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Contravariant reich type F-contraction in F-bipolar metric spaces with application

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Abstract

In this work, we explore fixed point (FP) results using the concept of contravariant Reich-type F-contraction within the framework of F-bipolar metric spaces (F-BMS). Our findings extend certain results from the existing literature. Additionally, we provide an example and an application to demonstrate the existence and uniqueness of the integral equation (IE).

Mathematics Subject Classification (2010): 47H10, 54H25, 47H09

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1. Introduction and preliminaries

In non-linear analysis, the theory of FP holds a conspicuous and esteemed position and has extensive applications across various fields. FP theory started in the early 19th century, when Poincare 1806, became the first person to work on FP theory.

The Banach Contraction Principle holds a significant position in the Complete Metric Space (CMS). This inaugural theorem in metric theory was first investigated in 1922. Over time, by modifying the metric function, many generalizations of CMS have been initiated [1, 2]. The Fixed Point (FP) is a significant tool for obtaining a unique solution to various types of problems. A mapping $\zeta : \mho \to \mho$

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has a FP if $\zeta(r) = r$. Researchers from various fields continue to derive benefits from utilizing the Banach contraction for the betterment of humanity. Kannan provided significant FP results in 1968, which widened the field of FP theory. Later, Reich integrated the principles of Banach contraction and Kannan's contraction and provided a result in 1971. Fisher [3] inaugurated rational expressions in contractive inequality and gave a result within CMS. For a comprehensive exploration of further study in this context, we provided the additional contribution mentioned in [4, 5]

Czerwik [6] introduced the concept of b-metric space (b-MS) by changing the triangular inequality with a non-negative constant $s \geq 1$. Branciari [7] further generalized the idea of MS by substituting the triangular inequality with a rectangular inequality, introducing the concept of a rectangular metric space (RMS). Berinde et al. [8] gave a concise overview of the advancements in FP theory, with a particular focus on b-MS and talked about some significant relevant characteristics. Brzdek [9] provided FP results for non-linear operators in a b-MS, for comprehensive detail see [10, 11]. In 2018, Jleli [12] initiated the notion of F-Metric Space (F-MS), a fascinating generalization of classical MS, b-MS and RMS. After that, Al-Mazrooei et al. [13] proved some FP results for rational inequality in the structure of F-MS.

In the generalized spaces mentioned above, we measure the distance between elements of the same set. However, a question arises: how can we find the distance between elements of different sets? To resolve this problem, Mutlu et al. [14] introduced the concept of Bipolar Metric Space (BMS). The refined concept of BMS has contributed to the advancement of FP results in FP theory. A substantial body of research has been devoted to explore the existence of FP for self and multivalued mappings within the context of BMS[15, 16]. In 2022, Rawat et al. [17] combined two progressive ideas, F-MS and BMS, to create a new concept called F-Bipolar Metric Space (F-BMS). They obtained some FP results from this integration. Recently, B. Alamri [18] utilized Reich and rational type contractions, while A.H. Albargi [19] employed (α,ψ) contraction in the structure of F-BMS to obtain FP results.

Motivated by the interesting setting of F-Bipolar Metric Space (F-BMS), we use F-Contraction to derive fixed point results with application to F-BMS in this manuscript.

Below are the definitions from existing literature that will be utilized in the remaining manuscript.

Definition 1.1 [4] Wardowski consider a nonlinear function $F:(0,\infty) \to (-\infty,0) \cup [0,+\infty)$ which fulfill the conditions given below

(F_1) F is strictly increasing i.e. $\wp < \chi \quad \Rightarrow \quad F(\wp) < F(\chi)$;

(F_2) For each sequence $\alpha_d \subset [0,\infty)$, one has $\lim_{d\to\infty}\alpha_d = 0 \quad \Leftrightarrow \quad \lim_{d\to\infty}F(\alpha_d) = -\infty$;

(F_3) There exist $M \in (0,1)$ such as $\lim_{d\to\infty}\alpha^M F(\alpha) = 0$.

