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On Huang-Samet multivalued p-contractions

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Abstract

This paper is devoted to prove the existence of fixed points for some classes of multivalued maps in the context of metric spaces. The obtained results generalize the recent theorems of Huang and Samet. Some examples are presented making effective our results.

Key words and phrases: complete metric space; multivalued mapping; lower semi-continuous; weakly Picard continuous; p-contraction; (ψ,Γ,α)-contraction; fixed point; Euler Gamma function.

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1. Introduction and Preliminaries

Denote by $CB(X)$ the set of all nonempty closed and bounded subsets of (X,d) a metric space (MS). For $\Lambda,\Omega \in CB(X)$,

$$H(\Lambda,\Omega) = \max\left\{\sup_{\zeta\in\Lambda}\delta(\zeta,\Omega), \sup_{\varsigma\in\Omega}\delta(\varsigma,\Lambda)\right\},$$

where $\delta(\omega,\Lambda) = \inf\{d(\omega,\zeta) \mid \zeta \in \Lambda\}$ is the distance from an element ω to the set Λ. The map H is known as Hausdorff metric induced by d. ω in X is termed as a fixed point (FP) of a multivalued mapping $T : X \to 2^X$ when $\omega \in T(\omega)$. Here, 2^X is the set of nonempty subsets of X. Also, $\omega \in X$ is a

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FP of $S: X \to X \times X$ if $S(\omega,\omega)=\omega$. Fix(S) denotes the set of FPs of S.Note that $T: X \to CB(X)$ is a contraction if for all $\ddot{\Re},\dot{b} \in X$,

$$H(T\ddot{\Re},T\dot{b}) \le rd(\ddot{\Re},\dot{b}),$$

where $r \in [0,1)$. The existence of FPs for multivalued mappings was explored in [5] by Nadler. His related result is in the following:

Theorem 1.1: [5] *Let $T: X \to CB(X)$ be a contraction mapping on a complete metric space (CMS) (X,d). Then, there T admits a FP.*

This theorem generalizes the Banach contraction principle [6] for multivalued maps. Moreover, the applications of this theorem are used in many fields as game theory, mathematical economics, computing homology of maps, differential inclusions, discontinuous differential equations, optimal control, digital imaging and computer assisted proofs in dynamics [1–4]. Later, this theorem has been generalized and extended in variant directions. For more details, see [7–13].

Recently, Huang and Samet [14] introduced new classes of contractions, termed as the class of $p-$contractions with respect to (wrt) a family of mappings (including classes of contractions) via a finite number of maps $S_i : X \to X$. They generalized the Banach FP theorem. Also, they introduced another class of maps called the class of (ψ,Γ,α)-contractions, including contractive maps $T: X \to X$ via the ratio $\psi(\frac{\Gamma(s+1)}{\Gamma(s+\alpha)})$, where Γ is the Euler Gamma function, $\psi:[0,\infty,)\to[0,\infty)$ is a function, and $\alpha \in (0,1)$ is a real.

Our work is to answer to an open problem suggested in [14], that is, to study the multivalued version of the main result in [14] by considering the class of multivalued mappings $T: X \to CB(X)$.

The next is needful in the sequel.

Definition 1.2: *We say that a mapping $T: X \to CB(X)$ is weakly Picard continuous on (X,d) if the following condition holds: If for each $\ddot{\Re},\dot{c}$ in X and for each $\{\breve{e}_n\}\subset X$ so that $\breve{e}_0=\ddot{\Re},\breve{e}_{n+1}\in T\breve{e}_n$ for $n\ge 0$ and*

$$\lim_{n\to\infty} d(\breve{e}_n,\dot{c})=0,$$

then there is a subsequence $\{\breve{e}_{n_j}\}$ of $\{\breve{e}_n\}$ such that

$$\lim_{j\to\infty} H(T\breve{e}_{n_j},T\dot{c})=0.$$

Remark 1.3: *If $T: X \to CB(x)$ is continuous on (X,d), then T is weakly Picard continuous (WPC) on (X,d). However, the converse is not necessarily true, as demonstrated in the next example.*

Example 1.4: *Let d be the Euclid metric on* $[0,1]$*, that is,*

$$d(\ddot{i},\dot{q})=|\ddot{i}-\dot{q}|,\quad \ddot{i},\dot{q}\in[0,1].$$

Take $T:[0,1]\to CB([0,1])$ as

$$T\ddot{i}=\begin{cases}\left\{\frac{\ddot{i}}{2},0\right\}, & \text{if } 0\le\ddot{i}<1,\\ \left\{\frac{1}{4},0\right\}, & \text{if } \ddot{i}=1.\end{cases}$$

