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A version of Hilbert's 13th problem for infinitely differentiable functions

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Abstract

It is famous that Hilbert's 13th problem, asking if there exists a continuous real-valued function of multivariables which cannot be represented as any finite-time nested superposition of several functions of fewer variables, was proved by Kolmororov and Arnold. Actually, it is well known that there exist some other versions having been derived from the original one and still remaining to be open such as the analytic function version and the infinitely differentiable function version.

In this paper, we discuss a version of Hilbert's 13th problem for the infinitely differentiable functions. Exactly speaking, we show an example of an infinitely differentiable function of three real variables which cannot be represented as finite-time nested superposition of several infinitely differentiable functions of two real variables.

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1. Introduction

It is famous that Hilbert's 13th problem, asking if there exists a continuous real-valued function of multivariables which cannot be represented as any finite-time nested superposition of several functions of fewer variables, was proved by Kolmororov and Arnold in 1957. Actually, it is well known that there exist some other versions having been derived from the original one and still remaining to be open such as the analytic function version and the infinitely differentiable function version.

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In this paper, we discuss s version of Hilbert's 13th problem for the infinitely differentiable functions. Exactly speaking, we show an example of an infinitely differentiable function of three real variables which cannot be represented as finite-time nested superposition of several infinitely differentiable functions of two real variables.

As for the original solution to the Hiblert's 13th problem, we can refer to Kolmogorv and Lorentz [5, 7]. As for the simplification of the superposition representation, we can refer to Arnol'd and Vituskin [6, 8]. Kolmogorv and Arnold solved a version of Hilbert's 13th problem for continuous functions, which is contrary to Hilbert's predicton. Therefore, as for the analytic function version, we can refer to Babenko and Akashi [1, 2], and these results are supporting Hilbert's prediction affirmatively. As for some relations with ε-entropy, we can refer to Erohin [3]. Finally, the authors are trying to give an entropy theoretic classification of the countably normed spaces and the countably inner product spaces such as rigged Hilbert spaces and nuclear spaces in terms of the interpolative metric space theory developed by Karapinar [4].

2. Preliminaries

Let $\mathbb{R}$ denote the set of all real numbers and let $[0,1]$ and $(0,1)$ denote the closed unit interval and the open unit one included by $\mathbb{R}$, respectively. Then, for any function $f(\cdot,\cdot)$ of two variables defined on $\mathbb{R}^2$ and for any point (x_0, y_0) belonging to $\mathbb{R}^2$, $f(\cdot,\cdot)$ is definied to be infinitely differentiable at (x_0, y_0), if, for any positive integer k, there exists a set of $(k+1)(k+2)/2$ real numbers, which is denoted by $\{c_{i,j}; 0 \le i, 0 \le j, 0 \le i+j \le k\}$ satisfying the following equality:

$$\lim_{\sqrt{\Delta x^2 + \Delta y^2} \to +0} \frac{f(x_0 + \Delta x, y_0 + \Delta y) - \sum_{0 \le i+j \le k} c_{ij} \Delta x^i \Delta y^j}{\sqrt{\Delta x^2 + \Delta y^2}^{\,k}} = 0.$$

Here, the set of all infinitely differentiable functions of two variales on $\mathbb{R}^2$ (resp. $[0,1]^2$ and $(0,1)^2$), which is denoted by $C^\infty(\mathbb{R}^2)$ (resp. $C^\infty([0,1]^2)$ and $C^\infty((0,1)^2)$), is defined as the set of all functions of two variables which are infinitely differentiable at all points belonging to $\mathbb{R}^2$ (resp. $[0,1]^2$ and $(0,1)^2$). Moreover, the set of the infinitely differentiable functions of three variales on $\mathbb{R}^3$ (resp. $[0,1]^3$ and $(0,1)^3$), which is denoted by $C^\infty(\mathbb{R}^3)$ (resp. $C^\infty([0,1]^3)$ and $C^\infty((0,1)^3)$), is defined as the same way stated as above.

Let $\phi(\cdot)$ be the function of one variable on $\mathbb{R}$ with values in $\mathbb{R}$ defined as the following:

$$\phi(x) = \begin{cases} 0, & x \le 0, \\ e^{\frac{-1}{x}}, & x > 0. \end{cases}$$

Then, we can obtain the following:

Proposition 2.1: *$\phi(\cdot)$ belongs to $C^\infty(\mathbb{R})$.*

Proof. Without loss of generality, it is sufficient to prove that $\phi(\cdot)$ is infinitely differentiable at $(0,0)$, and since, for any positive integer n, the following equality:

$$\lim_{\Delta x \to +0} \frac{e^{-1/\Delta x}}{\Delta x^n} = 0$$

holds, we can conclude the proof.

The above proposition shows that $\phi(\cdot)$ cannot be represented as any Taylor power series expansion. As for some relations of polynomials with superposition representation, we can exress as the following:

Remark 2.2: *For any positive integer* k*, there exists a certain polynomial* $h_k(\cdot,\cdot,\cdot)$ *defined on* $[0,1]^3$*, which cannot be represented as any* k*-time nested superposition of several polynomials of two variables defined on* $[0,1]^2$.

