Results in Nonlinear Analysis 8 (2025) No. 2, 207–221
https://doi.org/10.31838/rna/2025.08.02.017
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

Solving fuzzy system for Volterra-Fredholm integral equations of the second kind using homotopy perturbation method

Mahasin Thabet Younis, Waleed Al-Hayani

Department of Mathematics, College of Computer Science and Mathematics, University of Mosul, Iraq

Abstract

In the current research, the linear fuzzy system of the Volterra-Fredholm integral equations (FSVFIEs) is solved using the homotopy perturbation method (HPM). The Banach contraction fixed point is used to demonstrate the convergence of the solution under the established approximate scheme. The symmetric fuzzification solution is obtained by using convex symmetrical triangular fuzzy numbers.To verify the accuracy and efficiency of this method to handling FSVFIEs, the approximate and exact solutions are compared. Results from numerically solving examples of FSVFIEs are used to test the effectiveness of the suggested approach. These results show that the suggested strategy is highly effective and that the suggested method is easy to use.

Mathematics Subject Classification (2020): 03E72, 45D05, 45B05

Keywords: Fuzzy numbers; Fuzzy system of integral equations; Fuzzy system of Volterra-Fredholm integral equations; Homotopy perturbation method (HPM).

1. Introduction

The fields of fuzzy integral equations, which have garnered more attention, especially in connection with fuzzy control, have advanced quickly [1]. Volterra-Fredholm integral equations in the mathematical modeling of the spatiotemporal evolution of an epidemic, The sources include different physical

E-mail address: mahasin_thabet@uomosul.edu.iq (Mahasin Thabet Younis); waleedalhayani@uomosul.edu.iq, waleedalhayani@yahoo.es (Waleed Al-Hayani)

Received May 25, 2025; Accepted June 30, 2025; Online July 18, 2025

and biological models as well as parabolic boundary value problems [2, 3]. Additionally, analytical solutions to these problems can be discovered in earlier studies. Simultaneously, the utilization of numerical techniques is crucial in resolving these equations [6]. Integral equation systems are crucial to science and engineering [7].

Many scientists, engineers, and researchers have studied and successfully used the homotopy perturbation method (HPM) in recent years to solve differential and integral equations. Ji-Huan He developed and improved the HPM, which was first suggested in 1998 [8, 9]. A fundamental idea of topology, the homotopy method, and the classical perturbation approach are combined to form the HPM. A suitable method for obtaining an approximation or analytical solutions for many problems appearing in different scientific domains would be made possible by this coupling [10, 11]. One of the many benefits of the HPM is that it can yield near-perfect solutions with few iterations, and in many situations The homotopy analysis method's (HAM) increased applicability is due to its dependability and the fact that it requires less computational effort [12], the solution series converges quickly. A few years ago. Numerous scientists' papers have included applications of the HPM theory, demonstrating the approach's growth into a potent mathematical instrument [13]. Scientists and engineers have applied the HPM to nonlinear problems because it can be used to continuously deform an easy-to-solve simple problem into a challenging problem. This is particularly true for integral equations, where the method can be used to transform a challenging problem into an easy-to-solve simple problem [7, 14].

Through behavioral science, the field of Banach fixed point theory in functional analysis emerged as a crucial tool in non-linear sciences and engineering during the past few decades [15].

Fuzzy set theory is studied in the field of fuzzy mathematics. To demonstrate knowledge and analysis with nonstatistical uncertainty, Zadeh proposed the fuzzy set in 1965 [16]. The application of fuzzy or interval formulations in a variety of fields, such as artificial intelligence, topology, fractional calculus, fixed-point theory, integral inequalities, bifurcation, and consumer electronics [17]. Recently contains a review of nonlinear analysis techniques that have been developed recently. Many authors have utilized the HPM for various purposes [6]. The study of fuzzy IEs has become more and more important. In this work, we concentrate on first-order fuzzy Volterra integral equations given a fuzzy initial condition [18].

In recent years, fuzzy set theory has undergone numerous advancements and generalizations. Yusufoglu [13] provides a straightforward and efficient approach to solve these equations, which has been refined and enhanced by scientists and engineers. The authors give a strategy for resolving these equations and go over the convergence of the generated series in the HPM as well as convergence analysis for the linear mixed Volterra-Fredholm integral equations [4, 5, 6]. In order to obtain approximate solutions for fuzzy Volterra integral equations of linear and nonlinear with a separable kernel, the HAM is investigated, this method offers a dependable means of guaranteeing the convergence of the approximation series [19].

The main goal of this research is to study the convergence of the HPM using the Banach contraction fixed point theorem, and the sufficient conditions for convergence were examined. Additionally includes developing a more effective and efficient approach for solving the fuzzy system of Volterra-Fredholm Integral equations (FSVFIEs) using the HPM. Because convex symmetrical triangular fuzzy numbers are used, the fuzzy solution's lower and upper representations are symmetrical. The efficiency of the approach is demonstrated with numerical examples.

2. Preliminary

In this section, we will go over some basic concepts related to fuzzy number:

Let X be a nonempty set. The membership function $u: X \to [0,1]$ defines a fuzzy set u in X. Consequently, for each x in X, $u(x)$ is understood to be the degree of membership of element x in the fuzzy set u. Fuzzy sets with u on $\mathbb{R}$ are called convex if $u(\lambda x + (1-\lambda)y) \geq min\{u(x), u(y)\}$ for each $x, y \in \mathbb{R}$ and $\lambda \in [0,1]$. Upper semicontinuous sets are closed for every $\lambda \in [0,1]$, and normal sets are those where there are $x \in \mathbb{R} : u(x) = 1$. The fuzzy set u is support if $\{x \in \mathbb{R}, u(x) > 0\}$ [18, 20].

A fuzzy subset of the real line with a normal, convex, upper semicontinuous membership function of bounded support is called a fuzzy number u. Then, for each $\alpha \in [0,1]$, it is straightforward to prove that u is a fuzzy number if and only if $[u]^{\alpha}$ is compact convex subset of $\mathbb{R}$. In other words, if u is a fuzzy number, then $[u]^{\alpha} = [\underline{u}(\alpha), \bar{u}(\alpha)]$ where, for each $\alpha = [0,1]$, $\underline{u}(\alpha) = \min\{x : x \in [u]^{\alpha}\}$ and $\bar{u}(\alpha) = \max\{x : x \in [u]^{\alpha}\}$. The parametric form or $\alpha - cut$ representation of a fuzzy number u is represented by the symbol $[u]^{\alpha}$. We shall designate the set of fuzzy numbers on $\mathbb{R}$ by $\mathbb{R}$ [18].