Here, T is not continuous at $\ddot{i}=1$. Indeed,

$$H(T\ddot{i},T1)=\max\{\Delta(T\ddot{i},T1),\Delta(T1,T\ddot{i})\}=\frac{1}{4}\max\{\min\{1,|2\ddot{i}-1|\},\min\{2\ddot{i},|2\ddot{i}-1|\}\}$$
$$=\frac{1}{4}\max\{|2\ddot{i}-1|,\min\{2\ddot{i},|2\ddot{i}-1|\}\}.$$

Then

$$H(T\ddot{i},T1)=\begin{cases}\frac{\ddot{i}}{2}, & if\ 2\ddot{i}\leq|2\ddot{i}-1|,\\ \frac{1}{4}|2\ddot{i}-1|, & if\ 2\ddot{i}>|2\ddot{i}-1|.\end{cases}$$

Then

$$\lim_{\ddot{i}\to1}H(T\ddot{i},T1)=\frac{1}{4}>0.$$

For every $\ddot{i}\in X$, $\{\breve{e}_n\}$ defined by $\breve{e}_0=\ddot{i}$ and $\breve{e}_{n+1}\in T\breve{e}_n$, for all $n\geq0$ is given by

$$\breve{e}_n=0\text{or}\breve{e}_n=\frac{1}{2^n}\forall n\geq0.$$

Then

$$T\breve{e}_n=\{0\}\text{or}T\breve{e}_n=\{0,\frac{1}{2^n}\}\forall n\geq0.$$

Let $\dot{q}\in X$ be so that

$$\lim_{n\to\infty}d(\breve{e}_n,\dot{q})=0.$$

Thus, necessarily $\dot{q}=0.$ Hence,

$$\lim_{n\to\infty}H(T\breve{e}_n,T\dot{q})=\lim_{n\to\infty}H(T\breve{e}_n,\{0\})=0.$$

Thus, T is WPC on $([0,1],d)$.

Definition 1.5 Let (X,d) a MS. $f:X\to[0,\infty)$ is termed as lower semi-continuous (LSC) if for every $\ddot{i}\in X$ and sequence $\{\breve{e}_n\}\subset X$,

$$\lim_{n\to\infty}d(\breve{e}_n,\ddot{i})=0\Rightarrow f(\ddot{i})\leq\liminf_{n\to\infty}f(\breve{e}_n).$$

For $T:X\to CB(X)$, consider $f_T:X\to[0,\infty)$ as

$$f_T(\ddot{i})=d(\ddot{i},T\ddot{i})\text{forall}\ddot{i}\in X.$$

Remark 1.6: *Remark that if T is WPC on (X,d), then f_T is LSC. However, the converse is not necessarily true, as demonstrated in the next example.*

Example 1.7: *Let d be the Euclid metric on $X=[0,1]$. Let $T:X\to CB(X)$ be the multivalued map defined by*

$$T\varpi=\begin{cases}\{0,1\}, & if\ \varpi=0,\\ \{\frac{\varpi}{2}\}, & if\ 0<\varpi\leq1.\end{cases}$$

Obviously, T is not continuous at $\varpi = 0$. Indeed, for $0 < \varpi \leq 1$,

$$H(T\varpi, T0) = \max\{\Delta\left(\{0,1\}, \left\{\frac{\varpi}{2}\right\}\right), \Delta\left(\left\{\frac{\varpi}{2}\right\}, \{0,1\}\right)$$
$$= \max\left\{\max\left\{d\left(0, \frac{\varpi}{2}\right), d\left(1, \frac{\varpi}{2}\right)\right\}, \min\left\{d\left(0, \frac{\varpi}{2}\right), d\left(1, \frac{\varpi}{2}\right)\right\}\right\}$$
$$= \max\left\{\frac{\varpi}{2}, 1 - \frac{\varpi}{2}\right\} = 1 - \frac{\varpi}{2}.$$