Definition 1.2 [4] *Wardowaki defined that a mapping $\zeta : \mho \to \mho$, on a MS $(\mho,\eth)$ is termed as an $F-$ Contraction if there exist $\aleph > 0$ such that*

$$\aleph + F[\eth(\zeta\wp,\zeta\chi)] \quad \leq \quad F[\eth(\wp,\chi)] \quad \forall \quad \wp,\chi \in \mho.$$

Jleli et [12] gave an elongation to a MS in the given fashion in 2018.

Let $\daleth$ be a family of continuous functions and $T:(0,+\infty) \to (-\infty,0) \cup [0,+\infty)$ fulfilling the conditions given below

(T_1) T is non-decreasing

(T_2) For each sequence $\{\mathbf{t}_d\} \subseteq R^+, \lim_{d\to\infty}\gamma_d = 0 \Leftrightarrow \lim_{d\to\infty}T(\gamma_d) = -\infty$.

Definition 1.3 [12] *Let $\mho$ be a non-empty set and let $\eth = \mho \times \mho \to R^+ \cup \{0\}$, suppose that there be some $(T,\kappa) \in \daleth \times R^+ \cup \{0\}$ in such a manner*

(4_a) $\eth(\wp,\chi) \quad = \quad 0, \quad$ iff $\quad \wp = \chi$

(4_b) $\eth(\wp,\chi) \quad = \quad \eth(\chi,\wp)$

(4_c) For every $S \in N$, $S \geq 2$, and for every $(\mathbf{u}_d)_{d=1}^S \subset \mho$ with $(\mathbf{u}_1,\mathbf{u}_S) = (\wp,\chi)$, we get

$$\eth(\wp,\chi)>0 \quad implies \quad that \quad T(\sum_{d=0}^{S-1}\eth(\wp,\chi)) \;\leq\; T(\sum_{d=0}^{S-1}\eth(\mathbf{u}_d,\mathbf{u}_{d+1}))+\kappa$$

then $(\mho,\eth)$ is called F-MS.

Example 1 [12] *Let* $\mho = R$, *Define* $\eth:\mho\times\mho\to[0,+\infty)$ *by*

$$\eth(\wp,\chi)=\begin{cases}(\wp,\chi)^2, & if\ (\wp,\chi)\in[0,3]\times[0,3]\\ |\wp,\chi|^2, & if\ (\wp,\chi)\notin[0,3]\times[0,3]\end{cases}$$

along with T(t) = In(t) and $In(\kappa)=In(3)$, then $(\mho,\eth)$ is F-MS.

Definition 1.4 *[14] Let* $\mho$ *and* C *be non-empty sets and let* $\eth:\mho\times C\to R^{+}\cup\{0\}$ *be a function given, if the function* $\eth$ *verifies*

(5_a) $\eth(\wp,\chi) = 0$, iff $\wp=\chi$

(5_b) $\eth(\wp,\chi) = \eth(\chi,\wp)$, if $\wp,\chi\in\mho\cap C$

(5_c) $\eth(\wp,\chi) \leq \eth(\wp,\chi')+\eth(\wp',\chi')+\eth(\wp',\chi)$

for all $(\wp,\chi),(\wp',\chi')\in\mho\times C$, then, the triple $(\mho,C,\eth)$ is known as BMS.

Example 2 [14] Let the family of all singleton and compact subsets of R be $\mho$ and C respectively. Define $\eth:\mho\times C\to[o,\infty)$ by

$$\eth(\wp,\Lambda)=|\wp-inf(\Lambda)|+|\wp-sup(\Lambda)|$$

for $\{\wp\}\subseteq\mho$ and $\Lambda\subseteq C$, then $(\mho,C,\eth)$ is a BMS.

Definition 1.5 *[14] Let two bipolar metric spaces* $(\mho_1,\mathbf{C}_1,\eth_1)$ *and* $(\mho_2,\mathbf{C}_2,\eth_2)$ *be given. A mapping* $\zeta:\mho_1\cup\mathbf{C}_1\rightrightarrows\mho_2\cup\mathbf{C}_2$ *is known to be a covarient mapping, if* $\zeta\mho_1\subseteq\mho_2$ *and* $\zeta\mathbf{C}_1\subseteq\mathbf{C}_2$. *Similarly a mapping* $\zeta:\mho_1\cup\mathbf{C}_1\leftrightarrows\mho_2\cup\mathbf{C}_2$ *is said to be a contravarient mapping, if* $\zeta\mho_1\subseteq\mathbf{C}_2$ *and* $\zeta\mho_2\subseteq\mathbf{C}_1$.