Theorem 1 *Assume that $\bar{u} : [0,1] \to \mathbb{R}$, satisfies the following requirements: $\underline{u}(1) \le \bar{u}(1)$, $\bar{u}$ is a bounded decreasing function, and $\underline{u}$ is a bounded increasing function; $\underline{u}$ and $\bar{u}$ are left-hand continuous functions at $\alpha = k$ for every k that falls inside $(0,1]$; $\underline{u}$ and $\bar{u}$ are right-hand continuous functions at $\alpha = 0$ defined by*

$$u(x) = \sup\{\alpha : \underline{u}(1) \le x \le \bar{u}(1)\},$$

is a fuzzy number with parameterization given by $[\underline{u}(\alpha), \bar{u}(\alpha)]$.

For the proof see [20].

Theorem 2 *The fuzzy system of integral equation of the second kind which has the form:*

$$\tilde{u}_{n+1}(x,\alpha) = \tilde{f}_n(x,\alpha) + \lambda \int_0^x \int_a^b F(t, \tilde{u}_n(t,\alpha))\,dtds, \quad n \ge 0, \tag{1}$$

has a unique solution.

Proof. We defined an operator $T : C[a,b] \to C[a,b]$, where $C[a,b]$ is the set of all continuous function on $[a,b]$,

$$T\tilde{u}_n(x,\alpha) = \tilde{u}_0(x,\alpha) + \lambda \int_0^x \int_a^b F(t, \tilde{u}_n(t,\alpha))\,dtds,$$

$\|x\| = \sup_{t \in [a,b]} |x(t)|$ and $(X, \|.\|)$ is a Banach space, where $F(t, \tilde{u}_n(t,\alpha))$ is nonlinear function. We will prove that T has a unique solution, let $\{x_n\}$ be a sequence such that $X_n = TX_{n-1}$, $n \ge 1$, converges to the unique fixed point x^* of T. If T is the contraction then T has a unique fixed point x^*.Set T^n is the contraction for some sufficiently large $n \ge 1$. Now, if T has fixed point then $u = Tu$, also we assume that $F(t, \tilde{u}_n(t,\alpha))$ be continuous function defined on the domain $D \subset \mathbb{R}^2$, and satisfy the Lipchitz condition with respect to $u \in D$. Let $u_1, u_2 \in C[a,b]$

$$\begin{aligned}
|T\tilde{u}_1(x,\alpha) - T\tilde{u}_2(x,\alpha)| &= \left| \int_0^x \int_a^b F(t, \tilde{u}_1(t,\alpha))\,dtds \right. \\
&\quad \left. - \int_0^x \int_a^b F(t, \tilde{u}_2(t,\alpha))\,dtds \right| \\
&\le \alpha \int_0^x \int_a^b |\tilde{u}_1(t,\alpha) - \tilde{u}_2(t,\alpha)|\,dtds \\
&\le \alpha \int_0^x \int_a^b \sup_{t \in [a,b]} |\tilde{u}_1(t,\alpha) - \tilde{u}_2(t,\alpha)|\,dtds \\
&\le \alpha \int_0^x \int_a^b \|\tilde{u}_1 - \tilde{u}_2\|\,dtds \\
&= \alpha(b-a)\|\tilde{u}_1 - \tilde{u}_2\| x
\end{aligned}$$

$$
\begin{aligned}
\left|T^2\tilde{u}_1(x,\alpha)-T^2\tilde{u}_2(x,\alpha)\right| &= \left|T\left(T\tilde{u}_1(x,\alpha)\right)-T\left(T\tilde{u}_2(x,\alpha)\right)\right| \\
&= \left|\int_0^x\int_a^b TF\left(t,\tilde{u}_1(t,\alpha)\right)dtds \right. \\
&\quad \left. -\int_0^x\int_a^b TF\left(t,\tilde{u}_2(t,\alpha)\right)dtds\right| \\
&\le \alpha\int_0^x\int_a^b \left|T\tilde{u}_1(t,\alpha)-T\tilde{u}_2(t,\alpha)\right|dtds \\
&\le \alpha\int_0^x\int_a^b \alpha(b-a)\|\tilde{u}_1-\tilde{u}_2\|tdtds \\
&\le \alpha^2(b-a)\int_0^x\int_a^b \|\tilde{u}_1-\tilde{u}_2\|tdtds \\
&= \alpha^2(b-a)^2\|\tilde{u}_1-\tilde{u}_2\|\frac{x^2}{2}
\end{aligned}
$$

and

$$
\left|T^3\tilde{u}_1(x,\alpha)-T^3\tilde{u}_2(x,\alpha)\right| = \alpha^3(b-a)^3\|\tilde{u}_1-\tilde{u}_2\|\frac{x^3}{3!},
$$

and so on, by integration n times we get

$$
\left|T^n\tilde{u}_1(x,\alpha)-T^n\tilde{u}_2(x,\alpha)\right| = \alpha^n(b-a)^n\|\tilde{u}_1-\tilde{u}_2\|\frac{x^n}{n!}.
$$

So that,

$$
\begin{aligned}
\left\|T^n\tilde{u}_1-T^n\tilde{u}_2\right\| &= \sup_{x\in[0,1]}\left|T^n\tilde{u}_1(x,\alpha)-T^n\tilde{u}_2(x,\alpha)\right|, \\
&\le \frac{\alpha^n(b-a)^n}{n!}\|\tilde{u}_1-\tilde{u}_2\|.
\end{aligned}
$$

If n sufficiently large, then $0<\frac{\alpha^n(b-a)^n}{n!}\le 1$, this means that T^n is contraction for n large, which implise that T has a unique fixed point by generalized Banach contraction fixed point theorem. The unique fixed point has unique solution, then the unique solution converges to the unique fixed point. which complete the proof.

Theorem 3 *The fuzzy system of the integral equation that defined in Eq. (1) is converges iff*

$$
\frac{\|u_{n+1}\|}{\|u_n\|}\le 1.
$$

Proof. From Theorem 2 and by integration, we have

$$
\left\|T^{n+1}\tilde{u}_1-T^{n+1}\tilde{u}_2\right\| = \frac{\alpha^{n+1}(b-a)^{n+1}}{(n+1)!}\|\tilde{u}_1-\tilde{u}_2\|,
$$

we will prove that $\|u_{n+1}\|\le\|u_n\|$. Since

$$
\left\|T^{n+1}\tilde{u}_1-T^{n+1}\tilde{u}_2\right\| \le \left\|T^n\tilde{u}_1-T^n\tilde{u}_2\right\|,
$$

therefore,

$$\frac{\alpha^{n+1}(b-a)^{n+1}}{(n+1)!}\|\tilde{u}_1-\tilde{u}_2\| \le \frac{\alpha^n (b-a)^n}{n!}\|\tilde{u}_1-\tilde{u}_2\|,$$

by solving this inequality, we have

$$0<\frac{\alpha(b-a)}{n+1}\le 1,$$

since T has fixed point such that $u=Tu$, therefore $\|u_{n+1}\|\le\|u_n\|$, this means that $\frac{\|u_{n+1}\|}{\|u_n\|}\le 1$, which complete the proof.