Then

$$\lim_{\varpi \to 0^+} H(T\varpi, T0) = 1 > 0.$$

Moreover, for $\varpi = 0$, let $\{\breve{e}_n\} \subset X$ such that $\breve{e}_0 = 0, \breve{e}_1 = 1$ and $\breve{e}_{n+1} \in T\breve{e}_n$, for all $n \geq 0$. Then, we have for all $n = 1, 2 \cdots$

$$\breve{e}_n = \frac{1}{2^{n-1}},$$

which implies

$$\lim_{n \to \infty} \breve{e}_n = 0.$$

Suppose there is a subsequence $\{\breve{e}_{n_j}\}$ of $\{\breve{e}_n\}$ such that

$$\lim_{j \to \infty} H(T\breve{e}_{n_j}, T0) = 0. \tag{1}$$

Then,

$$\lim_{j \to \infty} H\left(\frac{1}{2^{n_j - 1}}, T0\right) = 0.$$

On the other hand,

$$H\left(\frac{1}{2^{n_j - 1}}, T0\right) = 1 - \frac{1}{2^{n_j - 1}}.$$

Then letting $j \to \infty$, we obtain

$$\lim_{j \to \infty} H(T\breve{e}_{n_j}, T0) = 1,$$

which is a contradiction with (1). Thus, T is not WPC.

Now, we prove tat f_T is LSC on X. Recall that

$$f_T(\varpi) = d(\varpi, T\varpi) = \begin{cases} \min\{d(0,0), d(0,1)\} = 0, if\ \varpi = 0, \\ d(\varpi, \frac{\varpi}{2}) = \frac{\varpi}{2}, if\ 0 \leq \varpi < 1 \end{cases}$$
$$= \frac{\varpi}{2} \text{ for all } \varpi \in X.$$

Then f_T is continuous on X, and so it is LSC.

We also have the following useful lemma, known as the Jensen inequality.

Lemma 1.8: ([15]) *Let* $J:[0,\infty)\to\mathbb{R}$ *be a convex function. Then, for every* $n=1,2\cdots$ *and* $\{\alpha_1,\alpha_2,\cdots,\alpha_n\}$, $\{t_1,t_2,\cdots,t_n\}\subset[0,\infty)$ *with* $\sum_{i=1}^{n}\alpha_i>0$*, we have*

$$J\left(\frac{\sum_{i=1}^{n}\alpha_i t_i}{\sum_{i=1}^{n}\alpha_i}\right)\le\frac{\sum_{i=1}^{n}\alpha_i J(t_i)}{\sum_{i=1}^{n}\alpha_i}.$$

2. The class of multivalued p-contractions wrt a family of mappings

First, we present the class of multivalued p-contractions wrt a collection of mappings, and study the existence of FPs. Let $m\in\mathbb{N}^*$, $p\ge 1$ be constants, and $\{S_i\}_{i=1}^{m}$ be a family of mappings $S_i:X\times X\to X$.

Definition 2.1: *A mapping* $T:X\to CB(X)$ *is termed as a multivalued* p*-contraction wrt* $\{S_i\}_{i=1}^{m}$ *if there is* $r\in[0,1)$ *so that:*

for all $\ddot{i},\dot{c}\in X$ and $\bar{t}\in T\ddot{i}$, there is $\hbar\in T\dot{c}$ satisfying

$$\begin{aligned}&d^p(\bar{t},S_1(\bar{t},\hbar))+\sum_{i=1}^{m-1}d^p(S_i(\bar{t},\hbar),S_{i+1}(\bar{t},\hbar))+d^p(S_m(\bar{t},\hbar),\hbar)\\&\le\xi\left[d^p(\ddot{i},S_1(\ddot{i},\dot{c}))+\sum_{i=1}^{m-1}d^p(S_i(\ddot{i},\dot{c}),S_{i+1}(\ddot{i},\dot{c}))+d^p(S_m(\ddot{i},\dot{c}),\dot{c})\right].\end{aligned}\tag{2}$$

Our first result is stated as follows:

Theorem 2.2: *Let* (X,d) *be a CMS and* $T:X\to CB(X)$ *a mapping. Assume that:*

- T is a multivalued p-contraction wrt $\{S_i\}_{i=1}^{m}$;
- f_T is LSC.

Then, T possesses a FP Υ_* in X.
In addition, assume that $Ta=\{a\}$ if a is a FP of T. Then, Υ_* is unique and $\Upsilon_*\in\bigcap_{i=1}^{m}Fix(S_i)$.

