\left(\mathcal{U}\right)(w,v,s) \le V\left(\mathcal{U}\right)(w,v,s)&+C\left((\mathcal{M}_{\mathbb{R}^n}\otimes id_{\mathbb{R}^m}\otimes id_{\mathbb{R}^1})\circ\sigma^{(1)}_{l,t}\right)(\mathcal{U})(w,v,s)\\&+C\left(id_{\mathbb{R}^n}\otimes\mathcal{M}_{\mathbb{R}^m}\otimes id_{\mathbb{R}^1})\circ\sigma^{(2)}_{l,t}\right)(\mathcal{U})(w,v,s)\\&+C\left(\mathcal{M}_{\mathbb{R}^n}\otimes\mathcal{M}_{\mathbb{R}^m}\otimes id_{\mathbb{R}^1})\circ\sigma^{(3)}_{l,t}\right)(\mathcal{U})(w,v,s)\end{aligned} \tag{13}$$

and

$$\begin{aligned}\sigma^*_{\Lambda}\left(\mathcal{U}\right)(w,v,s) \le V\left(\mathcal{U}\right)(w,v,s)&+2C\left((\mathcal{M}_{\mathbb{R}^n}\otimes id_{\mathbb{R}^m}\otimes id_{\mathbb{R}^1})\circ\sigma^{(1)}_{l,t}\right)(\mathcal{U})(w,v,s)\\&+2C\left(id_{\mathbb{R}^n}\otimes\mathcal{M}_{\mathbb{R}^m}\otimes id_{\mathbb{R}^1})\circ\sigma^{(2)}_{l,t}\right)(\mathcal{U})(w,v,s)\\&+2C\left(\mathcal{M}_{\mathbb{R}^n}\otimes\mathcal{M}_{\mathbb{R}^m}\otimes id_{\mathbb{R}^1})\circ\sigma^{(3)}_{l,t}\right)(\mathcal{U})(w,v,s),\end{aligned} \tag{14}$$

where $\mathcal{M}_{\mathbb{R}^\tau}$ indicates to the Hardy-Littlewood maximal function on $\mathbb{R}^\tau$. Therefore, by (11)-(14), the boundedness of $\mathcal{M}_{\mathbb{R}^\tau}$ and a bootstrapping argument, we obtain (10) which leads to (8). Finally, the proof of (9) comes directly from (8). Consequently, the proof of this lemma is complete.

Lemma 2.3 *Suppose that* $\nu \geq 2$, $g \in \nabla_{\kappa}\left(\mathbb{R}_{+} \times \mathbb{R}_{+}\right)$ *with* $\kappa > 1$, $\Theta \in L^{q}\left(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}\right)$ *with* $1 < q \leq 2$, *and* Λ *is given as in Theorem 1.1. Then, for any set of functions* $\{\mathcal{H}_{k,j}(\cdot,\cdot,\cdot), j,k \in \mathbb{Z}\}$ *on* $\mathbb{R}^{n} \times \mathbb{R}^{m} \times \mathbb{R}$, *we have*

$$\left\| \left(\sum_{j,k\in\mathbb{Z}} \int_{\nu^{j}}^{\nu^{j+1}} \int_{\nu^{k}}^{\nu^{k+1}} \left| \lambda_{l,t} * \mathcal{H}_{j,k} \right|^{2} \frac{dldt}{lt} \right)^{1/2} \right\|_{L^{p}(\mathbb{R}^{n}\times\mathbb{R}^{m}\times\mathbb{R})} \leq C_{p,g,\theta}(\ln \nu) \left\| \left(\sum_{j,k\in\mathbb{Z}} \left| \mathcal{H}_{j,k} \right|^{2} \right)^{1/2} \right\|_{L^{p}(\mathbb{R}^{n}\times\mathbb{R}^{m}\times\mathbb{R})} \tag{15}$$

for all $|1/2 - 1/p| < \min\{1,/2, 1/\kappa'\}$.

