



A study of bi-convex classes in Leaf-like domains using quantum calculus through subordination

Abdullah Alsoboh¹, Ala Amourah², Jamal Salah^{3*}

¹Department of Basic and Applied Sciences, College of Applied and Health Sciences, A'Sharqiyah University, Ibra, Sultanate of Oman;

²Mathematics Education Program, Faculty of Education and Arts, Sohar University, Sohar Oman; Jadara University Research Center, Jadara University, Jordan; ³Department of Basic and Applied Sciences, College of Applied and Health Sciences, A'Sharqiyah University, Ibra, Sultanate of Oman.

Abstract

This research explores properties of a bi-convex class of functions that are associated with a leaf-shaped region by utilizes the subordination principle and q -calculus. The study also analyzes limitations on coefficients, with a particular emphasis on $|a_2|$ and $|a_3|$. Furthermore, it evaluates Fekete Szegő inequalities for functions within the bi-convex class. The findings are supported by figures, examples, and references to relevant studies.

Mathematics Subject Classification (2020): 30Cxx

Key words and Phrases: q -calculus leaf-like domain analytic functions bi-univalent functions convex class subordination Fekete-Szegő problem.

1. Introduction and Definitions

Let f be a holomorphic function defined within the open unit disc \mathbb{U} , where $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. A function can be categorised as belonging to a specific class \mathcal{A} if it can be expressed in the following manner:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1)$$

Email addresses: abdullah.alsoboh@asu.edu.om, AAmourah@su.edu.om, dr.jamal@asu.edu.om (Abdullah Alsoboh, Ala Amourah, Jamal Salah)

Furthermore included in the collection \mathcal{S} are univalent functions satisfying the normalising requirements:

$$f(0) = 0 \text{ and } f'(0) = 1. \quad (2)$$

An analysis function \varkappa defined within the domain \mathbb{U} is referred to as a Schwarz function if it meets the conditions $\varkappa(z) < 1$ and $\varkappa(0) = 0$. In the context of a class \mathcal{A} comprising two functions, f and f it is stated that f is influenced by f denoted as $f \prec f$, when there exists a Schwarz function \varkappa that $\varkappa(z) = f(\varkappa(z))$, for every complex number ($z \in \mathbb{U}$).

The class \mathcal{P} is closely related to Carathéodory functions, which are described by Miller [27]. These functions are characterized by the following features:

$$\operatorname{Re}\{\delta(z)\} > 0, \text{ and } \delta(0) = 1 \quad \forall z \in \mathbb{U}.$$

A Maclaurin series expansion provides a way to represent a function $\delta(z)$, within the class \mathcal{P} . It can be written as;

$$\delta(z) = 1 + \sum_{n=1}^{\infty} \delta_n z^n, \quad (z \in \mathbb{U}). \quad (3)$$

In this expression the coefficients δ_n must satisfy the condition;

$$|\delta_n| \leq 2, \text{ for all } n \geq 1. \quad (4)$$

This concept aligns with Carathodory's Lemma, a known result discussed in the reference by Duren [20]. Essentially a function δ belongs to the class \mathcal{P} if and only if it satisfies the condition;

$$\delta(z) \prec \frac{1+z}{1-z}, \quad (z \in \mathbb{U}).$$

For any function f in the category \mathcal{S} , an inverse function denoted by f^{-1} exists. The inverse function is defined as follows:

$$z = f^{-1}(f(z)), \quad \varsigma = f(f^{-1}(\varsigma)) \quad \left(r_0(f) \geq \frac{1}{4}; \quad |\varsigma| < r_0(f); z \in \mathbb{U} \right),$$

where

$$f^{-1}(\varsigma) = g(\varsigma) = \varsigma - \varrho_2 \varsigma^2 + (2a_2^2 - a_3) \varsigma^3 - (a_4 + 5a_2^3 - 5a_3 a_2) \varsigma^4 + \dots. \quad (5)$$

A function $f(z) \in \mathcal{S}$ is called bi-univalent when its inverse function ($f^{-1}(\varsigma)$) is also bi-univalent. The set Σ includes all bi-univalent functions, in \mathbb{U} . Check out the table below for some examples of functions, in the class Σ and their respective inverse functions.

Table 1: Certain univalent functions along with their inverses.

$f_1(z) = \frac{z}{z+1}$	$f_1^{-1}(\varsigma) = \frac{\varsigma}{1-\varsigma}$
$f_2(z) = \frac{e^{2z} - 1}{e^{2z} + 1}$	$f_2^{-1}(\varsigma) = -\log(1 - \varsigma)$
$f_3(z) = \frac{1}{2} \log\left(\frac{1+z}{1-z}\right)$	$f_3^{-1}(\varsigma) = \frac{e^\varsigma - 1}{e^\varsigma}$

Ma and Minda [26] developed the class $K(\aleph)$ by use of subordination. They put up their method as follows:

$$K(\aleph) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \aleph(z), \quad (z \in \mathbb{U}) \right\}.$$

In this formula, $\aleph(z)$ denotes a function with a positive real component and is normalised in accordance with criteria (2). The table below illustrates the diverse methodologies employed by several writers to delineate subclasses of functions by selecting particular forms for \aleph .

Table 2: Certain categories of functions are characterized by subordination.