Proof. Let $x_0,x_1\in X$ such that $x_1\in Tx_0$. By (2), there exists $x_2\in Tx_1$ such that

$$\begin{aligned}&d^p(x_1,S_1(x_1,x_2))+\sum_{i=1}^{m-1}d^p(S_i(x_1,S_1(x_1,x_2)),S_{i+1}(x_1,S_1(x_1,x_2)))+d^p(S_m(x_1,x_2),x_2))\\&\le r\left[d^p(x_0,S_1(x_0,x_1))+\sum_{i=1}^{m-1}d^p(S_i(x_0,x_1),S_{i+1}(x_0,x_1))+d^p(S_m(x_0,x_1),x_1)\right].\end{aligned}$$

Again, by By (2), there is $x_3\in Tx_2$ so that

$$\begin{aligned}&d^p(x_2,S_1(x_2,x_3))+\sum_{i=1}^{m-1}d^p(S_i(x_2,S_1(x_2,x_3)),S_{i+1}(x_2,S_1(x_2,x_3)))+d^p(S_m(x_2,x_3),x_3))\\&\le r\left[d^p(x_1,S_1(x_1,x_2))+\sum_{i=1}^{m-1}d^p(S_i(x_1,x_2),S_{i+1}(x_1,x_2))+d^p(S_m(x_1,x_2),x_2)\right],\end{aligned}$$

Continuing in this fashion, we construct $\{x_n\}\subset X$ so that for all $n\ge 1$,

- $x_{n+1}\in Tx_n$;

$$\begin{aligned}&d^p(x_n,S_1(x_n,x_{n+1}))+\sum_{i=1}^{m-1}d^p(S_i(x_n,S_1(x_n,x_{n+1})),S_{i+1}(x_n,S_1(x_n,x_{n+1})))\\&+d^p(S_m(x_n,x_{n+1}),x_{n+1}))\\&\le r[d^p(x_{n-1},S_1(x_{n-1},x_n))+\sum_{i=1}^{m-1}d^p(S_i(x_{n-1},x_n),S_{i+1}(x_{n-1},x_n))]\\&+rd^p(S_m(x_{n-1},x_n),x_n).\end{aligned}$$

Moreover, by induction, one finds for all $n\ge 1$,

$$\begin{aligned}&d^p(x_n,S_1(x_n,x_{n+1}))+\sum_{i=1}^{m-1}d^p(S_i(x_n,S_1(x_n,x_{n+1})),S_{i+1}(x_n,S_1(x_n,x_{n+1})))\\&+d^p(S_m(x_n,x_{n+1}),x_{n+1}))\le\theta r^n,\end{aligned}\tag{3}$$

where

$$\theta=d^p(x_0,S_1(x_0,x_1))+\sum_{i=1}^{m-1}d^p(S_i(x_0,x_1),S_{i+1}(x_0,x_1))+d^p(S_m(x_0,x_1),x_1).$$

Now, by the triangle inequality, we obtain that, for all $n\ge 0$,

$$\begin{aligned}d^p(x_n,x_{n+1})\;&\le[d(x_n,S_1(x_n,x_{n+1}))+\sum_{i=1}^{m-1}d(S_i(x_n,S_1(x_n,x_{n+1})),S_{i+1}(x_n,S_1(x_n,x_{n+1})))\\&\quad+d(S_{m-1}(x_n,x_{n+1}),S_m(x_n,x_{n+1}))+d(S_m(x_n,x_{n+1}),x_{n+1})]^p\\&=(m+1)^p[\frac{d(x_n,S_1(x_n,x_{n+1}))}{m+1}+\sum_{i=1}^{m-1}\frac{d(S_i(x_n,x_{n+1}),S_{i+1}(x_n,x_{n+1}))}{m+1}\\&\quad+\frac{d(S_m(x_n,x_{n+1}),x_{n+1})}{m+1}]^p.\end{aligned}$$

Thus, since the function $t\mapsto t^p$ is convex on $[0,\infty)$, using Lemma 1.8, we have

$$\begin{aligned}&d^p(x_n,x_{n+1})\le(m+1)^{p-1}[d^p(x_n,S_1(x_n,x_{n+1}))\\&+\sum_{i=1}^{m-1}d^p(S_i(x_n,x_{n+1}),S_{i+1}(x_n,x_{n+1}))+d^p(S_m(x_n,x_{n+1}),x_{n+1})]\end{aligned}$$

which implies by (3), that for all $n\ge 0$,

$$d^p(x_n,x_{n+1})\le(m+1)^{p-1}\theta r^n.$$

Hence, for all $n\ge 0$,

$$d(x_n,x_{n+1})\le\left[(m+1)^{p-1}\theta\right]^{\frac{1}{p}}r_p^n,\tag{4}$$

where $r_p=r^{\frac{1}{p}}<1$. So, for all $k\ge 0$,

$$d(x_n,x_{n+k})\le d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+\cdots+d(x_{n+k-1},x_{n+k}).$$