The above two progressive ideas (BMS and F-MS) was unified by Rawat et al. and came with a new idea known as F-Bipolar Metric Space, as described below.

Definition 1.6 *[17] Let* $\mho$ *and* C *be non-empty sets and let* $\eth:\mho\times C\to[0,\infty)$ *be a function given. Suppose that there be some* $(T,\kappa)\in\daleth\times[0,+\infty)$ *such that*

(B_1) $\eth(\wp,\chi) = 0$, iff $\wp=\chi$

(B_2) $\eth(\wp,\chi) = \eth(\chi,\wp)$, if $\wp,\chi\in\mho\cap C$

(B_3) For all S∈ N, S ≥ 2 and for all $(\mathbf{u}_d)_{d=1}^{S}\subset\mho$ and $(\mathbf{v}_d)_{d=1}^{S}\subset C$ along with $(\mathbf{u}_1,\mathbf{v}_S)=(\wp,\chi)$, we have

$$\eth(\wp,\chi) \;>\; 0 \quad implies \quad that \quad T(\eth(\wp,\chi)) \;\leq\; T(\sum_{d=1}^{S-1}\eth(\mathbf{u}_{d+1},\mathbf{v}_d)+\sum_{d=1}^{S}\eth(\mathbf{u}_d,\mathbf{v}_d))+\kappa$$

then $(\mho,C,\eth)$ is called as F- BMS.

Example 3 *[17] Let* $\mho=\{1,2\}$ *and* $C=\{2,7\}$. *Define* $\eth:\mho\times C\to[0,+\infty)$ *by*

$$\eth(1,2)=6,\eth(1,7)=10,\eth(2,2)=0$$

here all the properties (B_1), (B_2) and (B_3) is fulfilled with T(t) = In(t) ∈ $\daleth$ and $\kappa>0$.

Hence $(\mho,C,\eth)$ is F-BMS.

2. Fixed point results

We start by introducing a key definition that forms the basis for the main results presented in this manuscript.

Definition 2.1 *Let* $(\mho, C, \eth)$ *be a F-BMS. A mapping* $\zeta : (\mho, C, \eth) \leftrightarrows (\mho, C, \eth)$ *is known to be a contravariant Reich-type* F*-contraction, if there exist constants* $\mu_1, \mu_2, \mu_3 \in [0,1)$*, where* $\mu_1 + \mu_2 + \mu_3 < 1$ *and*

$$\aleph + F[\eth(\zeta\wp, \zeta\chi)] \leq F[\mu_1 \eth(\wp, \chi) + \mu_2 \eth(\wp, \zeta\wp) + \mu_3 \eth(\chi, \zeta\chi)] \tag{1}$$

for all $(\wp, \chi) \in \mho \times C$.

Theorem 2.2 Let $(\mho, C, \eth)$ be a complete F-BMS and let $\zeta : (\mho, C, \eth) \leftrightarrows (\mho, C, \eth)$ be a contravariant Reich-type F-contraction, then there will be a unique FP of the mapping $\zeta : \mho \cup C \to \mho \cup C$, provided that the mapping $\zeta : (\mho, C, \eth) \leftrightarrows (\mho, C, \eth)$ is continuous.

Proof. Let $\wp_0$ be an arbitrary point in $\mho$. Define the bisequence $(\wp_d, \chi_d)$ in $(\mho, C, \eth)$ and

$$\chi_d = \zeta\wp_d \quad and \quad \wp_{d+1} = \zeta\chi_d \quad \therefore \quad \forall \quad d = 1,2,3...$$

Now by (1), we obtain

$$\begin{aligned} \aleph + F[\eth(\wp_d, \chi_d)] &= \aleph + F[\eth(\zeta\chi_{d-1}, \zeta\wp_d] \\ &\leq F[\mu_1 \eth(\wp_d, \chi_{d-1}) + \mu_2 \eth(\wp_d, \zeta\wp_d) + \mu_3 \eth(\chi_{d-1}, \zeta\chi_{d-1})]. \end{aligned}$$