Proof. We point out that for $\kappa \geq 2$, $\nabla_{2}\left(\mathbb{R}_{+} \times \mathbb{R}_{+}\right) \supseteq \nabla_{\kappa}\left(\mathbb{R}_{+} \times \mathbb{R}_{+}\right)$. So, the proof of this lemma will be given only whenever $\kappa \in (1,2]$. In this case, we have $|1/2 - 1/p| < 1/\kappa'$, which leads to $\frac{2\kappa}{3\kappa - 2} < p < \frac{2\kappa}{2-\kappa}$. If $2 \leq p < \frac{2\kappa}{2-\kappa}$, then by duality, there is a function $\mho$ belongs to the space $L^{(p/2)'}(\mathbb{R}^{n} \times \mathbb{R}^{m} \times \mathbb{R})$ such that $\|\mho\|_{L^{(p/2)'}(\mathbb{R}^{n}\times\mathbb{R}^{m}\times\mathbb{R})} \leq 1$ and

$$\begin{aligned} &\left\| \left(\sum_{j,k\in\mathbb{Z}} \int_{\nu^{j}}^{\nu^{j+1}} \int_{\nu^{k}}^{\nu^{k+1}} \left| \lambda_{l,t} * \mathcal{H}_{j,k} \right|^{2} \frac{dldt}{lt} \right)^{1/2} \right\|_{L^{p}(\mathbb{R}^{n}\times\mathbb{R}^{m}\times\mathbb{R})}^{2} \\ &= \iiint_{\mathbb{R}^{n}\times\mathbb{R}^{m}\times\mathbb{R}} \sum_{j,k\in\mathbb{Z}} \int_{\nu^{j}}^{\nu^{j+1}} \int_{\nu^{k}}^{\nu^{k+1}} \left| \lambda_{l,t} * \mathcal{H}_{j,k}(w,v,s) \right|^{2} \frac{dldt}{lt} \left| \mho(w,v,s) \right| dwdvds. \end{aligned}$$

Thanks to Schwartz's inequality, we have

$$\begin{aligned} \left| \lambda_{l,t} * \mathcal{H}_{j,k}(w,v,s) \right|^{2} &\leq C \|\Theta\|_{L^{q}(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})} \|g\|_{\nabla_{\kappa}(\mathbb{R}_{+}\times\mathbb{R}_{+})}^{\kappa} \int_{\frac{1}{2}t}^{t} \int_{\frac{1}{2}l}^{l} \iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}} \left| \Theta(\xi,\zeta) \right| \\ &\times \left| \mathcal{H}_{j,k}(u - l\xi, v - t\zeta, s - \Lambda(l,t)) \right|^{2} \left| g(l,t) \right|^{2-\kappa} d\sigma_{n}(\xi) d\sigma_{m}(\zeta) \frac{dldt}{lt}, \end{aligned}$$

which leads by Hölder's inequality to

$$\begin{aligned} &\left\| \left(\sum_{j,k\in\mathbb{Z}} \int_{\nu^{j}}^{\nu^{j+1}} \int_{\nu^{k}}^{\nu^{k+1}} \left| \lambda_{l,t} * \mathcal{H}_{j,k} \right|^{2} \frac{dldt}{lt} \right)^{1/2} \right\|_{L^{p}(\mathbb{R}^{n}\times\mathbb{R}^{m}\times\mathbb{R})}^{2} \leq C \|\Theta\|_{L^{q}(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})} \\ &\times \|g\|_{\nabla_{\kappa}(\mathbb{R}_{+}\times\mathbb{R}_{+})}^{\kappa} \left\| \sum_{j,k\in\mathbb{Z}} \left| \mathcal{H}_{j,k} \right|^{2} \right\|_{L^{(p/2)}(\mathbb{R}^{n}\times\mathbb{R}^{m}\times\mathbb{R})} \left\| \mathrm{M}_{|g|^{2-\kappa},\nu}(\tilde{\mho}) \right\|_{L^{(p/2)'}(\mathbb{R}^{n}\times\mathbb{R}^{m}\times\mathbb{R})} \\ &\leq C(\ln \nu)^{2} \|\Theta\|_{L^{q}(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})} \|g\|_{\nabla_{\kappa}(\mathbb{R}_{+}\times\mathbb{R}_{+})}^{\kappa} \left\| \left(\sum_{j,k\in\mathbb{Z}} \left| \mathcal{H}_{j,k} \right|^{2} \right)^{1/2} \right\|_{L^{p}(\mathbb{R}^{n}\times\mathbb{R}^{m}\times\mathbb{R})}^{2} \times \left\| \lambda^{*}_{|g|^{2-\kappa}}(\tilde{\mho}) \right\|_{L^{(p/2)'}(\mathbb{R}^{n}\times\mathbb{R}^{m}\times\mathbb{R})} \\ &\leq C_{p,g,\Theta}^{2} (\ln \nu)^{2} \left\| \left(\sum_{j,k\in\mathbb{Z}} \left| \mathcal{H}_{j,k} \right|^{2} \right)^{1/2} \right\|_{L^{p}(\mathbb{R}^{n}\times\mathbb{R}^{m}\times\mathbb{R})}^{2}, \end{aligned}$$