3. HPM applied to FSVFIEs

In this work will consider FSVFIEs of the second kind, which have the following form:

$$\tilde{u}_i(x,\alpha)=\tilde{f}_i(x,\alpha)+\sum_{j=1}^{2}\lambda_{ij}\int_0^x\int_a^b K_{ij}(s,t)\tilde{u}_j(t,\alpha)\,dtds,\quad i=1,2 \tag{2}$$

where $\tilde{f}_i(x,\alpha)\in C[a,b]$ and $K_{ij}(s,t)$ are continuous on

$$D=\{(s,t): a\le t\le b \text{and} a\le s\le x\le b\},$$

while $\tilde{u}_i(x,\alpha)$ are unknown continuous functions to be found.

Recall Eq. (2) and define the operator L as follows

$$L(\tilde{u}_i^*(x,\alpha))=\tilde{u}_i^*(x,\alpha)-\tilde{f}_i(x,\alpha)-\sum_{j=1}^{2}\lambda_{ij}\int_0^x\int_\Omega K_{ij}(s,t)\tilde{u}_j^*(t,\alpha)\,dtds=0, \tag{3}$$

give the solutions $\tilde{u}_i^*(x,\alpha)=\tilde{u}_i(x,\alpha)$. A HPM convex homotopy is defined $H(\tilde{u}_i^*,p):\mathbb{R}\times[0,1]\to\mathbb{R}$ by

$$H(\tilde{u}_i^*,p)=(1-p)F(\tilde{u}_i^*)+pL(\tilde{u}_i^*)=0, \tag{4}$$

where $F(\tilde{u}_i^*)=\tilde{u}_i^*(x,\alpha)-\tilde{f}_i(x,\alpha)$, functions as an operator, $p\in[0,1]$ is the homotopy parameter, and $\tilde{u}_{i,0}(x,\alpha)$ defines the initial solutions of Eq. (2). From Eq. (4), we have

$$H(\tilde{u}_i^*,0)=F(\tilde{u}_i^*),\quad H(\tilde{u}_i^*,1)=L(\tilde{u}_i^*), \tag{5}$$

where $H(\tilde{u}_i^*,p)$ from the trivial problem is all that is $H(\tilde{u}_i^*,p)$ changed when the imbedding parameter p is changed from 0 to 1. $H(\tilde{u}_i^*,0)=F(\tilde{u}_i^*)=0$ to the original problem $H(\tilde{u}_i^*,1)=L(\tilde{u}_i^*)=0$. In topology, it is called deformation while $H(\tilde{u}_i^*,0)$ and $H(\tilde{u}_i^*,1)$ are called homotopic. By using $L(\tilde{u}_i^*)$ and $F(\tilde{u}_i^*)$ the homotopy operator of the equation under consideration will be obtained, as defined above:

$$\begin{aligned}H(\tilde{u}_i^*,p)=(1-p)F(\tilde{u}_i^*)+p\Big(&\tilde{u}_i^*(x,\alpha)-\tilde{f}_i(x,\alpha)\\&-\sum_{j=1}^{2}\lambda_{ij}\int_0^x\int_\Omega K_{ij}(s,t)\tilde{u}_j^*(t,\alpha)\,dtds\Big).\end{aligned} \tag{6}$$

So that

$$H\left(\tilde{u}_i^*,p\right)=\tilde{u}_i^*\left(x,\alpha\right)-\tilde{f}_i\left(x,\alpha\right) \\ +p\left(-\sum_{j=1}^{2}\lambda_{ij}\int_0^x\int_\Omega K_{ij}\left(s,t\right)\tilde{u}_j^*\left(t,\alpha\right)dtds\right)=0. \tag{7}$$

The approach allows power series to be used

$$\tilde{u}_i^*\left(x,\alpha\right)=\sum_{n=0}^{\infty}p^n\tilde{u}_{i,n}^*\left(x,\alpha\right), \tag{8}$$

If Eq. (8) has a convergence radius of one or more, and the series $\sum_{n=0}^{\infty}\tilde{u}_{i,n}^*\left(x,\alpha\right)$ converges absolutely, then the approximate solutions of Eq. (2) are

$$\tilde{u}_i^*\left(x,\alpha\right)=\lim_{p\to1}\sum_{n=0}^{\infty}p^n\tilde{u}_{i,n}^*\left(x,\alpha\right)=\sum_{n=0}^{\infty}\tilde{u}_{i,n}^*\left(x,\alpha\right). \tag{9}$$

Substituting Eq. (8) in Eq. (7) gives

$$\sum_{n=0}^{\infty}p^n\tilde{u}_{i,n}^*\left(x,\alpha\right)=\tilde{f}_i\left(x,\alpha\right)+p\left(\sum_{j=1}^{2}\lambda_{ij}\int_0^x\int_\Omega K_{ij}\left(s,t\right)\sum_{n=0}^{\infty}p^n\tilde{u}_{j,n}^*\left(t,\alpha\right)dtds\right),$$

and obtaining the recurrence relations that ultimately lead to the approximate solutions by equating the same power terms of the embedding parameter p:

$$p^0:\tilde{u}_{i,0}^*\left(x,\alpha\right)=\tilde{f}_i\left(x,\alpha\right), \\ p^n:\tilde{u}_{i,n+1}^*\left(x,\alpha\right)=\sum_{j=1}^{2}\lambda_{ij}\int_0^x\int_\Omega K_{ij}\left(s,t\right)\tilde{u}_{j,n}^*\left(t,\alpha\right)dtds,\quad n=1,2,... \tag{10}$$

Assuming that the series (8) is convergent, the aforementioned relations are obtained.

4. Application Problems

The HPM for solving FSVFIEs is demonstrated in the next two problems in this section. The maximum errors are defined as follows to demonstrate the exceptional accuracy of the solution results when compared to the exact solution.

$$MAE_L=\left\|\underline{u}_{i,Exact}\left(x,\alpha\right)-\underline{u}_{i,n}\left(x,\alpha\right)\right\|_\infty \\ MAE_U=\left\|\overline{u}_{i,Exact}\left(x,\alpha\right)-\overline{u}_{i,n}\left(x,\alpha\right)\right\|_\infty ,n\geq1,$$

where

$ll\left(\underline{u}_{i,Exact}\left(x,\alpha\right),\overline{u}_{i,Exact}\left(x,\alpha\right)\right)$: Lower and Upper Exact solution,

$\left(\underline{u}_{i,n}\left(x,\alpha\right),\overline{u}_{i,n}\left(x,\alpha\right)\right)$: Lower and Upper-Approximate solution

Thirty-digit precision was achieved in the computations related to the problems using a Maple 22 package.