As F is strictly increasing so,

$$\begin{aligned} \eth(\wp_d, \chi_d) &\leq \mu_1 \eth(\wp_d, \chi_{d-1}) + \mu_2 \eth(\wp_d, \zeta\wp_d) + \mu_3 \eth(\chi_{d-1}, \zeta\chi_{d-1}) \\ \eth(\wp_d, \chi_d) &\leq \mu_1 \eth(\wp_d, \chi_{d-1}) + \mu_2 \eth(\wp_d, \chi_d) + \mu_3 \eth(\wp_d, \chi_{d-1}) \end{aligned}$$

implies that

$$\eth(\wp_d, \chi_d) \leq \frac{\mu_1 + \mu_3}{1 - \mu_2} \eth(\wp_d, \chi_{d-1}). \tag{2}$$

Moreover,

$$\begin{aligned} \eth(\wp_d, \chi_{d-1}) &= \eth[\zeta\chi_{d-1}, \zeta\wp_{d-1}] \\ &\leq \mu_1 \eth(\wp_{d-1}, \chi_{d-1}) + \mu_2 \eth(\wp_{d-1}, \zeta\wp_{d-1}) + \mu_3 \eth(\chi_{d-1}, \zeta\chi_{d-1}) \\ &\leq \mu_1 \eth(\wp_{d-1}, \chi_{d-1}) + \mu_2 \eth(\wp_{d-1}, \chi_{d-1}) + \mu_3 \eth(\wp_d, \chi_{d-1}). \end{aligned}$$

Which implies that,

$$\eth(\wp_d, \chi_{d-1}) \leq \frac{\mu_1 + \mu_2}{1 - \mu_3} \eth(\wp_{d-1}, \chi_{d-1}). \tag{3}$$

Let $\gamma = max[\frac{\mu_1 + \mu_3}{1 - \mu_2}, \frac{\mu_1 + \mu_2}{1 - \mu_3}] < 1$. It can be easily seen by (2) and (3) that,

$$\eth(\wp_d, \chi_d) \leq \gamma^{2d} \eth(\wp_0, \chi_0). \tag{4}$$

Similarly we have,

$$\eth(\wp_{d+1}, \chi_d) \leq \gamma^{2d+1} \eth(\wp_0, \chi_0). \quad \therefore \quad \forall \quad d = 1,2,3... \tag{5}$$

Let $(T, \kappa) \in \daleth \times [0, \infty)$ provided that (B_3) is fulfilled. Let $\varepsilon > 0$ be fixed.
Via (T_2), there exist $\delta > 0$ such as

$$0 < t < \delta implies T(t) < T(\varepsilon) - \kappa. \tag{6}$$

Now by (4) and (5), we gain

$$\sum_{i=d}^{p-1}\eth(\wp_{i+1},\chi_i)+\sum_{i=d}^{p}\eth(\wp_i,\chi_i)\leq (\gamma^{2d}+\gamma^{2d+2}+\ldots+\gamma^{2p})\eth(\wp_0,\chi_0)+(\gamma^{2d+1}+\gamma^{2d+3}+\ldots+\gamma^{2p-1})\eth(\wp_0,\chi_0).$$

$$\leq \gamma^{2d}\sum_{d=0}^{\infty}\gamma^{d}\eth(\wp_0,\chi_0)=\frac{\gamma^{2d}}{1-\gamma}\eth(\wp_0,\chi_0).$$

For $p>d$. Since $\lim_{d\to\infty},\frac{\gamma^{2d}}{1-\gamma}\eth(\wp_0,\chi_0)$. So there exist $\mathbf{d}_0\in N$ such that

$$0<\frac{\gamma^{2d}}{1-\gamma}\eth(\wp_0,\chi_0)<\delta,$$

for $d\geq\mathbf{d}_0$. So, for $S>d>\mathbf{d}_0$, utilizing (T_1) and (6), we have

$$T(\sum_{i=d}^{S-1}\eth(\wp_{i+1},\chi_i)+\sum_{i=d}^{S}\eth(\wp_i,\chi_i))\leq T(\frac{\gamma^{\mathbf{2d}}}{1-\gamma}\eth(\wp_0,\chi_0)) < T(\varepsilon)-\kappa. \tag{7}$$