where $\tilde{\mho}(-w,-v,-s) = \mho(w,v,s)$. The last inequality is obtained by employing Lemma 2.2 with $|g|^{2-\kappa} \in \nabla_{\frac{\kappa}{2-\kappa}}(\mathbb{R}_{+} \times \mathbb{R}_{+})$.

Now, if $\frac{2\kappa}{3\kappa-2}<p<2$, then we deduce by duality that there is a collection of functions $f_{j,k}(w,v,s,l,t)$ on $\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+$ which satisfies

$$\left\|\left\|\| f_{j,k}\|_{L^2([\nu^k,\nu^{k+1}]\times[\nu^j,\nu^{j+1}],\frac{dldt}{lt})}\right\|_{l^2}\right\|_{L^{p'}(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})}\leq 1$$

and

$$\begin{aligned}&\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\nu^j}^{\nu^{j+1}}\int_{\nu^k}^{\nu^{k+1}}\left|\lambda_{l,t}*\mathcal{H}_{j,k}\right|^2\frac{dldt}{lt}\right)^{1/2}\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})}\\&=\iiint_{\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}}\sum_{j,k\in\mathbb{Z}}\int_{\nu^j}^{\nu^{j+1}}\int_{\nu^k}^{\nu^{k+1}}\left(\lambda_{l,t}*\mathcal{H}_{j,k}(w,v,s)\right)f_{j,k}(w,v,s,l,t)\frac{dldt}{lt}dwdvds\\&\leq C_p(\ln\nu)\left\|\mathcal{Q}(f_{j,k})\right\|^{1/2}_{L^{(p'/2)}(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})}\left\|\left(\sum_{j,k\in\mathbb{Z}}\left|\mathcal{H}_{j,k}\right|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})},\end{aligned}\tag{16}$$

where

$$\mathcal{Q}(f_{j,k})(w,v,s)=\sum_{j,k\in\mathbb{Z}}\int_{\nu^j}^{\nu^{j+1}}\int_{\nu^k}^{\nu^{k+1}}\left|\lambda_{l,t}*f_{j,k}(w,v,s,l,t)\right|^2\frac{dldt}{lt}.$$

As $p'>2$, the duality gives that there is a function $\mathcal{B}$ lies in $L^{(p'/2)'}(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})$, which satisfies $\|\mathcal{B}\|_{L^{(p'/2)'}(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})}\leq 1$ and

$$\begin{aligned}&\left\|\mathcal{Q}(f_{j,k})\right\|_{L^{(p'/2)}(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})}\\&=\sum_{j,k\in\mathbb{Z}}\iiint_{\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}}\int_{\nu^j}^{\nu^{j+1}}\int_{\nu^k}^{\nu^{k+1}}\left|\lambda_{l,t}*f_{j,k}(w,v,s,l,t)\right|^2\frac{dldt}{lt}\mathcal{B}(w,v,s)dwdvds\\&\leq C\left\|\Theta\right\|_{L^q(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})}\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\nu^j}^{\nu^{j+1}}\int_{\nu^k}^{\nu^{k+1}}\left|f_{j,k}(w,v,s,l,t)\right|^2\frac{dldt}{lt}\right)\right\|_{L^{(p'/2)}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R})}\\&\times\|g\|^{\kappa}_{\nabla_\kappa(\mathbb{R}_+\times\mathbb{R}_+)}\left\|\lambda^*_{|g|^{2-\kappa}}(\mathcal{B})\right\|_{L^{(p'/2)'}(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})}\leq C^2_{g,\Theta}.\end{aligned}\tag{17}$$

Therefore, the last inequality and (16) yield (15) for $\frac{2\kappa}{3\kappa-2}<p<2$. This completes the proof of this lemma.