Problem 1. We consider the FSVFIEs (2) with

$$\begin{aligned} &\lambda_{11} = \frac{1}{2}, \quad \lambda_{12} = \frac{1}{4}, \quad K_{11}(s,t) = s - t, \qquad K_{12}(s,t) = st, \\ &\lambda_{21} = \frac{1}{3}, \quad \lambda_{22} = \frac{1}{4}, \quad K_{21}(s,t) = st + 1, \quad K_{22}(s,t) = s, \end{aligned} \tag{11}$$

and $a = 0,\ b = 1,$ where $\tilde{f}_i(x,\alpha) = \left[\underline{f}_i(x,\alpha), \overline{f}_i(x,\alpha)\right]$ are chosen such that the exact solutions will be $\tilde{u}_1(x,\alpha) = \left[(2\alpha - 1)e^x, (2 - \alpha)e^x\right]$ and $\tilde{u}_2(x,\alpha) = \left[\alpha x^2, (3 - 2\alpha)x^2\right]$.

The HPM is then used recursively to calculate the lower iterations (L), as explained below:

$$\begin{aligned} &p^0: \quad \underline{u}_{i,0}(x,\alpha) = \underline{f}_i(x,\alpha), \\ &p^n: \quad \underline{u}_{i,n+1}(x,\alpha) = \sum_{j=1}^{2} \lambda_{ij} \int_0^x \int_0^1 K_{ij}(s,t)\, \underline{u}_{i,n}(t,\alpha)\, dt ds, n \ge 0 \end{aligned} \tag{12}$$

and the upper iterations (U) are

$$\begin{aligned} &p^0: \quad \overline{u}_{i,0}(x,\alpha) = \overline{f}_i(x,\alpha), \\ &p^n: \quad \overline{u}_{i,n+1}(x,\alpha) = \sum_{j=1}^{2} \lambda_{ij} \int_0^x \int_0^1 K_{ij}(s,t)\, \overline{u}_{i,n}(t,\alpha)\, dt ds, n \ge 0. \end{aligned} \tag{13}$$

Through systemic solution (12) and (13), we obtain the iterations $\tilde{u}_{i,0}(x,\alpha), \tilde{u}_{i,1}(x,\alpha), \ldots, \tilde{u}_{i,n}(x,\alpha)$. Consequently, the series form approximations of the solutions are

$$\tilde{u}_i(x,\alpha) = \tilde{u}_{i,0}(x,\alpha) + \sum_{n=1}^{10} \tilde{u}_{i,n}(x,\alpha).$$

At each point $x_i = 0.1l,\ l = 0,1,\ldots,10$ within the interval $0 < \alpha \le 1$, show the maximum errors for the lower and upper between the exact solution and approximate solution using the HPM ($m = 10$) in Table 1. However, we list the absolute errors (AE) when $\alpha = 1$ on the interval $0 \le x \le 1$, in addition to a comparison with the absolute errors of the fixed point method's numerical treatment [21] in Table 2.

Table 1: The maximum errors for Problem 1.

$0 < \alpha \le 1$	$\tilde{u}_1(x,\alpha)$		$\tilde{u}_2(x,\alpha)$	
x_i	MAE_L	MAE_U	MAE_L	MAE_U
0.0	0.0	0.0	0.0	0.0
0.1	$1.665E-11$	$9.033E-12$	$1.807E-11$	$9.807E-12$
0.2	$3.171E-11$	$1.720E-11$	$3.553E-11$	$1.927E-11$
0.3	$4.519E-11$	$2.451E-11$	$5.236E-11$	$2.839E-11$
0.4	$5.707E-11$	$3.096E-11$	$6.856E-11$	$3.718E-11$
0.5	$6.737E-11$	$3.655E-11$	$8.414E-11$	$4.562E-11$
0.6	$7.607E-11$	$4.128E-11$	$9.909E-11$	$5.372E-11$
0.7	$8.319E-11$	$4.515E-11$	$1.134E-10$	$6.149E-11$
0.8	$8.872E-11$	$4.816E-11$	$1.271E-10$	$6.890E-11$
0.9	$9.265E-11$	$5.030E-11$	$1.401E-10$	$7.598E-11$
1.0	$9.500E-11$	$5.159E-11$	$1.526E-10$	$8.272E-11$

Table 2: Comparison the absolute errors for Problem 1.

$\alpha=1$	$\tilde{u}_1(x,\alpha)$		$\tilde{u}_2(x,\alpha)$	
x_i	AE	[21]	AE	[21]
0.0	0.0	0.0	0.0	0.0
0.1	$9.0336E-12$	$9.0336E-12$	$9.807E-12$	$9.808E-12$
0.2	$1.7206E-11$	$1.7206E-11$	$1.927E-11$	$1.927E-11$
0.3	$2.4517E-11$	$2.4518E-11$	$2.839E-11$	$2.840E-11$
0.4	$3.0968E-11$	$3.0969E-11$	$3.718E-11$	$3.718E-11$
0.5	$3.6558E-11$	$3.6558E-11$	$4.562E-11$	$4.563E-11$
0.6	$4.1286E-11$	$4.1287E-11$	$5.372E-11$	$5.373E-11$
0.7	$4.5154E-11$	$4.5155E-11$	$6.149E-11$	$6.149E-11$
0.8	$4.8161E-11$	$4.8161E-11$	$6.890E-11$	$6.891E-11$
0.9	$5.0306E-11$	$5.0306E-11$	$7.598E-11$	$7.599E-11$
1.0	$5.1591E-11$	$5.1592E-11$	$8.272E-11$	$8.273E-11$

Figures 1–6 below display exact solutions $\left(\tilde{u}_{i,Exact}(x,\alpha)\right)$ and the fuzzy approximate solutions by the HPM $\left(\tilde{u}_{i,10}(x,\alpha)\right)$ of the system (11) are in the form of fuzzy numbers for any $\alpha\in(0,1]$ at $x=0.5$.