From (B_3) and inequality (7),we locate that $\eth(\wp_d,\chi_p)>0$, implies

$$T(\eth(\wp_d,\chi_p)) \leq T(\sum_{i=d}^{S-1}\eth(\wp_{i+1},\chi_i)+\sum_{i=d}^{S}\eth(\wp_i,\chi_i))+\kappa < T(\varepsilon).$$

Similarly, for $d>S\geq\mathbf{d}_0,\eth(\wp_d,\chi_S)>0$ implies

$$T(\eth(\wp_d,\chi_S)) \leq T(\sum_{i=S}^{d-1}\eth(\wp_{i+1},\chi_i)+\sum_{i=S}^{d}\eth(\wp_i,\chi_i)+\kappa < T(\varepsilon).$$

Thus by (T_1), $\eth(\wp_d,\chi_S)<\varepsilon$, $\forall$ $S,d\leq\mathbf{d}_0$. Therefore, $(\wp_d,\chi_d)$ is a cauchy bisequence in $(\mho,C,\eth)$. As $(\mho,C,\eth)$ is complete, so $(\wp_d,\chi_d)$ biconverges to a point $\rho\in\mho\cap C$. So, $\wp_d\to\rho$ and $\chi_d\to\rho$. Furthermore, seeing that the contravarient mapping ζ is continuous , so we possess

$$(\wp_d)\to\rho \quad implies \quad that \quad (\chi_d)=(\zeta\wp_d)\to\zeta\rho.$$

Additionally, since (χ_d) has a unique limit ρ in $\mho\cap C$. Hence, $\zeta\rho=\rho$, so ζ has a fixed point.
Now if there is another and distinct fixed point ρ^* in ζ, then $\zeta\rho^*=\rho^*$ yields that $\rho^*\in\mho\cap C$. Then,

$$\aleph+F[\eth(\rho,\rho^*)] = \aleph+F[\eth(\zeta\rho,\zeta\rho^*)] \leq F[\mu_1\eth(\rho,\rho^*)+\mu_2\eth(\rho,\zeta\rho)+\mu_3\eth(\rho^*,\zeta\rho^*)].$$

As F is increasing so,

$$\begin{aligned}\eth(\rho,\rho^*) &= \eth(\zeta\rho,\zeta\rho^*) \leq \mu_1\eth(\rho,\rho^*)+\mu_2\eth(\rho,\zeta\rho)+\mu_3\eth(\rho^*,\zeta\rho^*).\\ \eth(\rho,\rho^*) &\leq \mu_1\eth(\rho,\rho^*).\end{aligned}$$

Which is a contradiction, so $\rho=\rho^*$. As a result ζ has a unique FP .
The above result yields the following corollaries which can be easily proved

Corollary 1 Let $(\mho,C,\eth)$ be a complete F-BMS and let $\zeta:(\mho,C,\eth)\rightleftarrows(\mho,C,\eth)$ be a contravarient mapping. If there exist some constant $\mu\in[0,1)$ and

$$\aleph+F[\eth(\zeta\wp,\zeta\chi)] \leq F[\mu\ \eth(\wp,\chi)]$$

for all $(\wp,\chi)\in\mho\times C$, then the mapping $\zeta:\mho\cup C\to\mho\cup C$ has a unique FP, provided that the mapping $\zeta:(\mho,C,\eth)\rightleftarrows(\mho,C,\eth)$ is continuous.

Corollary 2 Let $(\mho, C, \eth)$ be a complete F-BMS and let $\zeta : (\mho, C, \eth) \rightleftarrows (\mho, C, \eth)$ be a contravarient mapping. If there exist some constant $\eta < \frac{1}{3}$ and

$$\aleph + F[\eth(\zeta\wp, \zeta\chi)] \leq F[\eta \{\eth(\wp, \chi) + \eth(\wp, \zeta\wp) + \eth(\chi, \zeta\chi)\}]$$

for all $(\wp, \chi) \in \mho \times C$, then the mapping $\zeta : \mho \cup C \to \mho \cup C$ has a unique FP, provided that the mapping $\zeta : (\mho, C, \eth) \rightleftarrows (\mho, C, \eth)$ is continuous.