$\aleph(z)$	Author/s	Reference	Year
$\sqrt{1+z}$	Sokol and Stankiewicz	[33]	1996
$\frac{z}{1-z}$	Piejko and Sokół	[29]	2012
$z + \sqrt{1+z^2}$	Priya and Sharma	[30]	2018
$z + \sqrt[3]{1+z^3}$	Singh and Kaur	[32]	2021

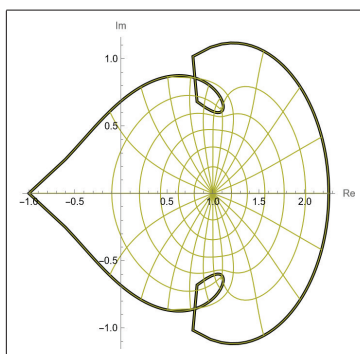


Figure 1: The image of $\aleph(\mathbb{U})$ is displayed in a leaf-shaped region. Here, $\aleph(z;1) = z + \sqrt[3]{1+z^3}$.

Quantum calculus, also known as q -calculus, extends beyond the conventional framework of ordinary calculus by incorporating the parameter $q \in (0,1)$, thereby generalizing classical analytical techniques. This field has garnered significant interest due to its deep connections with physics, quantum mechanics, and Geometric Function Theory (GFT). A foundational resource for understanding q -difference calculus and its diverse applications is the work of Gasper and Rahman [24], which provides a comprehensive exposition on the subject. Central to the study of analytic functions within this framework is the q -difference operator ∂_q , which plays a crucial role in function theory. Notable advancements in this area include the work of Seoudy and Aouf [31], who extended q -calculus to functions within the unit disc, further enriching GFT. For further exploration, numerous classical and contemporary studies provide valuable insights, including [1, 3, 6–13, 17, 18, 25, 28, 35].

Recently, Alsoboh and Oros [7] investigated a particular class of bi-univalent functions connected to the leaf-like domain associated with q -calculus. The study concentrated on the definition that follows:

Definition 1.1: A function $f \in \mathcal{S}$ is considered a member of the class $\Sigma_q(\lambda, \aleph(z;q))$ if it has the form (1) and satisfies the following subordinations:

$$(1-\lambda)\frac{f(z)}{z} + \lambda(\partial_q f(z)) \prec \aleph(z;q), \quad (z \in \mathbb{U}), \quad (6)$$

where $\aleph(0; q) = 1$, $\lambda \geq 1$, and

$$\aleph(z; q) = \frac{(1+q)z}{2+(1-q)z} + \sqrt[3]{1 + \left(\frac{(1+q)z}{2+(1-q)z} \right)^3}, \quad (7)$$

where

$$\partial_q f(z) = \begin{cases} \frac{f(z) - f(qz)}{z - qz}, & \text{if } z \neq 0, \\ f'(z), & \text{if } q \rightarrow 1^-, z \neq 0, \\ f'(0) & \text{if } z = 0 \end{cases}$$

Recent works by Amourah et al. [14] established the starlike class $\mathcal{S}^*(q; \aleph(z; q))$, which pertains to the leaf-like domain within \mathbb{U} . This class is defined as follows:

Definition 1.2: A function $f \in \mathcal{S}$ is considered a member of the class $\mathcal{S}^*(q; \aleph(z; q))$ if it has the form (1) and satisfies the following subordinations:

$$\frac{z \partial_q f(z)}{\psi(z)} \prec \aleph(z; q), \quad (z \in \mathbb{U}), \quad (8)$$

where $\aleph(z; q)$ of the form (7).

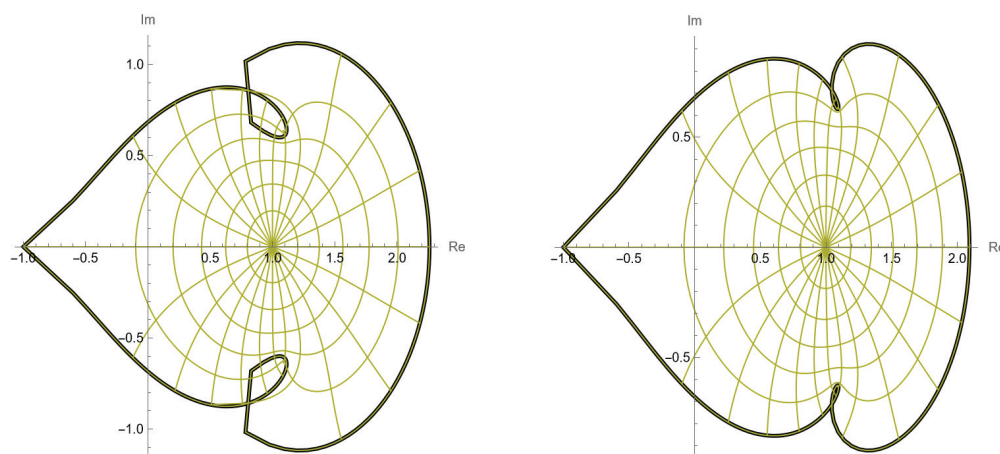
Inspired by the articles of Soboh and Oros [7] and Amourah et al. [14], we apply the discussed ideas and make use of the subordinating idea q -calculus. We thereby derive a bi-convex class connected with the leaf-like domain inside the open unit disc \mathbb{U} .

Definition 1.3: A function $f \in \mathcal{S}$ is considered a member of the class $\mathcal{C}(q; \aleph(z; q))$ if it has the form (1) and satisfies the following subordinations:

$$1 + \frac{z \partial_q^2 f(z)}{\partial_q f(z)} \prec \aleph(z; q), \quad (z \in \mathbb{U}), \quad (9)$$

where $\aleph(z; q)$ of the form (7).

Remark 1.4: The unit disk \mathbb{U} undergoes a transformation into a region with a leaf-like shape using the analytic and univalent function $\aleph(z; q)$. This function exhibits symmetry with respect to the real axis and fulfills the conditions $\aleph(0; q) = \partial_q \aleph(0; q) = 1$.



(a) The image of $\aleph(\mathbb{U}; q)$ with $q \rightarrow 1^-$

(b) The image of $\aleph(\mathbb{U}; 0.89)$

Figure 2: Continues.