Problem 2. We consider the FSVFIEs (2) with

$$\begin{aligned}&\lambda_{11}=0,\quad \lambda_{12}=\frac{1}{4},\quad K_{11}(s,t)=0,\quad K_{12}(s,t)=st-1,\\&\lambda_{21}=0,\quad \lambda_{22}=\frac{1}{4},\quad K_{21}(s,t)=0,\quad K_{22}(s,t)=s-t,\end{aligned}\tag{14}$$

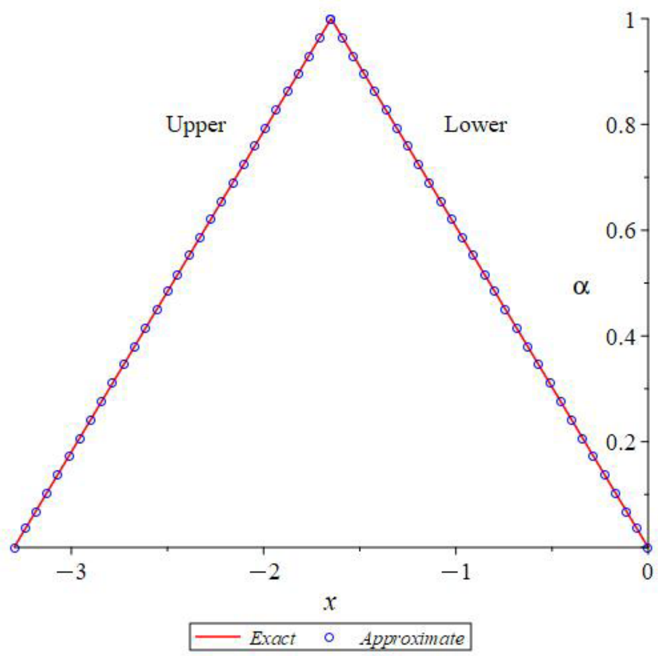

Figure 1: Graph of $\tilde{u}_{1,Exact}(x,\alpha)$ and $\tilde{u}_{1,10}(x,\alpha)$

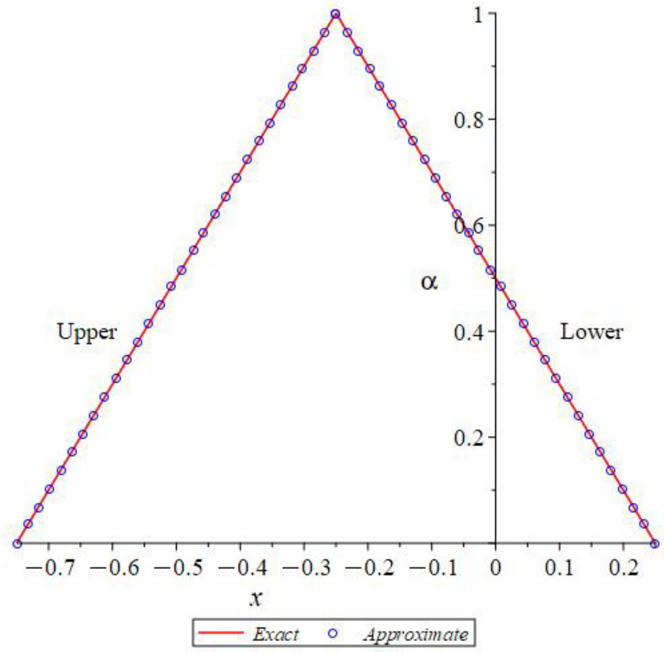

Figure 2: Graph of $\tilde{u}_{2,Exact}(x,\alpha)$ and $\tilde{u}_{2,10}(x,\alpha)$

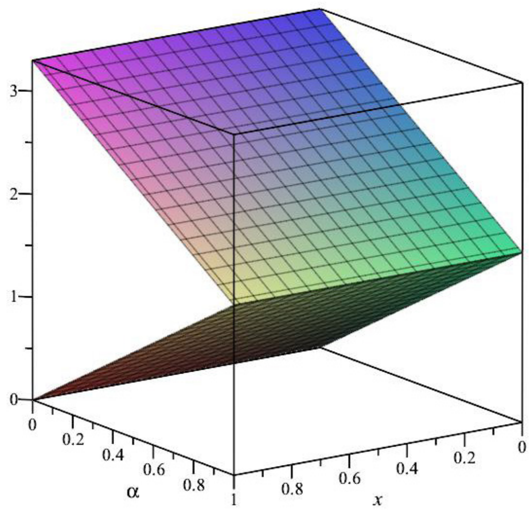

Figure 3: Graph of $\underline{u}_{1,Exact}(x,\alpha)$ and $\overline{u}_{1,Exact}(x,\alpha)$

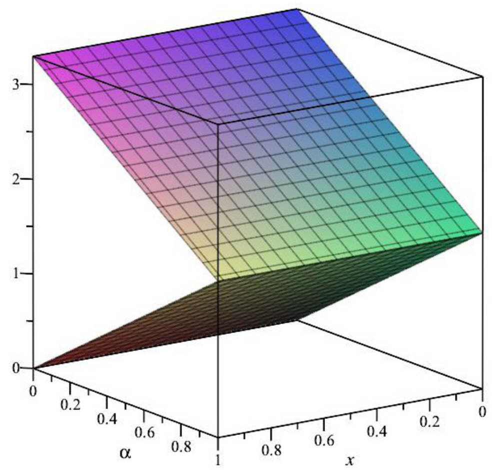

Figure 4: Graph of $\underline{u}_{1,10}(x,\alpha)$ and $\overline{u}_{1,10}(x,\alpha)$

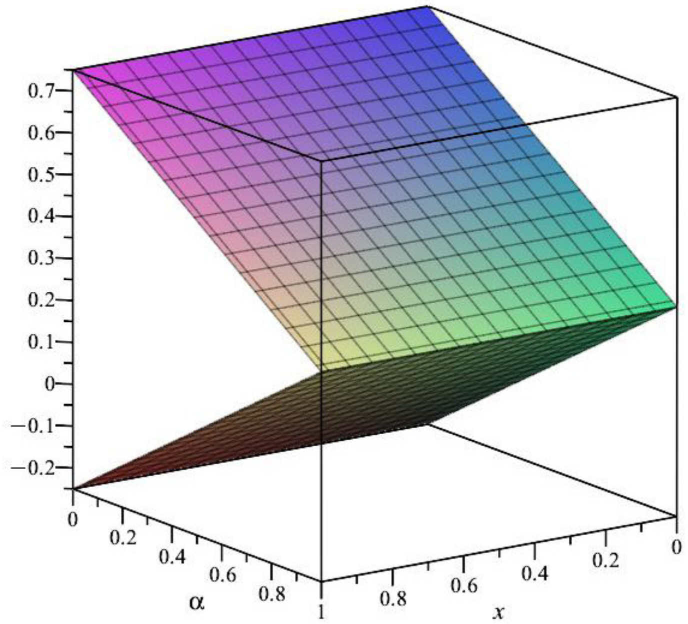

Figure 5: Graph of $\underline{u}_{2,Exact}(x,\alpha)$ and $\bar{u}_{2,Exact}(x,\alpha)$