Using (4), we write

$$d(x_n, x_{n+k}) \le \left[(m+1)^{p-1}\theta\right]^{\frac{1}{p}} \sum_{i=n}^{n+k-1} r_p^i$$
$$\le \left[(m+1)^{p-1}\theta\right]^{\frac{1}{p}} \sum_{i=n}^{\infty} r_p^i \to 0 \text{as} n \to \infty.$$

This implies that $\{x_n\}$ is Cauchy. The completness of (X,d) yields that $\{x_n\}$ is convergent to some $\Upsilon_* \in X$. As f_T is LSC, one writes

$$d(\Upsilon_*, T\Upsilon_*) = f_T(\Upsilon_*) \le \liminf_{n\to\infty} d(x_n, Tx_n) \le \liminf_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Finally, we get $d(\Upsilon_*, T\Upsilon_*) = 0$, that is, $\Upsilon_* \in \overline{T\Upsilon_*} = T\Upsilon_*$. Then, Υ_* is a FP of T.
Now, if $\varsigma^* \in Fix(T)$, as $\Upsilon_* \in T\Upsilon_*$, it follows by (2), there is $\dot{\Theta} \in T\varsigma^*$ so that

$$d^p(\Upsilon_*, S_1(\Upsilon_*, \dot{\Theta})) + \sum_{i=1}^{m-1} d^p(S_i(\Upsilon_*, \dot{\Theta}), S_{i+1}(\Upsilon_*, \dot{\Theta})) + d^p(S_m(\Upsilon_*, \dot{\Theta}), \dot{\Theta}))$$
$$\le r[d^p(\Upsilon_*, S_1(\Upsilon_*, \varsigma^*)) + \sum_{i=1}^{m-1} d^p(S_i(\Upsilon_*, \varsigma^*), S_{i+1}(\Upsilon_*, \varsigma^*)) + d^p(S_m(\Upsilon_*, \varsigma^*), \varsigma^*)].$$

As $T\varsigma^* = \{\varsigma^*\}$, we get $\dot{\Theta} = \varsigma^*$. Then we obtain

$$d^p(\Upsilon_*, S_1(\Upsilon_*, \varsigma^*)) + \sum_{i=1}^{m-1} d^p(S_i(\Upsilon_*, \varsigma^*), S_{i+1}(\Upsilon_*, \varsigma^*)) + d^p(S_m(\Upsilon_*, \varsigma^*), \Upsilon_*))$$
$$\le r[d^p(\Upsilon_*, S_1(\Upsilon_*, \varsigma^*)) + \sum_{i=1}^{m-1} d^p(S_i(\Upsilon_*, \varsigma^*), S_{i+1}(\Upsilon_*, \varsigma^*)) + d^p(S_m(\Upsilon_*, \varsigma^*), \varsigma^*)].$$

Hence, by the fact $r \in [0,1)$, we get

$$d^p(\Upsilon_*, S_1(\Upsilon_*, \varsigma^*)) = \sum_{i=1}^{m-1} d^p(S_i(\Upsilon_*, \varsigma^*), S_{i+1}(\Upsilon_*, \varsigma^*)) = d^p(S_m(\Upsilon_*, \varsigma^*), \varsigma^*)) = 0.$$

Finally,

$$\Upsilon_* = S_1(\Upsilon_*, \varsigma^*) = S_2(\Upsilon_*, \varsigma^*) = \cdots = S_m(\Upsilon_*, \varsigma^*) = \varsigma^*,$$

that is, $\Upsilon_* = \varsigma^*$. Furthermore, as $\Upsilon_* \in T\Upsilon_*$, it follows from (2), that there is $\dot{\Theta} \in T\Upsilon_*$ so that

$$d^p(\Upsilon_*, S_1(\Upsilon_*, \dot{\Theta})) + \sum_{i=1}^{m-1} d^p(S_i(\Upsilon_*, \dot{\Theta}), S_{i+1}(\Upsilon_*, \dot{\Theta})) + d^p(S_m(\Upsilon_*, \dot{\Theta}), \dot{\Theta}))$$
$$\le r[d^p(\Upsilon_*, S_1(\Upsilon_*, \Upsilon_*)) + \sum_{i=1}^{m-1} d^p(S_i(\Upsilon_*, \Upsilon_*), S_{i+1}(\Upsilon_*, \Upsilon_*)) + d^p(S_m(\Upsilon_*, \Upsilon_*), \Upsilon_*)].$$