3. Proof of main results

Let $\Theta\in L^q\left(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}\right)$ for some $1<q\leq 2$ and $g\in\nabla_\kappa\left(\mathbb{R}_+\times\mathbb{R}_+\right)$ for some $\kappa>1$. Set $\nu=2^{q'\kappa'}$. By Minkowski's inequality, we obtain

$$\begin{aligned}&\mathcal{M}_{\Theta,\Lambda,g}(\mathcal{U})(w,v,s)\\&\leq \sum_{j,k=0}^{\infty}\left(\iint_{\mathbb{R}_+\times\mathbb{R}_+}\left|\frac{1}{l^{\rho_1}t^{\rho_2}}\int_{2^{-j-1}t<|\zeta|\leq 2^{-j}t}\int_{2^{-k-1}l<|\xi|\leq 2^{-k}l}K_{\Theta,g}(\xi,\zeta)\right.\right.\\&\left.\left.\times\mathcal{U}(w-\xi,v-\zeta,s-\Lambda(|\xi|,|\zeta|))d\xi d\zeta\right|^2\frac{dldt}{lt}\right)^{1/2}\\&\leq \frac{2^{c_1+c_2}}{(2^{c_1}-1)(2^{c_2}-1)}\left(\iint_{\mathbb{R}_+\times\mathbb{R}_+}\left|\lambda_{l,t}*\mathcal{U}(w,v,s)\right|^2\frac{dldt}{lt}\right)^{1/2}.\end{aligned}\tag{18}$$

For $\alpha\in\mathbb{Z}$, let $\{\Phi_\alpha\}$ be a set of smooth partition of unity over $(0,\infty)$, which is adapted to the interval $[\nu^{-\alpha-1},\nu^{-\alpha+1}]\equiv\mathcal{I}_\alpha$ and satisfyes the following:

$$\Phi_\alpha\in C^\infty,\ 0\leq\Phi_\alpha\leq 1,\ \sum_{\alpha\in\mathbb{Z}}\Phi_\alpha(l)=1,$$

$$supp\ (\Phi_\alpha)\subseteq\mathcal{I}_\alpha\ and\ \left|\frac{d^\beta\Phi_\alpha(l)}{dl^\beta}\right|\leq\frac{C_\beta}{l^\beta},$$

where C_β does not depend on the lacunary sequence $\{\nu^\beta;\beta\in\mathbb{Z}\}$.

Define the multiplier operator $\{\Psi_{j,k}\}$ in $\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}$ by $(\widehat{\Psi_{j,k}(\mathcal{U})})(x,y,z)=\Phi_j(|x|)\Phi_k(|y|)\hat{\mathcal{U}}(x,y,z)$. Thus, for any $\mathcal{U}\in C_0^\infty(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})$, Minkowski's inequality gives that

$$\left(\iint_{\mathbb{R}_+\times\mathbb{R}_+}\left|\lambda_{l,t}*\mathcal{U}(w,v,s)\right|^2\frac{dldt}{lt}\right)^{1/2}\leq C\sum_{r,\mu\in\mathbb{Z}}\mathcal{F}_{\mu,r}(\mathcal{U})(w,v,s),\tag{19}$$

where

$$\mathcal{F}_{\mu,r}(\mathcal{U})(w,v,s)=\left(\iint_{\mathbb{R}_+\times\mathbb{R}_+}\left|\mathcal{J}_{\mu,r}(\mathcal{U})(w,v,s,l,t)\right|^2\frac{dldt}{lt}\right)^{1/2},$$

$$\mathcal{J}_{\mu,r}(\mathcal{U})(w,v,s,l,t)=\sum_{j,k\in\mathbb{Z}}\lambda_{l,t}*\Psi_{j+\mu,k+r}*\mathcal{U}(w,v,s)\chi_{_{\nu^k,\nu^{k+1})\times\nu^j,\nu^{j+1})}}(l,t).$$

To prove Theorem 1.1, it suffices to show that a real number $\varepsilon>0$ exists such that

$$\left\|\mathcal{F}_{\mu,r}(\mathcal{U})\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R})}\leq C_{p,g,\Theta}(\ln\nu)2^{-\frac{\varepsilon}{2}(|r|+|\mu|)}\left\|\mathcal{U}\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})}.\tag{20}$$

for all $|1/p-1/2|<\min\{1/\kappa',1/2\}$.