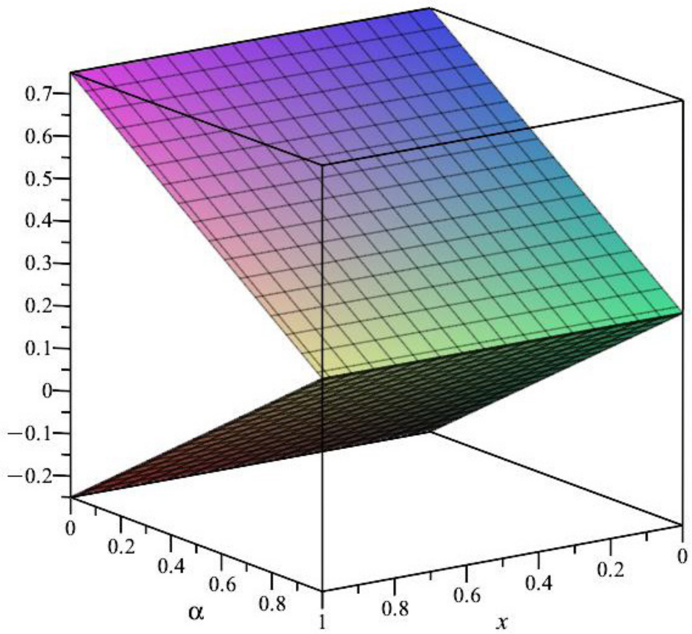

Figure 6: Graph of $\underline{u}_{2,10}(x,\alpha)$ and $\bar{u}_{2,10}(x,\alpha)$

and $a=0$, $b=\frac{\pi}{2}$, where $\tilde{f}_i(x,\alpha)=\left[\underline{f}_i(x,\alpha),\bar{f}_i(x,\alpha)\right]$ are chosen such that the exact solutions will be $\tilde{u}_1(x,\alpha)=\left[\alpha\cos(x),(2-\alpha)\cos(x)\right]$ and $\tilde{u}_2(x,\alpha)=\left[\frac{(\alpha+1)}{2}\sin^2(x),(3-2\alpha^3)\sin^2(x)\right]$.

The lower (L) and upper (U) iterations are given in Eqs. (12) and (13) respectively. Solving the systems of lower and upper, we get the iterations $\tilde{u}_{i,0}(x,\alpha),\tilde{u}_{i,1}(x,\alpha),\ldots,\tilde{u}_{i,n}(x,\alpha)$. Consequently, the series form approximations of the solutions are

$$\tilde{u}_i(x,\alpha)=\tilde{u}_{i,0}(x,\alpha)+\sum_{n=1}^{10}\tilde{u}_{i,n}(x,\alpha).$$

In Table 3 present the maximum errors for the lower and upper between the exact solution and approximate solution using the HPM $(m=10)$ at each point $x_i = 0.1l\pi$, $l = 0,1,\ldots,5$ within the interval $0 < \alpha \le 1$. But, in Table 4 we list the absolute errors (AE) when $\alpha = 1$ and $0 \le x \le \frac{\pi}{2}$, and a comparison with the fixed point method's numerical treatment's absolute errors [21].

Figures 7–12 below display exact solutions $\left(\tilde{u}_{i,Exact}(x,\alpha)\right)$ and the fuzzy approximate solutions by the HPM $\left(\tilde{u}_{i,10}(x)\right)$ of the system (14) are in the form of fuzzy numbers for any $\alpha \in (0,1]$ at $x = \frac{\pi}{4}$.

Table 3: The maximum errors for Problem 2.

$0 < \alpha \le 1$	$\tilde{u}_1(x,\alpha)$		$\tilde{u}_2(x,\alpha)$	
x_i	MAE_L	MAE_U	MAE_L	MAE_U
0.0	0.0	0.0	0.0	0.0
0.1π	$2.337E-11$	$8.346E-12$	$1.747E-11$	$6.239E-12$
0.2π	$3.993E-11$	$1.426E-11$	$2.653E-11$	$9.475E-12$
0.3π	$4.968E-11$	$1.774E-11$	$2.717E-11$	$9.706E-12$
0.4π	$5.262E-11$	$1.879E-11$	$1.941E-11$	$6.933E-12$
0.5π	$4.875E-11$	$1.741E-11$	$3.238E-12$	$1.156E-12$

Table 4: Comparison the absolute errors for Problem 2.

$\alpha = 1$	$\tilde{u}_1(x,\alpha)$		$\tilde{u}_2(x,\alpha)$	
x_i	AE	[21]	AE	[21]
0.0	0.0	0.0	0.0	0.0
0.1π	$8.3464E-12$	$1.52318E-10$	$6.2396E-12$	$8.2427E-11$
0.2π	$1.4260E-11$	$2.55728E-10$	$9.4750E-12$	$1.3157E-10$
0.3π	$1.7742E-11$	$3.10231E-10$	$9.7064E-12$	$1.4742E-10$
0.4π	$1.8793E-11$	$3.15826E-10$	$6.9335E-12$	$1.3003E-10$
0.5π	$1.7410E-11$	$2.72513E-10$	$1.1566E-12$	$7.9341E-11$

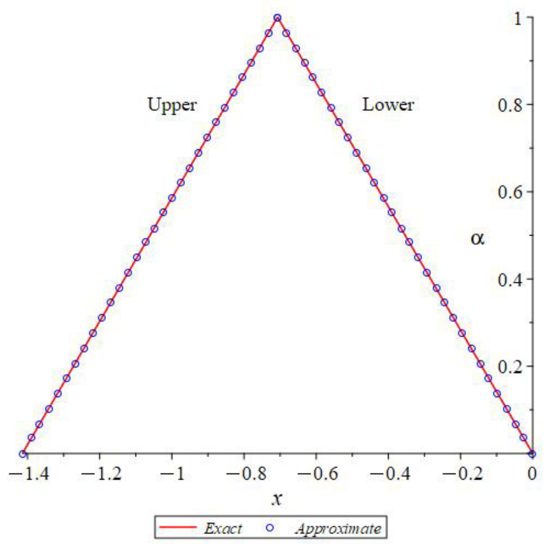

Figure 7: Graph of $\tilde{u}_{1,Exact}(x,\alpha)$ and $\tilde{u}_{1,10}(x,\alpha)$

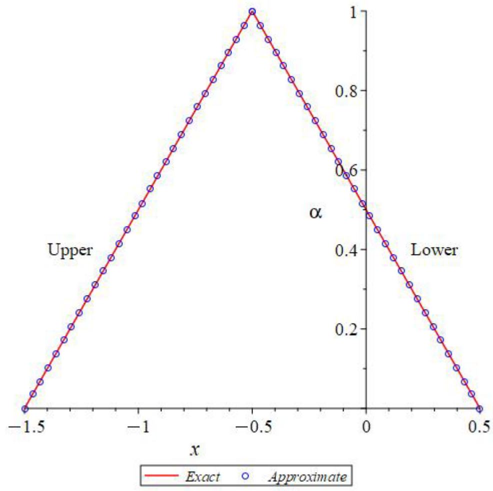

Figure 8: Graph of $\tilde{u}_{2,Exact}(x,\alpha)$ and $\tilde{u}_{2,10}(x,\alpha)$