If we apply Weierstrass approximation theorem to the above remark, then we can obtain the following:

Remark 2.3: *For any positive integer* k*, there exists a certain polynomial* $h_k(\cdot,\cdot,\cdot)$ *belonging to* $C^\infty((0,1)^3)$*, which cannot be represented as any* k*-time nested superposition of several elements belonging to* $C^\infty((0,1)^2)$.

3. Main results

In this section, we show an exmaple of a certain function belonging to $C^\infty((0,1)^3)$, which cannot be represented as any finite-time superposition of several functions $C^\infty((0,1)^2)$. For any positive integer k, let $h_k(\cdot,\cdot,\cdot)$ be a polynomial belonging to $C^\infty((0,1)^3)$ and being unable to be represented as k-time nested superposition of $C^\infty((0,1)^2)$, and let $\Phi(\cdot)$ be the function defined on the closed unit interval $[0,1]$ with values in $[0,1]$ as the following:

$$\Phi(x)=\begin{cases} 0, & x \le 0, \\ e^{\left(\frac{-1}{x}+\frac{-1}{1-x}\right)}, & 0 < x < 1, \\ 0, & x \ge 1. \end{cases}$$

Moreover, for any positive integer n, let $p_n(\cdot)$ and $q_n(\cdot)$ be the functions definied on $\mathbb{R}$ as the following:

$$p_n(x) = (1-x)\left(1-\frac{1}{2^{n-1}}\right)+x\left(1-\frac{1}{2^n}\right),$$
$$q_n(x) = p_n^{-1}(x),$$

respectively. It can be remarked that $p_n([0,1]) = [1-1/(2^{n-1}),1-1/2^n]$ and $q_n([1-1/(2^{n-1}),1/2^n]) = [0,1]$ hold. Then, as for the function $g(\cdot,\cdot,\cdot)$ which is defined as the following:

$$g(x,y,z)=\sum_{k=1}^{\infty} h_k(q_k(x),q_k(y),q_k(z))\Phi(q_k(x))\Phi(q_k(y))\Phi(q_k(z)), \quad 0 < x,y,z < 1,$$

we can obtain the following:

Theorem 3.1: $g(\cdot,\cdot,\cdot)$ *is an infinitely differentiable function defined on* $(0,1)^3$ *and, for any positive integer* k*, cannot be represented as* k*-time nested superposition of several functions of two variables belonging to* $C^\infty((0,1)^2)$.

Proof. If we assume that, for a certain positive integer k, $g(\cdot,\cdot,\cdot)$ can be represented as a k-time nested superposition of several functions of two variables belonging to $C^\infty((0,1)^2)$, then, for any positive integer n, it can be proved that the following function:

$$h_n(x,y,z)=\frac{g(p_n(x),p_n(y),p_n(z))}{\Phi(x)\Phi(y)\Phi(z)}, \quad 0 < x,y,z < 1,$$

should be represented as a $(k+3)$-time nested superposition of several functions of two variables belonging to $C^\infty((0,1)^2)$. Actually, this result contradicts the former assumption that, for any positive integer n, $h_n(\cdot,\cdot,\cdot)$ cannot be represented as n-time nested superposition, if we take $k+4$ as a value of n.

4. Application to multidimensional numerical nonlinear approximation

For any positive integers k and m, let $w_3^{m,k}(\cdot,\cdot,\cdot)$ be a polynomial of the m-th degree defined on $[0,1]^3$ which is assumed to be represented as k-time nested superposition of $2^{k+1}-1$ polynomials of the m-th degree defined on $[0,1]^2$, which are denoted $w_{2,1}^{m,k}(\cdot,\cdot),\cdots,w_{2,2^{k+1}-1}^{m,k}(\cdot,\cdot)$. Then, the cardinal number of the set of all coefficients accompanying $w_3^{m,k}(\cdot,\cdot)$, which is denoted by $card(w_3^{m,k})$, is exactly equal to ${}_{m+3}C_3$, while, for any interger i satisfying $1 \le i \le 2^{k+1}-1$, the cardinal number of the set of all coefficients accompanying $w_{2,i}^{m,k}(\cdot,\cdot)$, which is denoted by $card(w_{2,i}^{m,k})$, is exactly equal to ${}_{m+2}C_2$. Therefore, if we compare the $card(w_3^{m,k})$ with the total sum of $card(w_{2,1}^{m,k}),\cdots,card(w_{2,2^{k+1}-2}^{m,k})$ and $card(w_{2,2^{k+1}-1}^{m,k})$, then we can obtain the following equalities:

$$\lim_{m\to\infty}\frac{card(w_3^{m,k})}{\sum_{i=1}^{2^{k+1}-1} card(w_{2,i}^{m,k})} = \lim_{m\to\infty}\frac{{}_{m+3}C_3}{(2^{k+1}-1)\,{}_{m+2}C_2} = \infty.$$

This result shows that, the larger the degree of a polynomial of three variables is taken, the more difficult is it to find the corresponding superposition representation by several polynomials of two variables. This is the reason why multidimensional database cannot be easily compressed numerically under the condition that reproducibility should be assured.

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