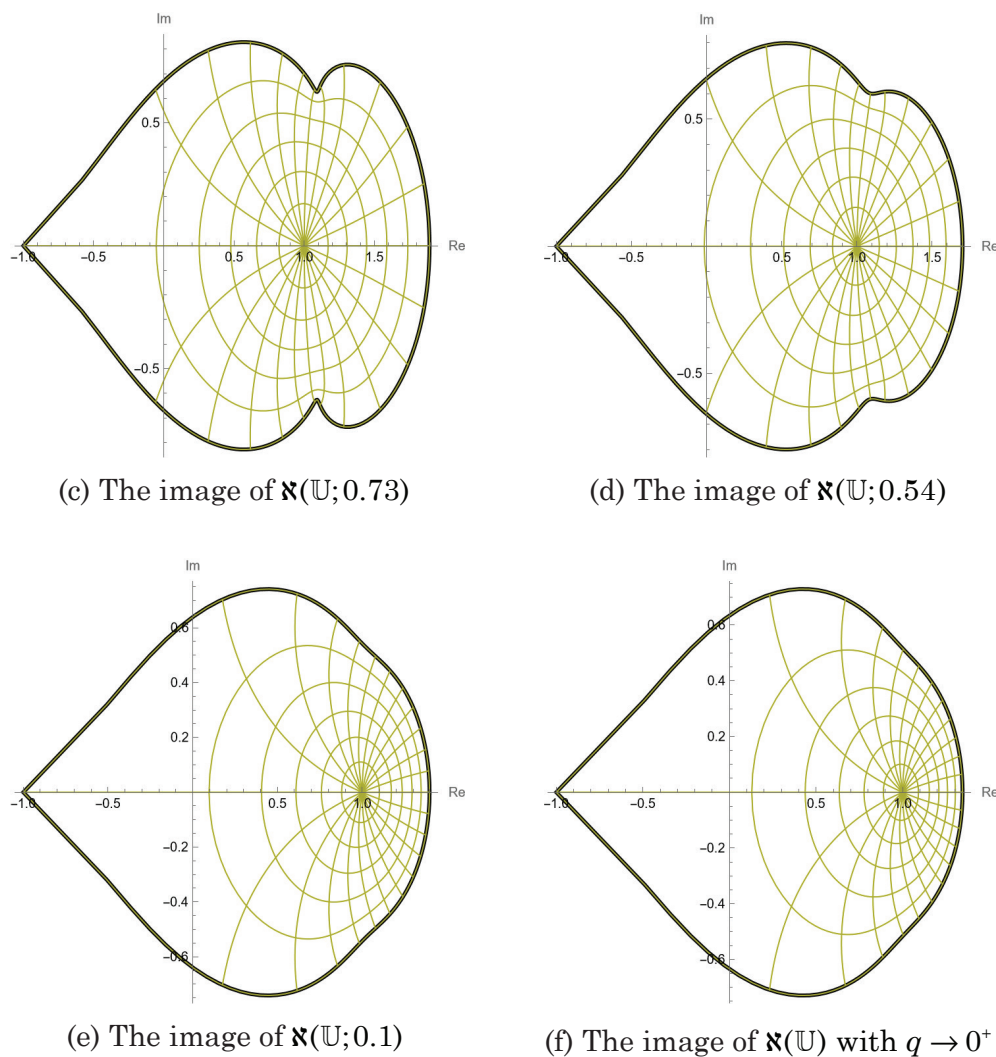


Figure 2: The figure shows the leaf-shaped region $\mathfrak{K}(\mathbb{U}; q)$.

The primary objective of this work is to investigate the properties of bi-univalent functions associated with the leaf-like domain in \mathbb{U} . This section, along with supporting examples, provides a clear definition of the class under consideration. In Section 3, we derive coefficient estimates and analyze the Fekete-Szegő functional for the newly introduced class. Furthermore, leveraging the theorems established in the preceding sections, we present corollaries that align with the specific scenarios under discussion.

2. The Bi-Convex Class $\Sigma_{\kappa}(\mathfrak{K}(z; q))$

In this section, we employ the q -difference operator and the subordination principle between analytic functions, which were previously mentioned, to present a rigorous mathematical characterization of the newly introduced class $\Sigma_{\kappa}(\mathfrak{K}(z; q))$ of bi-univalent functions affiliated with a leaf-like domain.

Definition 2.1: A function f of the type (1) belongs to the class $\Sigma_{\kappa}(\mathfrak{K}(z; q))$ if it satisfies the specified subordination conditions:

$$\Phi(z; q) = 1 + \frac{z \partial_q^2 f(z)}{\partial_q f(z)} \prec \mathfrak{K}(z; q), \quad (z \in \mathbb{U}), \quad (10)$$

and

$$\chi(\varsigma; q) = 1 + \frac{\varsigma \partial_q^2 g(\varsigma)}{\partial_q g(\varsigma)} \prec \aleph(\varsigma; q) \quad (\varsigma \in \mathbb{U}), \quad (11)$$

with $\aleph(z; q)$ of the form (7) and $g = f^{-1}$.