As $T\Upsilon_* = \{\Upsilon_*\}$, we get $\dot{\Theta} = \Upsilon_*$. Then we obtain

$$d^p(\Upsilon_*, S_1(\Upsilon_*, \Upsilon_*)) + \sum_{i=1}^{m-1} d^p(S_i(\Upsilon_*, \Upsilon_*), S_{i+1}(\Upsilon_*, \Upsilon_*)) + d^p(S_m(\Upsilon_*, \Upsilon_*), \Upsilon_*))$$
$$\le r[d^p(\Upsilon_*, S_1(\Upsilon_*, \Upsilon_*)) + \sum_{i=1}^{m-1} d^p(S_i(\Upsilon_*, \Upsilon_*), S_{i+1}(\Upsilon_*, \Upsilon_*)) + d^p(S_m(\Upsilon_*, \Upsilon_*), \Upsilon_*)].$$

Hence, since $r \in [0,1)$, one gets

$$d^p(\Upsilon_*, S_1(\Upsilon_*, \Upsilon_*)) = \sum_{i=1}^{m-1} d^p(S_i(\Upsilon_*, \Upsilon_*), S_{i+1}(\Upsilon_*, \Upsilon_*)) = d^p(S_m(\Upsilon_*, \Upsilon_*), \Upsilon_*)) = 0.$$

Finally,

$$\Upsilon_* = S_1(\Upsilon_*, \Upsilon_*) = S_2(\Upsilon_*, \Upsilon_*) = \cdots = S_m(\Upsilon_*, \Upsilon_*) = \Upsilon_*,$$

that is, $\Upsilon_* \in \bigcap_{i=1}^m Fix(S_i)$.

If T is weak Picard continuous on (X,d), then we deduce from Theorem 2.2 the next result.

Corollary 2.3 *Let (X,d) be a CMS and $T: X \to CB(X)$ a mapping. Assume that:*

- T is a multivalued p-contraction wrt $\{S_i\}_{i=1}^m$;
- T is weak Picard continuous on (X,d).

Then, T possesses a FP Υ_* in X.

In addition, assume that $Ta = \{a\}$ if a is a FP of T. Then, Υ_* is unique and $\Upsilon_* \in \bigcap_{i=1}^m Fix(S_i)$.

If we take $m = 1$ in Theorem 2.2, we get the next result.

Corollary 2.4 *Let (X,d) be a CMS, $p \geq 1$ be a constant, and $S: X \times X \to X$ and $T: X \to CB(X)$ be mappings. Assume that:*

- There is $r \in [0,1)$ so that:
  for all $\ddot{i}, \dot{c} \in X$ and $\bar{t} \in T\ddot{i}$, there is $\hbar \in T\dot{c}$ verifying

$$d^p(\bar{t}, S(\bar{t}, \hbar)) + d^p(S(\bar{t}, \hbar), \hbar) \leq r[d^p(\ddot{i}, S(\ddot{i}, \dot{c})) + d^p(S(\ddot{i}, \dot{c}), \dot{c})];$$

- f_T is LSC.
  Then, T possesses a FP Υ_* in X.
  In addition, assume that, $Ta = \{a\}$ if a is a FP of T. Then, Υ_* is unique and $\Upsilon_* \in Fix(S)$.
  If we take $p = 1$ in Corollary 2.4, we obtain the next result.

Corollary 2.5 *Let (X,d) be a CMS, $p \geq 1$ be a constant, and $S: X \times X \to X$ be a mapping. Suppose that $T: X \to CB(X)$ is such that:*

- There is $r \in [0,1)$ such that:
  for all $\ddot{i}, \dot{c} \in X$ and $\bar{t} \in T\ddot{i}$, there is $\hbar \in T\dot{c}$ verifying

$$d(z, S(\bar{t}, \hbar)) + d(S(\bar{t}, \hbar), \hbar) \leq r[d(\ddot{i}, S(\ddot{i}, \dot{c})) + d(S(\ddot{i}, \dot{c}), \dot{c})];$$

- f_T is LSC.
  Then, T possesses a FP Υ_* in X.
  In addition, assume that, $Ta = \{a\}$ if a is a FP of T. Then, Υ_* is unique and $\Upsilon_* \in Fix(S)$.
  If we take $S(\ddot{i}, \dot{c}) = \ddot{i}$ for all $\ddot{i}, \dot{c} \in X$ in Corollary 2.5, we get the next result.