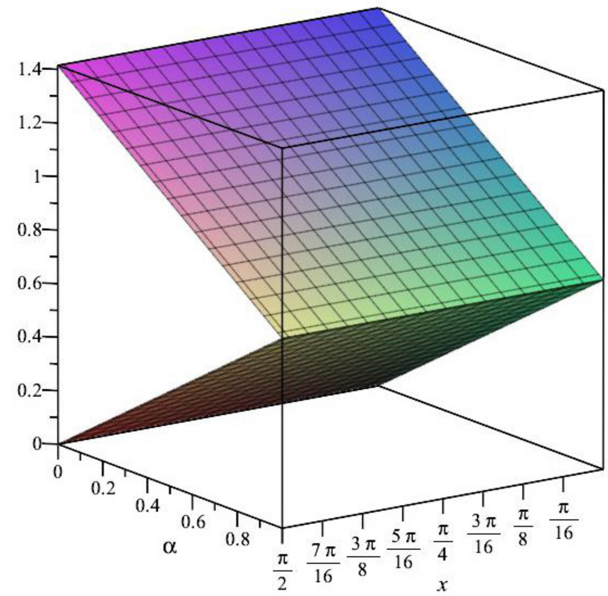

Figure 9: Graph of $\underline{u}_{1,Exact}(x,\alpha)$ and $\overline{u}_{1,Exact}(x,\alpha)$

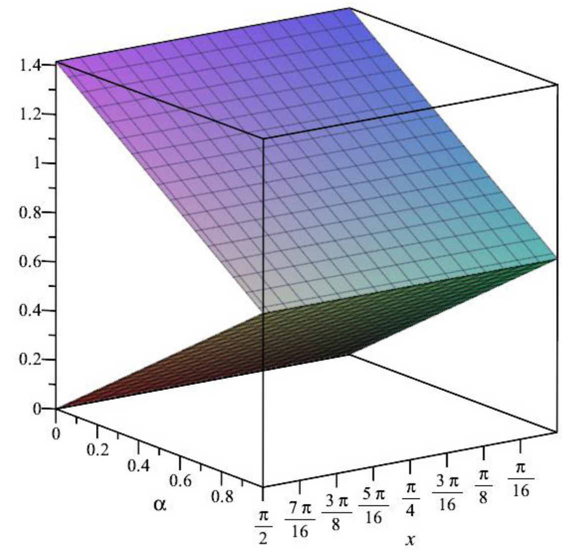

Figure 10: Graph of $\underline{u}_{1,10}(x,\alpha)$ and $\overline{u}_{1,10}(x,\alpha)$

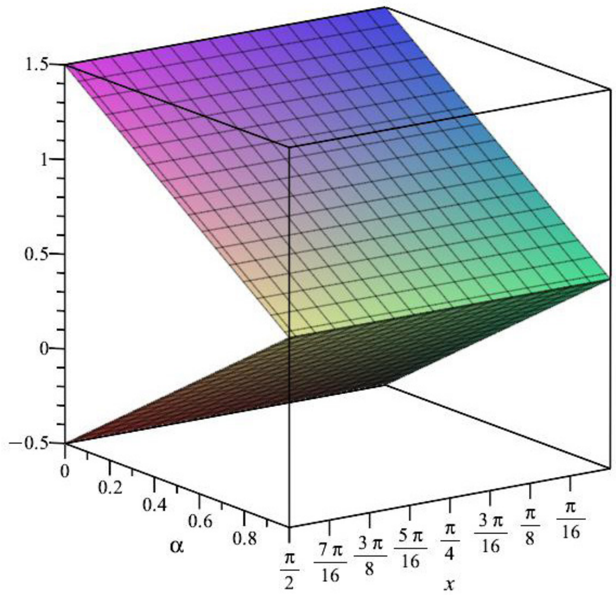

Figure 11: Graph of $\underline{u}_{2,Exact}(x,\alpha)$ and $\overline{u}_{2,Exact}(x,\alpha)$

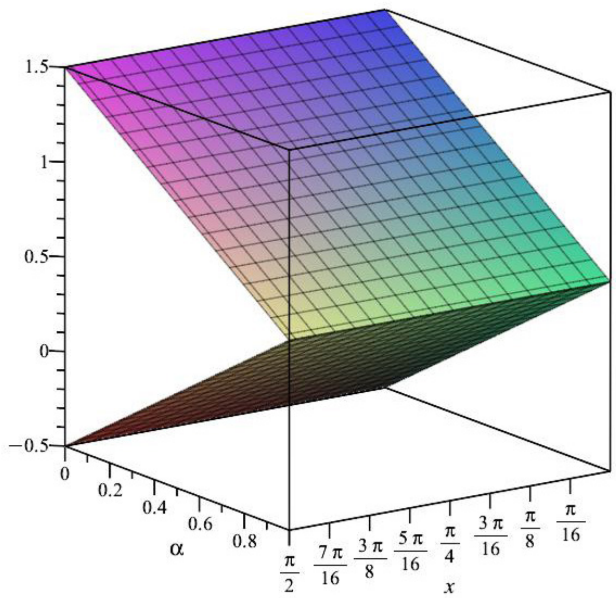

Figure 12: Graph of $\underline{u}_{2,10}(x,\alpha)$ and $\overline{u}_{2,10}(x,\alpha)$

5. Discussion

The main focus of this research is solving FSVFIEs of the second kind by using HPM, which also compares the convergence with [22]. The author in [23] has been converted into a system by using parametric form. The convergence of the Volterra equation by using the variational iteration method using Hilbert space, which is completely different from HPM by using contraction fixed point Banach space.

6. Conclusions

In this research, two main objectives were addressed: first, the convergence of the HPM was investigated using the Banach contraction fixed point theorem, and the sufficient conditions for convergence were examined. Secondly, the approximate solutions of the FSVFIEs are obtained through the use of the HPM. The provided examples illustrate the potential of the method.

Furthermore, the convergence of the solutions was proven, and the findings from numerical examples signify the accuracy and efficiency in this study. To determine the ideal value of the convergence-control parameter, the convergence of this method was qualitatively examined. The above examples demonstrate the method's potential. Numerical results and graphs show that the linear fuzzy system of the Volterra- Fredholm integral equations are well approximated by the method. A comparative analysis is conducted between the exact solutions and the numerical outcomes derived from the absolute errors of the numerical application of the fixed point method with the HPM for $\alpha = 1$ as well as the numerical solution for the given FSVFIEs to achieve the least amount of computation. The error decreases when the number of iterations m increases and gives faster convergence. The FSVFIEs are also illustrated through numerical results and graphs. When compared to other approximation or numerical approaches, This semi-analytical method demonstrated the HPM superior performance.