Remark 2.2: We wish to underscore that the class $\Sigma_\kappa(\aleph(z; q))$ is non-empty. To elaborate, let us examine the functions defined by:

$$f_*(z) = \frac{z}{1 - \vartheta z}, \quad |\vartheta| < 0.55. \quad (12)$$

It is evident that $f_* \in \mathcal{S}$ and, furthermore, $f_* \in \Sigma$ together with its inverse:

$$f_*^{-1}(\varsigma) = \frac{\varsigma}{1 + \vartheta \varsigma}, \quad |\vartheta| < 0.55, \quad (13)$$

via means of the notations defined in equations (10) and (11), we may determine, via a basic computation, that:

$$\begin{aligned} \Phi(f_*(z); q) &= \frac{\vartheta}{1 - q^2 \vartheta z} + 1, \quad |\vartheta| < 0.55 \\ \chi(f_*^{-1}(\varsigma); q) &= \frac{\vartheta}{1 + q^2 \vartheta \varsigma} + 1, \quad |\vartheta| < 0.55. \end{aligned} \quad (14)$$

Also, for all $z \in \mathbb{U}$, $\Phi(-\vartheta z; q) = \chi(\vartheta z; q)$, which implies that $\Phi(\mathbb{U}; q) = \chi(\mathbb{U}; q)$.

We utilize GeoGebra Classic 6 to generate visual representations of the boundary denoted as $\partial\mathbb{U}$, employing the functions \aleph and χ , as presented in Table 3. This approach is particularly applicable in situations where the conditions $|\vartheta| \leq 0.55$ and $q \in (0, 1)$ are satisfied. It is essential to acknowledge that \aleph is a univalent function in \mathbb{U} . Consequently, the relationships $\Phi(z; q) \prec \aleph(z; q)$ and $\chi(z; q) \prec \aleph(z; q)$ are valid. These relationships are substantiated by the following facts: $\Phi(0; q) = \chi(0; q) = \aleph(0; q) = 1$, $\Phi(\mathbb{U}; q) \subset \aleph(\mathbb{U}; q)$, and $\chi(\mathbb{U}; q) \subset \aleph(\mathbb{U}; q)$. For further clarification, please refer to Figure 3.

Example 2.3: if $q \rightarrow 1^-$, then $\Sigma_\kappa(\aleph(z; q))$ is reduced to $\Sigma_\kappa\left(z + \sqrt[3]{1 + z^3}\right)$ defined by

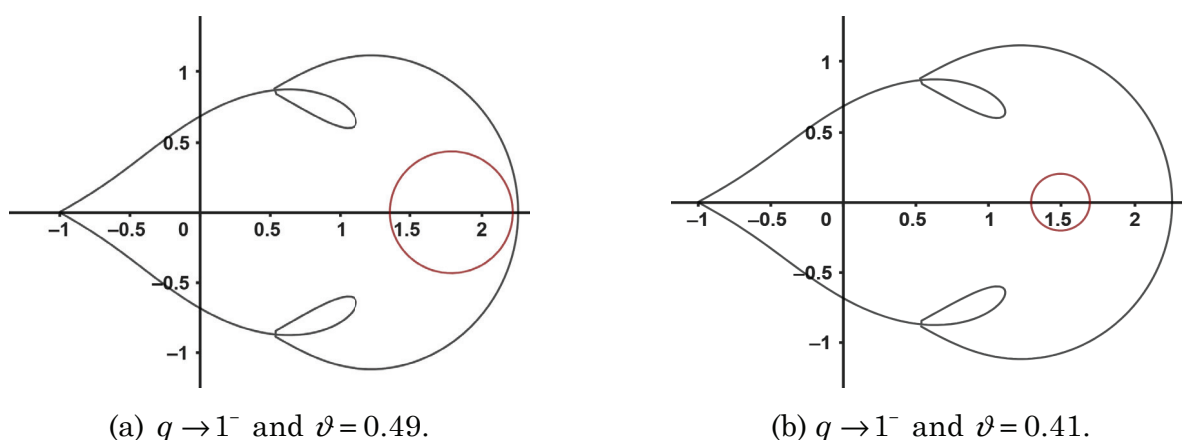


Figure 2: Continues.

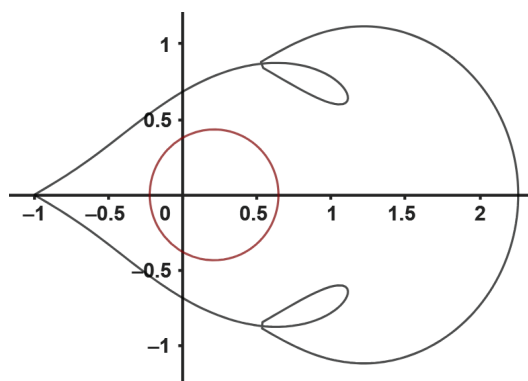
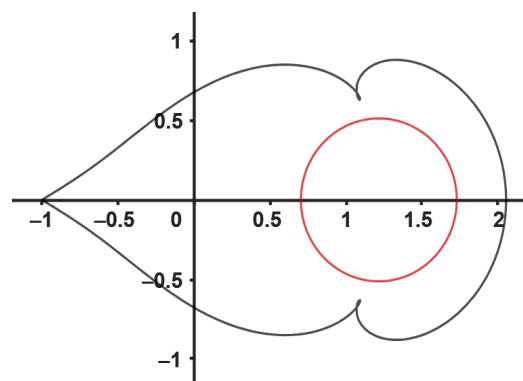
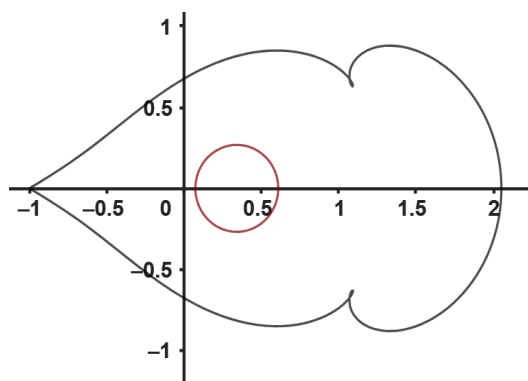
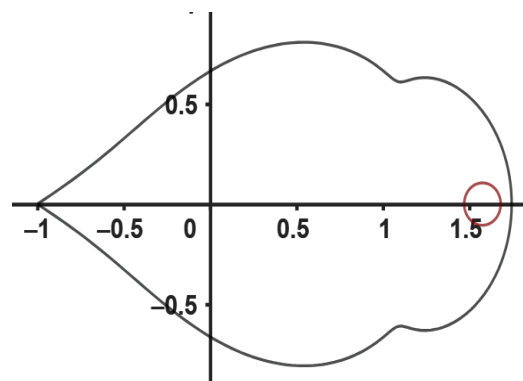
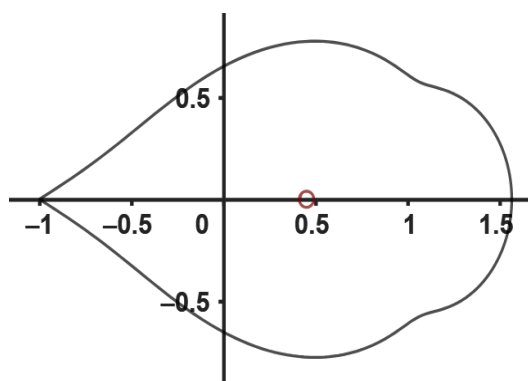
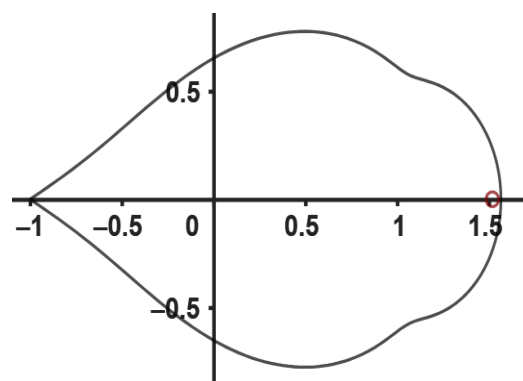