Corollary 2.6 *Let (X,d) be a CMS. Suppose that $T: X \to CB(X)$ is a multivalued mapping satisfying the following conditions:*

- There is $r \in [0,1)$ so that:
  for all $\ddot{i}, \dot{c} \in X$ and $\bar{t} \in T\ddot{i}$, there is $\hbar \in T\dot{c}$ verifying

$$d(\bar{t}, \hbar) \leq r d(\ddot{i}, \dot{c});$$

- f_T is LSC.

Then, T possesses a FP Υ_* in X.

In addition, assume that $Ta = \{a\}$ if a is a FP of T. Then, Υ_* is unique.

We now show that Theorem 2.2 includes Theorem 2.6 of Huang and Samet [14].

Corollary 2.7: *Let (X,d) be a CMS, $m \in \mathbb{N}^*, p \geq 1$ be constants, and $\{S_1, \cdots, S_m\}$ be a family of mappings such that $S_i : X \times X \to X$. Let $T : X \to X$ be a mapping. Assume that:*

- There is $r \in [0,1)$ so that

$$d^p(T\ddot{\imath}, S_1(T\ddot{\imath}, T\dot{c})) + \sum_{i=1}^{m-1} d^p(S_i(T\ddot{\imath}, T\dot{c}), S_{i+1}(T\ddot{\imath}, T\dot{c})) + d^p(S_m(T\ddot{\imath}, T\dot{c}), T\dot{c})$$
$$\leq r\left[d^p(\ddot{\imath}, S_1(\ddot{\imath}, \dot{c})) + \sum_{i=1}^{m-1} d^p(S_i(\ddot{\imath}, \dot{c}), S_{i+1}(\ddot{\imath}, \dot{c})) + d^p(S_m(\ddot{\imath}, \dot{c}), \dot{c}) \right]$$

for all $\ddot{\imath}, \dot{c} \in X$;

- T is WPC on (X,d).

Then:

- For every $x_0 \in X$, $\{T^n x_0\}$ is convergent to a FP of T;
- T has a unique FP Υ_* in X;
- $\Upsilon_* \in \bigcap_{i=1}^{m} \text{Fix}(S_i)$.

Proof. Assume that T is WPC on (X,d). Let $\ddot{\imath} \in X$ and $\{x_n\} \subset X$ so that $\lim_{n\to\infty} d(x_n, \ddot{\imath}) = 0$. Then there is a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ so that

$$\lim_{k\to\infty} d(Tx_{n_k}, T\ddot{\imath}) = 0.$$

Furthermore,

$$d(\ddot{\imath}, T\ddot{\imath}) \leq d(\ddot{\imath}, x_{n_k}) + d(x_{n_k}, Tx_{n_k}) + d(Tx_{n_k}, T\ddot{\imath}).$$

Hence, we get

$$f_T(\ddot{\imath}) = d(\ddot{\imath}, T\ddot{\imath}) \leq \liminf_{k\to\infty} d(\ddot{\imath}, x_{n_k}) + \liminf_{k\to\infty} d(x_{n_k}, Tx_{n_k}) + \liminf_{k\to\infty} d(Tx_{n_k}, T\ddot{\imath})$$
$$= \liminf_{k\to\infty} d(x_{n_k}, Tx_{n_k}) = \liminf_{n\to\infty} d(x_n, Tx_n).$$

Finally, f_T is LSC.

We provide below the following example illustrating our obtained results, where the Nadler FP theorem [5] is not applicable.

Example 2.8: Let $X = \{1,2,3\}$ endowed with the metric

$$d(\ddot{\imath}, \dot{q}) = \begin{cases} 1, & if\ \ddot{\imath} \neq \dot{q}, \\ 0, & if\ \ddot{\imath} = \dot{q}. \end{cases}$$

Notice that (X,d) is a CMS. Consider the mapping $T : X \to CB(X)$ defined by

$$T1 = T3 = \{1\} \text{ and } T2 = \{1,3\}.$$

We point out that T is not a contraction via Nadler [5]. Indeed,

$$H(T1, T2) = 1 = d(1,2).$$

Let $S: X \times X \to X$ be defined by

$$S(1,1) = S(2,3) = S(3,2) = S(1,3) = S(1,2) = S(2,2) = 1$$

and $S(3,1) = S(2,1) = S(3,3) = 3$.