First, we estimate the L^2-norm of $\mathcal{F}_{\mu,r}(\mathcal{U})$ as follows: Parseval–Plancherel identity, Fubini's Theorem and Lemma 2.1 produce

$$\begin{aligned}&\left\|\mathcal{F}_{\mu,r}(\mathcal{U})\right\|^2_{L^2(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})}\\&\leq\sum_{j,k\in\mathbb{Z}}\iiint_{D_{j+\mu,k+r}}\left(\int_{\nu^j}^{\nu^{j+1}}\int_{\nu^k}^{\nu^{k+1}}\left|\hat{\lambda}_{l,t}(x,y,z)\right|^2\frac{dldt}{lt}\right)\left|\hat{\mathcal{U}}(x,y,z)\right|^2dxdydz\\&\leq C^2_{p,g,\Theta}(\ln\nu)^2\sum_{j,k\in\mathbb{Z}}\iiint_{D_{j+\mu,k+r}}\left|x\nu^k\right|^{\pm\frac{2\eta}{q'\varepsilon'}}\left|y\nu^j\right|^{\pm\frac{2\eta}{q'\varepsilon'}}\left|\hat{\mathcal{U}}(x,y,z)\right|^2dxdydz\\&\leq C^2_{p,g,\Theta}(\ln\nu)^2 2^{-\varepsilon(|r|+|\mu|)}\sum_{j,k\in\mathbb{Z}}\iiint_{D_{j+s,k+r}}\left|\hat{\mathcal{U}}(x,y,z)\right|^2dxdydz\\&\leq C^2_{p,g,\Theta}(\ln\nu)^2 2^{-\varepsilon(|r|+|\mu|)}\left\|\mathcal{U}\right\|^2_{L^2(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})},\end{aligned}\tag{21}$$

where $\varepsilon \in (0,1)$ and $D_{j,k} = \left\{(x,y,z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} : (|x|,|y|) \in \mathcal{I}_j \times \mathcal{I}_k\right\}$. Next, we estimate the L^p-norm of $\mathcal{F}_{\mu,r}(\mathcal{U})$ as follows: By employing Lemma 2.3 and Littlewood–Paley theory, we obtain

$$\begin{aligned}
&\left\|\mathcal{F}_{\mu,r}(\mathcal{U})\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})} \\
&\le C\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\nu^j}^{\nu^{j+1}}\int_{\nu^k}^{\nu^{k+1}}\left(\left|\lambda_{l,t} * \Psi_{j+\mu,k+r} * \mathcal{U}\right|\right)^2 \frac{dldt}{lt}\right)^{1/2}\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})} \\
&\le C_{p,g,\Theta}(\ln\nu)\left\|\left(\sum_{j,k\in\mathbb{Z}}\left|\Psi_{j+\mu,k+r} * \mathcal{U}\right|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})} \\
&\le C_{p,g,\Theta}\,\frac{\kappa}{(\kappa-1)(q-1)}\left\|\mathcal{U}\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})}.
\end{aligned} \tag{22}$$

Consequently, interpolate (21) with (22), we get (20), which in turn with (18)-(19) finishes the proof of Theorem 1.1.

4. Conclusions

In this work, we established sharp L^p estimates for the operator $\mathcal{M}_{\Theta,\Lambda,g}$ whenever $g \in \nabla_\kappa(\mathbb{R}_+ \times \mathbb{R}_+)$ with $\kappa > 1$, $\Theta \in L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ with $1 < q \le 2$ and $\Lambda(l,t) = f(t\ell)$, where f is C^1 function, f' is convex and increasing mapping with $f'(0) = 0$. The obtained estimates allows us to utilize the extrapolation argument of Yano to show that $\mathcal{M}_{\Theta,\Lambda,g}$ is still bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R})$ under a weaker assumption on Θ; that is Θ lies either in $B_q^{(0,0)}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ or in $L(\log L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$. These assumptions are considered the weakest among their particular classes. In addition, we obtained L^p boundedness of $\mathcal{M}_{\Theta,\Lambda,g}$ for the full range of $p \in (1,\infty)$ provided that $\kappa > 2$. Our results generalize and improve several known results as the results in [1, 4, 5, 7, 11, 12, 14, 17].

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