 (c) $q \rightarrow 1^-$ and $\vartheta = -0.55$.

 (d) $q = 0.86$ and $\vartheta = 0.49$.

 (e) $q = 0.86$ and $\vartheta = 0.3$.

 (f) $q = 0.58$ and $\vartheta = 0.49$.

 (g) $q = 0.36$ and $\vartheta = -0.55$.

 (h) $q = 0.36$ and $\vartheta = 0.51$.

 Figure 3: The image of $\mathfrak{K}(e^{i\vartheta}; q)$ (black color) and $\Phi(e^{i\vartheta}; q)$ (red color) with several values of q , ϑ and $\theta \in [0, 2\pi)$.

$$\left\{ f \in \Sigma : \frac{zf''(z)}{f'(z)} + 1 \prec z + \sqrt[3]{1+z^3}, \quad (z \in \mathbb{U}) \right\}, \quad (15)$$

and

$$\left\{ f \in \Sigma : \frac{\zeta (f^{-1})''(\zeta)}{f'(\zeta)} + 1 \prec \zeta + \sqrt[3]{1+\zeta^3}, \quad (\zeta \in \mathbb{U}) \right\}. \quad (16)$$

3. The bounds of the coefficients and the Fekete-Szegő functional within the class $\Sigma_{\kappa}(\mathfrak{N}(z; q))$

Theorem 3.1: Let $f \in \Sigma$ of the form (1) be included in the class $\Sigma_{\kappa}(\mathfrak{N}(z; q))$. Then

$$|a_2| \leq \frac{1+q}{\sqrt{8[2]_q([3]_q(1+q) - 4[2]_q)}},$$

and

$$|a_3| \leq \frac{1+q}{4[2]_q[3]_q} + \frac{(1+q)^2}{16([2]_q)^2}.$$

proof: If $f \in \Sigma_{\kappa}(\mathfrak{N}(z; q))$, then, as per Definition 2.1, the existence of certain analytic functions \varkappa_1 and \varkappa_2 may be established. These functions fulfil the requirements $\varkappa_1(0) = \varkappa_2(0) = 0$, and $|\varkappa_1(z)| < 1$, $|\varkappa_2(\varsigma)| < 1$ for every $z, \varsigma \in \mathbb{U}$.

$$\delta_1(z) = \frac{1 + \varkappa_1(z)}{1 - \varkappa_1(z)} = 1 + \ell_1 z + \ell_2 z^2 + \dots, \quad (z \in \mathbb{U}),$$

and

$$\delta_2(\varsigma) = \frac{1 + \varkappa_2(\varsigma)}{1 - \varkappa_2(\varsigma)} = 1 + j_1 \varsigma + j_2 \varsigma^2 + \dots, \quad (\varsigma \in \mathbb{U}),$$

then $\delta_1, \delta_2 \in P$. From the above relations, we get

$$\varkappa_1(z) = \frac{\delta_1(z) - 1}{\delta_1(z) + 1}, \quad (z \in \mathbb{U}),$$

and

$$\varkappa_2(\varsigma) = \frac{\delta_2(\varsigma) - 1}{\delta_2(\varsigma) + 1}, \quad (\varsigma \in \mathbb{U}).$$