Corollary 3 Let $(\mho, C, \eth)$ be a complete F-BMS and let $\zeta : (\mho, C, \eth) \rightleftarrows (\mho, C, \eth)$ be a contravarient mapping. If there exist a constant $\eta < \frac{1}{2}$ and

$$\aleph + F[\eth(\zeta\wp, \zeta\chi)] \leq F[\eta \{\eth(\wp, \zeta\wp) + \eth(\chi, \zeta\chi)\}].$$

For all $(\wp, \chi) \in \mho \times C$, then the mapping $\zeta : \mho \cup C \to \mho \cup C$ has a unique FP, provided that the mapping $\zeta : (\mho, C, \eth) \rightleftarrows (\mho, C, \eth)$ is continuous.

Remark 1 If we set $F(t) = t$ in Theorem 2.2. We obtained FP results of Almari [18].

Theorem 2.3 Let $(\mho, C, \eth)$ be a complete F-BMS and let $\zeta : (\mho, C, \eth) \leftrightarrows (\mho, C, \eth)$ be a contravarient mapping,and if there exist $\eta \in (0,1)$ such that

$$\aleph + F[\eth(\zeta\wp, \zeta\chi)] \leq F[\eta \; Max\{\eth(\wp, \chi), \eth(\wp, \zeta\wp), \eth(\chi, \zeta\chi)\}]. \tag{8}$$

Then the mapping $\zeta : \mho \cup C \to \mho \cup C$ has a unique FP, given that the mapping $\zeta : (\mho, C, \eth) \leftrightarrows (\mho, C, \eth)$ is continuous.

Proof. Let $\wp_0$ be any arbitrary point in $\mho$. Define the bisequence $(\wp_n, \chi_d)$ in $(\mho, C, \eth)$ by

$$\chi_d = \zeta\wp_d \quad and \quad \wp_{d+1} = \zeta\chi_d$$

for all d=1,2,3..., Now by (8), we obtain

$$\begin{aligned}\aleph + F[\eth(\wp_d, \chi_d] &= F[\eth(\zeta\chi_{d-1}, \zeta\wp_d)] \\ &\leq F[\eta \; Max\{\eth(\wp_d, \chi_{d-1}), \eth(\wp_d, \zeta\wp_d), \eth(\chi_{d-1}, \zeta\chi_{d-1})\}].\end{aligned}$$

As F is strictly increasing so,

$$\begin{aligned}&\leq \eta \; Max\{\eth(\wp_d, \chi_{d-1}), \eth(\wp_d, \zeta\wp_d), \eth(\chi_{d-1}, \zeta\chi_{d-1})\}. \\ &\leq \eta \; Max\{\eth(\wp_d, \chi_{d-1}), \eth(\wp_d, \chi_d), \eth(\wp_d, \chi_{d-1})\}. \\ &\leq \eta \; Max\{\eth(\wp_d, \chi_{d-1}), \eth(\wp_d, \chi_d)\}.\end{aligned} \tag{9}$$

Now if $Max\{\eth(\wp_d, \chi_d), \eth(\wp_d, \chi_{d-1})\} = \eth(\wp_d, \chi_d)$, then we have,

$$\eth(\wp_d, \chi_d) \leq \eta \; \eth(\wp_d, \chi_d)$$

Which is a contradiction. Thus, $Max\{\eth(\wp_d, \chi_d), \eth(\wp_d, \chi_{d-1})\} = \eth(\wp_d, \chi_{d-1})$. Hence by (9), we have,

$$\eth(\wp_d, \chi_d) \leq \eta \; \eth(\wp_d, \chi_{d-1}). \tag{10}$$

Similarly,

$$\begin{aligned}\eth(\wp_d, \chi_{d-1}) &= \eth(\zeta\chi_{d-1}, \zeta\wp_{d-1}) \\ &\leq \eta \; Max\{\eth(\wp_{d-1}, \chi_{d-1}), \eth(\wp_d, \zeta\wp_{d-1}), \eth(\zeta\chi_{d-1}, \chi_{d-1})\}. \\ &\leq \eta \; Max\{\eth(\wp_{d-1}, \chi_{d-1}), \eth(\wp_{d-1}, \chi_{d-1}), \eth(\wp_d, \chi_{d-1})\}. \\ &\leq \eta \; Max\{\eth(\wp_{d-1}, \chi_{d-1}), \eth(\wp_d, \chi_{d-1})\}.\end{aligned}$$