Acknowledgements

The author is extremely appreciative to one of the reviewers for their very helpful recommendations.

Conflict of interest

The authors declare no conflict of interest in this paper.

Funding

No funding was received in this paper.

References

[1] A. A. Hamoud, and K. P. Ghadle, "Homotopy analysis method for the first order fuzzy Volterra-Fredholm integro-differential equations", *Indonesian Journal of Electrical Engineering and Computer Science*, vol. 11, no. 3, pp. 857–867, 2018.

[2] Abdul-Majid Wazwaz, *Linear and Nonlinear Integral Equations: Methods and Applications*, Springer, 2011.

[3] K. Maleknejad, and Y. Mahmoudi, "Taylor polynomial solution of high-order nonlinear Volterra-Fredholm integro-differential equations", Applied Mathematics and Computation, vol. 145, no. 2-3, pp. 641–653, 2003.

[4] A. Ebadian, Sh. Najafzadeh, F. Farahrooz, and A. A. Khajehnasiri, " On the convergence and numerical computation of two-dimensional fuzzy Volterra–Fredholm integral equation by the homotopy perturbation method" , SeMA Journal, vol. 75, pp. 17–34, 2018.

[5] A. Georgieva, and A. Alidema, Convergence of Homotopy Perturbation Method for Solving of Two-Dimensional Fuzzy Volterra Functional Integral Equations. In: K. Georgiev et al., (eds), Advanced Computing in Industrial Mathematics, Studies in Computational Intelligence, vol 793, Springer Nature Switzerland AG, 2019.

[6] Pakhshan M. A. Hasan, and Nejmaddin A. Sulaiman, "Homotopy perturbation method and convergence analysis for the linear mixed Volterra-Fredholm integral equations", *Iraqi Journal of Science*, vol. 61, no. 2, pp. 409–415, 2020.

[7] Mahasin Thabet Younis, and Waleed Al-Hayani, "A numerical study for solving the systems of fuzzy Fredholm integral equations of the second kind using the Adomian decomposition method", *Iraqi Journal of Science*, vol. 64, no. 7, pp. 3507–3530, 2023.

[8] Shijun Liao, *Beyond Perturbation*: *Introduction to the Homotopy Analysis Method*. CRC Press LLC, 2003.

[9] Shijun Liao, *Homotopy Analysis Method in Nonlinear Differential Equations*, Beijing and Springer-Verlag Berlin Heidelberg, 2012.

[10] Ji Huan He, and Yusry O. El-Dib, "Homotopy perturbation method with three expansions", *Journal of Mathematical Chemistry*, vol. 59, no. 4, pp. 1139–1150, 2021.

[11] Waleed Al-Hayani, and Mahasin Thabet Younis, "Solving fuzzy system of boundary value problems by homotopy perturbation method with green's function", *European Journal of Pure and Applied Mathematics*, vol. 16, no. 2, pp. 1236–1259, 2023.

[12] A. A. Hamoud, and K. P. Ghadle, "Usage of the homotopy analysis method for solving fractional Volterra-Fredholm integro-differential equation of the second kind", Tamkang Journal of Mathematics, vol. 49, no. 4, pp. 301–315, 2018.

[13] Elçin Yusufoglu, "A homotopy perturbation algorithm to solve a system of Fredholm–Volterra type integral equations", *Mathematical and Computer Modelling*, vol. 47, no. 11-12, 1099–1107, 2008.

[14] Ahmed A. M. Fawze, Borhan F. Juma'a, and Waleed Al-Hayani, "Modified of homotopy perturbation technique and its application to system of nonlinear Fredholm integral equations of 2nd Kind", *Iraqi Academics Syndicate International Conference for Pure and Applied Sciences*, *Journal of Physics: Conference Series,* (vol. 1818, no. 1, pp. 012057, 2021). IOP Publishing.

[15] Amjad Ali, Eskandar Ameer, Suhad Subhi Aiadi, Muhammad Tariq, Muhammad Arshad, Nabil Mlaiki, and Wasfi Shatanawi, "New extension to fuzzy dynamic system and fuzzy fixed point results with an application", *AIMS Mathematics*, vol. 8, no. 1, pp. 1208–1229, 2022.

[16] Shazia Kanwal, Mohammed Shehu Shagari, Hassen Aydi, Aiman Mukheimer, and Thabet Abdeljawad, "Common fixed-point results of fuzzy mappings and applications on stochastic Volterra integral equations", *Journal of Inequalities and Applications*, vol. 2022, no. 1, p. 110, 2022.

[17] Mamta Kapoor, Nasser Bin Turki, and Nehad Ali Shah, "Approximate analytical solution of fuzzy linear Volterra integral equation via Elzaki ADM", *Fractal and Fractional*, vol. **7**, no. 9, p. 650, 2023.

[18] Omar Abu Arqub, Shaher Momani, Saleh Al-Mezel, and Marwan Kutbi, "Existence, Uniqueness, and Characterization Theorems for Nonlinear Fuzzy Integro differential Equations of Volterra Type", *Mathematical Problems in Engineering*, vol. 2015, no. 1, pp. 835891, 2015.

[19] Ali. F. Jameel, N. R. Anakira, A. K. Alomari, and Noraziah H. Man, "Solution and analysis of the fuzzy Volterra integral equations via homotopy analysis method", *Computer Modeling in Engineering & Sciences*, vol. 127, no. 3, pp. 875–899, 2021.

[20] Rawaa Ibrahim Esa, Rasha H. Ibraheem, and Ali F. Jameel, "Numerical treatment for solving fuzzy Volterra integral equation by sixth order Runge-Kutta method", *Mathematics and Statistics*, vol. 9, no. 3, pp. 350–356, 2021.

[21] Pakhshan M. Hasan, Nejmaddin A. Sulaiman, Fazlollah Soleymani, and Ali Akgül, "The existence and uniqueness of solution for linear system of mixed Volterra-Fredholm integral equations in Banach space", *AIMS Mathematics*, vol. 5, no. 1, pp. 226–235, 2019.

[22] Hossein Attari, and Allahbakhsh Yazdani, "A computational method for fuzzy Volterra-Fredholm integral equations", *Fuzzy Information and Engineering*, vol. 3, no. 2, pp. 147–156, 2011.

[23] Zaid M. Odibat, "A study on the convergence of variational iteration method", *Mathematical and Computer Modelling*, vol. 51, no. 9–10, pp. 1181–1192, 2010.