From (10) and (11) it follows that

$$\begin{aligned} \mathfrak{N}(\varkappa_1(z); q) &= \frac{(1+q)(\delta_1(z)-1)}{1+3\delta_1(z)+q(1-\delta_1(z))} + \sqrt[3]{1 + \left(\frac{(1+q)(\delta_1(z)-1)}{1+3\delta_1(z)+q(1-\delta_1(z))} \right)^3} \\ &= 1 + \frac{1+q}{4} \ell_1 z + \frac{1+q}{4} \left(\ell_2 - \frac{(3-q)}{4} \ell_1^2 \right) z^2 + \dots, \quad (z \in \mathbb{U}), \end{aligned} \quad (17)$$

and

$$\begin{aligned} \mathfrak{N}(\varkappa_2(\varsigma); q) &= \frac{(1+q)(\delta_2(\varsigma)-1)}{1+3\delta_2(\varsigma)+q(1-\delta_2(\varsigma))} + \sqrt[3]{1 + \left(\frac{(1+q)(\delta_2(\varsigma)-1)}{1+3\delta_2(\varsigma)+q(1-\delta_2(\varsigma))} \right)^3} \\ &= 1 + \frac{1+q}{4} j_1 \varsigma + \frac{1+q}{4} \left(j_2 - \frac{(3-q)}{4} j_1^2 \right) \varsigma^2 + \dots, \quad (\varsigma \in \mathbb{U}). \end{aligned} \quad (18)$$

Also,

$$1 + \frac{z \partial_q^2 f(z)}{\partial_q f(z)} = 1 + \frac{1+q}{4} \ell_1 z + \frac{1+q}{4} \left(\ell_2 - \frac{(3-q)}{4} \ell_1^2 \right) z^2 + \dots, \quad (19)$$

and

$$1 + \frac{\partial_q^2 g(\zeta)}{\partial_q g(\zeta)} = 1 + \frac{1+q}{4} j_1 \zeta + \frac{1+q}{4} \left(j_2 - \frac{(3-q)}{4} j_1^2 \right) \zeta^2 + \dots, \quad (20)$$

By comparing the relevant coefficients in equation (19) and equation (20), we can deduce the following.

$$[2]_q a_2 = \frac{1+q}{4} \ell_1, \quad (21)$$

$$-[2]_q a_2 = \frac{1+q}{4} j_1, \quad (22)$$

$$[2]_q [3]_q a_3 - [2]_q^2 a_2^2 = \frac{1+q}{4} \left(\ell_2 - \frac{(3-q)}{4} \ell_1^2 \right), \quad (23)$$

and

$$[2]_q (2[3]_q - [2]_q) a_2^2 - [2]_q [3]_q a_3 = \frac{1+q}{4} \left(j_2 - \frac{(3-q)}{4} j_1^2 \right). \quad (24)$$

It follows from (21) and (22) that

$$\ell_1 = -j_1 \quad \text{and} \quad \ell_1^2 = j_1^2, \quad (25)$$

and

$$2([2]_q)^2 a_2^2 = \frac{(1+q)^2}{16} (\ell_1^2 + j_1^2) a_2^2 = \frac{(1+q)^2}{32([2]_q)^2} (\ell_1^2 + j_1^2) \Leftrightarrow \ell_1^2 + j_1^2 = \frac{32([2]_q)^2}{(1+q)^2} a_2^2.$$

Adding (23) and (24), with doing some calculations, we get

$$2[2]_q ([3]_q - [2]_q) a_2^2 = \frac{1+q}{4} \left[(\ell_2 + j_2) + \frac{(3-q)}{4} (\ell_1^2 + j_1^2) \right].$$

Substituting the value of $(\ell_1^2 + j_1^2)$ from (26), we obtain

$$2[2]_q \left[[3]_q - \left(1 + \frac{(3-q)}{(1+q)} \right) [2]_q \right] a_2^2 = \frac{1+q}{8} (\ell_2 + j_2).$$

Moreover,

$$a_2^2 = \frac{1+q}{16[2]_q \left[[3]_q - \left(1 + \frac{(3-q)}{(1+q)} \right) [2]_q \right]} (\ell_2 + j_2). \quad (27)$$

Applying (4) for the coefficients ℓ_2 and j_2 , we obtain

$$|a_2| \leq \frac{1+q}{\sqrt{8[2]_q \left[[3]_q (1+q) - 4[2]_q \right]}},$$

By subtracting (24) from (23), and using $\ell_1^2 = j_1^2$, we get

$$2[2]_q [3]_q (a_3 - a_2^2) = \frac{1+q}{4} (\ell_2 - j_2). \quad (28)$$

Then, in view of (26), the (28) becomes

$$\alpha_3 = \frac{1+q}{8[2]_q[3]_q}(\ell_2 - j_2) + \frac{(1+q)^2}{32([2]_q)^2}(\ell_1^2 + j_1^2).$$

Thus applying (4), we conclude that

$$|\alpha_3| \leq \frac{1+q}{4[2]_q[3]_q} + \frac{(1+q)^2}{16([2]_q)^2}.$$

□

Theorem 3.2: For any $\mu \in \mathbb{R}$, consider f defined as in equation (1), which belongs to the class $\Sigma_{\kappa}(\mathfrak{K}(z; q))$. Then,

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{(1+q)}{4[2]_q[3]_q}, & |1 - \mu| \leq 2 \left| 1 - \frac{4[2]_q}{[3]_q(1+q)} \right|, \\ \frac{(1+q)}{4[2]_q} |\mathcal{Y}(\mu)|, & |1 - \mu| \geq 2 \left| 1 - \frac{4[2]_q}{[3]_q(1+q\%) } \right|. \end{cases}$$

where

$$\mathcal{Y}(\mu) = \frac{(1-\mu)(1+q)}{2([3]_q(1+q) - 4[2]_q)}.$$

proof: If $f \in \mathcal{K}(\mathfrak{K}(z; q))$ is expressed in the manner (1), then from (27) and (28), we obtain

$$\begin{aligned} \alpha_3 - \mu \alpha_2^2 &= \frac{(1-\mu)(1+q)^2}{16[2]_q([3]_q(1+q) - 4[2]_q)}(\ell_2 + j_2) + \frac{1+q}{8[2]_q[3]_q}(\ell_2 - j_2) \\ &= \frac{(1+q)}{8[2]_q} \left[\left(\mathcal{Y}(\mu) + \frac{1}{[3]_q} \right) \ell_2 + \left(\mathcal{Y}(\mu) - \frac{1}{[3]_q} \right) j_2 \right] \end{aligned}$$

where

$$\mathcal{Y}(\mu) = \frac{(1-\mu)(1+q)}{2([3]_q(1+q) - 4[2]_q)}.$$

Then, we conclude that

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{(1+q)}{4[2]_q[3]_q}, & |\mathcal{Y}(\mu)| \leq \frac{1}{[3]_q}, \\ \frac{(1+q)}{4[2]_q} |\mathcal{Y}(\mu)|, & |\mathcal{Y}(\mu)| \geq \frac{1}{[3]_q}. \end{cases}$$