If $Max\{\eth(\wp_{d-1},\chi_{d-1}),\eth(\wp_d,\chi_{d-1})\}=\eth(\wp_d,\chi_{d-1})$, then we have,

$$\eth(\wp_d,\chi_{d-1}) \leq \eta \quad \eth(\wp_d,\chi_{d-1})$$

Which is contradiction. So, $Max\{\eth(\wp_{d-1},\chi_{d-1}),\eth(\wp_d,\chi_{d-1})\}=\eth(\wp_{d-1},\chi_{d-1})$. Hence,

$$\eth(\wp_d,\chi_{d-1}) \leq \eta \quad \eth(\wp_{d-1},\chi_{d-1}). \tag{11}$$

Now by (10) and (11), we can easily see that

$$\eth(\wp_d,\chi_d) \leq \eta^{2d}\eth(\wp_0,\chi_0).$$

Similarly,

$$\eth(\wp_{d+1},\chi_d) \leq \eta^{2d+1}\eth(\wp_0,\chi_0).$$

The remaining proof is same as that of Theorem 2.2.

Example 4 *Let* $\mho=\{4,8,10,20\}$ *and* $C=\{3,5,8,11\}$. *Define a metric* $\eth:\mho\times C\rightarrow[0,\infty)$ *by*

$$\eth(\wp,\chi)=3^{|\wp-\chi|}$$

Then, $(\mho,C,\eth)$ is a complete F-BMS. Define the contravarient mapping $\zeta:\mho\cup C\leftrightarrows\mho\cup C$ by

$$\zeta(\wp)=\begin{cases}8, & if\ \wp\in\mho\cup\{11\}\\ 4, & otherwise.\end{cases}$$

Then all the condition of Theorem (2.2) are satisfied with $\mu_1=\frac{1}{3},\mu_2=\frac{1}{3}$ and $\mu_3=\frac{1}{4}$ Thus by Theorem (2.2), '8' is the unique FP of ζ.

2. Application

2.1 Integral Equation

The metric FP theory stands as a prominent and impactful framework employed to analyze both differential and integral equations. Recognizing that a abundance of real world challenges can be reformulated as problems involving differentials and integral equations, we can conclude the significance of the metric FP theory in qualitative science and technology. Particularly, differential and integral equations appear in several scientific areas to include the different developments in optimal control, engineering, game theory, economy, and so on. In this part of manuscript, the Reich-type F-contraction is applied to the integral equation to showed existence and uniqueness of it's solution.

Theorem 2.4 Let's consider the IE

$$\psi(\wp) = h(\wp)+\int_{\mho\cup C}(M(\wp,\chi,\psi(\wp))\eth\chi$$

Where $\mho\cup C$ is a Lebesgue measurable set. Assume that

1. $M:(\mho^2\cup C^2)\times[0,\infty)\rightarrow[0,\infty)$ and $f\in A^\infty(\mho)\cup A^\infty(C)$.
2. There is a continuous function $\Psi:\mho^2\cup C^2\rightarrow[0,\infty)$ such that

$$\begin{aligned}&|M(\wp,\chi,\psi(\wp))-M(\wp,\chi,\phi(\wp))|\\ \leq\ & \frac{1}{3}\Psi(\wp,\chi)\{\mu_1|\psi(\chi)-\phi(\chi)|+\mu_2|\phi(\chi)-T\phi(\chi)|+\mu_3|\psi(\chi)-T\psi(\chi)|\}\end{aligned}$$

for all $\wp,\chi\in(\mho^2\cup C^2)$, and $T:A^\infty(\mho)\cup A^\infty(C)\rightarrow A^\infty(\mho)\cup A^\infty(C)$.

3. $\| \int_{\mho \cup C} \Psi(\wp, \chi) \eth\chi \| \leq 1$, *i.e.*, $\mathbf{Sup}_{\wp \in \mho \cup C} \int |\Psi(\wp, \chi) \eth\chi \leq 1$.

Then the IE is a unique solution in $A^{\infty}(\mho) \cup A^{\infty}(C)$.