This finishes the proof of Theorem 3.2. □

Theorems 3.1 and 3.2 yield the following corollary, which is typically associated with Example 2.3.

Corollary 3.3: For any $\mu \in \mathbb{R}$, consider f defined as in equation (1), which belongs to the class $\Sigma_{\kappa}(\mathfrak{K}(z; q))$. Then, we can state the following

$$|a_2| \leq \frac{1}{2\sqrt{2}}, \quad |a_3| \leq \frac{7}{48}, \quad \text{and} \quad |a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{12}, & |1 - \mu| \leq \frac{2}{3}, \\ \frac{1}{8}|1 - \mu|, & |1 - \mu| \geq \frac{2}{3}. \end{cases}$$

4. Conclusion

This paper investigates coefficient issues pertaining to a newly established subclass of bi-univalent functions within the domain \mathbb{U} , as specified in Definition 2.1. This subclass, represented by $K(\mathfrak{K}(z; q))$, has been extensively analysed. We have established bounds for the second and third Taylor-Maclaurin coefficients, $|a_2|$ and $|a_3|$, for functions inside this class. Furthermore, we have supplied estimations for the Fekete-Szegő functional, therefore augmenting our comprehension of the geometric and analytic characteristics of these functions. These findings establish a foundation for additional research into coefficient inequalities and associated extremal issues within the theory of bi-univalent functions.

In future studies, one could investigate the maximum bounds of the Zalcman conjecture and analyze Hankel determinants for the classes of bi-convex and bi-close-to-convex functions. These directions present promising opportunities for novel discoveries and deeper exploration in the field.

References

- [1] Ahuja O., Bohra N., Cetinkaya A., Kumar S. Univalent functions associated with the symmetric points and cardioid-shaped domain involving (p, q) -calculus. *Kyungpook Math. J.* **2021**, 61(1), 75–98.
- [2] Agarwal R. P. Certain fractional q -integrals and q -derivatives. *Proc. Cambridge Philos.* **1969**, 66, 365–370.
- [3] Alsoboh A., Amourah A., Darus M., Rudder C. A. Investigating new subclasses of bi-univalent functions associated with q -Pascal distribution series using the subordination principle. *Symmetry* **2023**, 15, 1109. <https://doi.org/10.3390/sym15051109>
- [4] Alsoboh A., Çağlar M., Buyankara M. Fekete–Szegő inequality for a subclass of bi-univalent functions linked to q -Ultraspherical polynomials. *Contemporary Mathematics* **2024**, 2366–2380.
- [5] Amourah A., Alsoboh A., Salah J., Al Kalbani K. Bounds on initial coefficients for bi-univalent functions linked to q -analog of Le Roy-type Mittag-Leffler function. *WSEAS Transactions on Mathematics* **2024**, 23, 714–722.
- [6] Alsoboh A., Amourah A., Sakar F. M., Ogilat O., Gharib G. M., Zomot N. Coefficient estimation utilizing the Faber polynomial for a subfamily of bi-Univalent functions. *Axioms* **2023**, 12, 512. <https://doi.org/10.3390/axioms12060512>
- [7] Alsoboh A., Oros G. I. A class of bi-univalent functions in a Leaf-like domain defined through subordination via q -Calculus. *Mathematics* **2024**, 12, 1594. <https://doi.org/10.3390/math12101594>
- [8] Alsoboh A., Tayyah A. S., Amourah A., Al-Maqbali A. A., Al Mashraf K., Sasa T. Hankel determinant estimates for bi-Bazilevič-type functions involving q -Fibonacci numbers. *European Journal of Pure and Applied Mathematics* **2025**, 18(3), 6698–6698.
- [9] Ahmed M., Alsoboh A., Amourah A., Salah J. On the fractional q -differintegral operator for subclasses of bi-univalent functions subordinate to q -ultraspherical polynomials. *European Journal of Pure and Applied Mathematics* **2025**, 18(3), Article ID: 6586.
- [10] Alsoboh A., Amourah A., Darus M., Rudder C. A. Studying the harmonic functions associated with quantum calculus. *Mathematics* **2023**, 11(10), 2220.
- [11] Alsoboh A., Darus M. New subclass of analytic functions defined by q -differential operator with respect to k -symmetric points. *International Journal of Mathematics and Computer Science* **2019**, 14, 761–773.
- [12] Alsoboh A., Amourah A., Alnajar O., Ahmed M., Seoudy T. M. Exploring q -Fibonacci Numbers in Geometric Function Theory: Univalence and Shell-like Starlike Curves. *Mathematics* **2025**, 13(8), 1294. <https://doi.org/10.3390/math13081294>
- [13] Alsoboh A., Darus M. On Fekete–Szegő problems for certain subclasses of analytic functions defined by differential operator involving q -Ruscheweyh operator. *Journal of Function Spaces* **2020**, 2020(1), 8459405. <https://doi.org/10.1155/2020/8459405>
- [14] Amourah A., Alsoboh A., Breaz D., El-Deeb S. M. A bi-starlike class in a Leaf-like domain defined through subordination via q -calculus. *Mathematics* **2024**, 12, 1735. <https://doi.org/10.3390/math12111735>