Proof. Suppose that $\mho$ and C are Lebesgue measurable sets, where $\prod = A^{\infty}(\mho)$ and $\coprod = A^{\infty}(C)$ be two normed linear spaces and $m(\mho \cup C) < \infty$. Let $\eth: \prod \times \coprod \to [0,\infty)$ be given as

$$\eth(x,\varsigma) = \| \mathbf{x} - \varsigma \|_{\infty},$$

for all $x,\varsigma \in \prod \times \coprod$. Then, $(\prod, \coprod, \eth)$ is a F-BMS. Define $T: \prod \cup \coprod \to \prod \cup \coprod$ by

$$T(\psi(\wp)) = h(\wp) + \int_{\mho \cup C} (M(\wp, \chi, \psi(\wp)) \eth\chi$$

for $\wp \in \mho \cup C$. Now we obtain

$$\begin{aligned}
\eth(T(\psi(\wp)), T(\phi(\wp))) &= \| T(\psi(\wp)) - T(\phi(\wp)) \| \\
&= | \int_{\mho \cup C} (M(\wp, \chi, \psi(\wp)) \eth\chi - \int_{\mho \cup C} (M(\wp, \chi, \phi(\wp)) \eth\chi | \\
&\leq \int_{\mho \cup C} | (M(\wp, \chi, \psi(\wp)) - (M(\wp, \chi, \phi(\wp)) | \eth\chi \\
&\leq \int_{\mho \cup C} \frac{1}{3} \Psi(\wp, \chi) \{ \mu_1 |\psi(\chi) - \phi(\chi)| + \mu_2 |\phi(\chi) - T\phi(\chi)| + \mu_3 |\psi(\chi) - T\psi(\chi)| \} \eth\chi \\
&\leq \frac{1}{3} \{ \mu_1 \|\psi(\chi) - \phi(\chi)\| + \mu_2 \|\phi(\chi) - T\phi(\chi)\| + \mu_3 \|\psi(\chi) - T\psi(\chi)\| \} \int_{\mho \cup C} |\Psi(\wp, \chi)| \eth\chi \\
&\leq \frac{1}{3} \{ \mu_1 \|\psi(\chi) - \phi(\chi)\| + \mu_2 \|\phi(\chi) - T\phi(\chi)\| + \mu_3 \|\psi(\chi) - T\psi(\chi)\| \} \mathbf{Sup}_{\wp \in \mho \cup C} \int_{\mho \cup C} |\Psi(\wp, \chi)| \eth\chi \\
&\leq \frac{1}{3} \{ \mu_1 \|\psi(\chi) - \phi(\chi)\| + \mu_2 \|\phi(\chi) - T\phi(\chi)\| + \mu_3 \|\psi(\chi) - T\psi(\chi)\| \} \\
&\leq \frac{1}{3} \{ \mu_1 \|\psi(\chi) - \phi(\chi)\| + \mu_2 \|\phi(\chi) - T\phi(\chi)\| + \mu_3 \|\psi(\chi) - T\psi(\chi)\| \} \\
&\leq \frac{1}{3} \{ \mu_1 \eth(\psi(\chi) - \phi(\chi) + \mu_2 \eth(\phi(\chi) - T\phi(\chi)) + \mu_3 \eth(\psi(\chi) - T\psi(\chi)) \} \\
&= F\{ \mu_1 \eth(\psi(\chi) - \phi(\chi) + \mu_2 \eth(\phi(\chi) - T\phi(\chi)) + \mu_3 \eth(\psi(\chi) - T\psi(\chi)) \}
\end{aligned}$$

Here $F(t) = \frac{1}{3} t$ and hence by Theorem (2.2), T possesses a unique FP in $\prod \cup \coprod$.

3. Conclusion

In this paper we obtained FP results using F-Contraction in the notion of F-BMS. Our results are the extension of some FP results which is already proven in the recent studies. We also provided a non-trivial example and application to showed the existence and uniqueness of the solution of IE to strengthen our result.

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Availability of Data and Materials

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Ethics approval and consent to participate

Not applicable.

Consent for publication

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Competing interest

The authors declare that they have no competing interests concerning the publication of this article.

Author's contributions

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