- [15] Al-Hawary T., Amourah A., Alsoboh A., Ogilat O., Harny I., Darus M. Applications of q -Ultraspherical polynomials to bi-univalent functions defined by q -Saigo's fractional integral operators. *AIMS Mathematics* **2024**, 9(7), 17063–17075.
- [16] Al-Hawary T., Amourah A., Alsoboh A., Freihat A. M., Ogilat O., Harny I., Darus M. Subclasses of Yamakawa-type bi-starlike functions subordinate to Gegenbauer polynomials associated with quantum calculus. *Results in Nonlinear Analysis* **2024**, 7(4), 75–83.
- [17] Alsoboh A., Amourah A., Alnajjar O., Ahmed M., Seoudy T. M. Exploring q -Fibonacci numbers in geometric function theory: Univalence and shell-like starlike curves. *Mathematics* **2025**, 13, 1294. <https://doi.org/10.3390/math13081294>
- [18] Andrei L., Caus V. A. Subordinations results on a q -derivative differential operator. *Mathematics* **2024**, 12(2), 208.
- [19] Al-Salam W. A. Some fractional q -integrals and q -derivatives. *Proc. Edinburgh Math. Soc.* **1966**, 15(2), 135–140.
- [20] Duren P. L. *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Band 259; Springer: New York, NY, USA; Berlin/Heidelberg, Germany; Tokyo, Japan, 1983.
- [21] Hadi S. H., Shaba T. G., Madhi Z. S., Darus M., Lupaş A. A., Tchier F. Boundary values of Hankel and Toeplitz determinants for q -convex functions. *MethodsX* **2024**, 13, Article 102842.
- [22] Hadi S. H., Darus M., Alamri B., Altinkaya Ş., Alatawi A. On classes of ζ -uniformly q -analogue of analytic functions with some subordination results. *Applied Mathematics in Science and Engineering* **2024**, 32(1), Article 2312803.
- [23] Fekete M., Szegő G. Eine Bemerkung über ungerade schlichte Funktionen. *J. Lond. Math. Soc.* **1933**, 1, 85–89.
- [24] Gasper G., Rahman M. *Basic Hypergeometric Series*, 2nd ed.; Encyclopedia of Mathematics and Its Applications; Cambridge University Press: Cambridge, MA, USA, 2004; Volume 96.
- [25] Khan M. F., AbaOud M. New applications of fractional q -calculus operator for a new subclass of q -Starlike functions related with the Cardioid domain. *Fractal Fract.* **2024**, 8(1), 71.
- [26] Ma W., Minda D. A unified treatment of some special classes of univalent functions, in: *Proc. Conf. Comp. Anal. Tianjin China*, **1992**, 157–169.
- [27] Miller S. S. Differential inequalities and Carathéodory functions. *Bull. Am. Math. Soc.* **1975**, 81, 79–81.
- [28] Khan B., Srivastava H. M., Khan N., Darus M., Tahir M., Ahmad Q. Z. Coefficient estimates for a subclass of analytic functions associated with a certain Leaf-like Domain. *Mathematics* **2020**, 8, 1334. <https://doi.org/10.3390/math8081334>
- [29] Piejko R. K., Sokół J. On the convolution and subordination of convex functions. *Applied Mathematics Letters*. **2012**, 25(3), 448–453.
- [30] Priya M. H., Sharma R. B. On a class of bounded turning functions subordinate to a leaf-like domain. *J. Phys. Conf. Ser.* **2018**, 1000, 012056.
- [31] Seoudy T. M., Aouf M. K. Coefficient estimates of new classes of q -starlike and q -convex functions of complex order. *J. Math. Inequal.* **2016**, 10.1, 135–145.
- [32] Singh G., Kaur C. Starlike and convex functions subordinate to Leaf-like domain. *Turkish Journal of Computer and Mathematics Education*. **2021**, 12(14), 6098–6102.
- [33] Sokół J., Stankiewicz J. Radius of convexity of some subclasses of strongly starlike functions. *Zeszyty Nauk. Politech. Rzeszowskiej Mat.* **1996**, 19, 101–105.
- [34] Srivastava H. M. Operators of basic (or q -) calculus and fractional q -calculus and their applications in geometric function theory of complex analysis. *Iran. J. Sci. Technol. Trans. A Sci.* **2020**, 44(1), 327–344.
- [35] Srivastava H. M., Mostafa A. O., Aouf M. K., Zayed H. M. Basic and fractional q -calculus and associated Fekete-Szegő problem for p -valently q -starlike functions and p -valently q -convex functions of complex order. *Miskolc Mathematical Notes*. **2019**, 20.1, 489–509.
- [36] Uçar H. E. Ö. Coefficient inequality for q -starlike functions. *App. Math. Comput.* **2016**, 276, 122–126.
- [37] Shakir Q. A., Tayyab A. S., Breaz D., Cotirlă L. I., Rapeanu E., Sakar F. M. Upper bounds of the third Hankel determinant for bi-univalent functions in crescent-shaped domains. *Symmetry* **2024**, 16(10), 1281.
- [38] Tayyab A. S., Atshan W. G. Starlikeness and bi-starlikeness associated with a new Carathéodory function. *Journal of Mathematical Sciences* **2025**, 1–25.
- [39] Tayyab A. S., Atshan W. G., Oros G. I. Third-order differential subordination results for meromorphic functions associated with the inverse of the Legendre Chi function via the Mittag-Leffler identity. *Mathematics* **2025**, 13(13), 2089.