We claim the hypotheses in Corollary 2.5 hold for $r \in [\frac{1}{2}, 1)$. To check this, we have the following cases:

Case 1. $\ddot{i} = \dot{c} = 1$. In this case, let $\bar{t} = 1 \in T1$ and for $\hbar = 1 \in T1$, so

$$\begin{aligned} d(\bar{t}, S(\bar{t}, \hbar)) + d(S(\bar{t}, \hbar), \hbar) &= d(1, S(1,1)) + d(S(1,1),1) = d(1,1) + d(1,1) \\ &= 0 \le r[d(\ddot{i}, S(\ddot{i}, \dot{c})) + d(S(\ddot{i}, \dot{c}), \dot{c})]. \end{aligned}$$

Case 2. $\ddot{i} = 1, \dot{c} = 2$. Let $\bar{t} = 1 \in T1$ and for $\hbar = 1 \in T2$,

$$\begin{aligned} d(\bar{t}, S(\bar{t}, \hbar)) + d(S(\bar{t}, \hbar), \hbar) &= d(1, S(1,1)) + d(S(1,1),1) = d(1,1) + d(1,1) \\ &= 0 \le r[d(\ddot{i}, S(\ddot{i}, \dot{c})) + d(S(\ddot{i}, \dot{c}), \dot{c})]. \end{aligned}$$

Case 3. $\ddot{i} = 1, \dot{c} = 3$. Let $\bar{t} = 1 \in T1$ and for $\hbar = 1 \in T3$,

$$\begin{aligned} d(\bar{t}, S(\bar{t}, \hbar)) + d(S(\bar{t}, \hbar), \hbar) &= d(1, S(1,1)) + d(S(1,1),1) = d(1,1) + d(1,1) \\ &= 0 \le r[d(\ddot{i}, S(\ddot{i}, \dot{c})) + d(S(\ddot{i}, \dot{c}), \dot{c})]. \end{aligned}$$

Case 4. $\ddot{i} = 2, \dot{c} = 1$. Let $\bar{t} = 1 \in T2$ and for $\hbar = 1 \in T3$,

$$\begin{aligned} d(\bar{t}, S(\bar{t}, \hbar)) + d(S(\bar{t}, \hbar), \hbar) &= d(1, S(1,1)) + d(S(1,1),1) = d(1,1) + d(1,1) \\ &= 0 \le r[d(\ddot{i}, S(\ddot{i}, \dot{c})) + d(S(\ddot{i}, \dot{c}), \dot{c})]. \end{aligned}$$

Also, let $\bar{t} = 3 \in T1$ and for $\hbar = 1 \in T3$,

$$\begin{aligned} d(\bar{t}, S(\bar{t}, \hbar)) + d(S(\bar{t}, \hbar), \hbar) &= d(3, S(3,1)) + d(S(3,1),1) = d(3,3) + d(3,1) \\ &= 1 \le 2r = r[d(2,3) + d(3,1)] = r[d(2, S(2,1)) + d(S(2,1),1)]. \end{aligned}$$

Case 5. $\ddot{i} = \dot{c} = 2$. Let $\bar{t} = 1 \in T2$ and for $\hbar = 1 \in T2$,

$$\begin{aligned} d(\bar{t}, S(\bar{t}, \hbar)) + d(S(\bar{t}, \hbar), \hbar) &= d(1, S(1,1)) + d(S(1,1),1) = d(1,1) + d(1,1) \\ &= 0 \le r[d(\ddot{i}, S(\ddot{i}, \dot{c})) + d(S(\ddot{i}, \dot{c}), \dot{c})]. \end{aligned}$$

Also, let $\bar{t} = 3 \in T2$ and for $\hbar = 3 \in T2$, we have

$$\begin{aligned} d(\bar{t}, S(\bar{t}, \hbar)) + d(S(\bar{t}, \hbar), \hbar) &= d(3, S(3,3)) + d(S(3,3),3) = d(3,3) + d(3,3) \\ &= 0 \le r[d(\ddot{i}, S(\ddot{i}, \dot{c})) + d(S(\ddot{i}, \dot{c}), \dot{c})]. \end{aligned}$$

Case 6. $\ddot{i} = 2, \dot{c} = 3$. Let $\bar{t} = 1 \in T2$ and for $\hbar = 1 \in T3$,

$$\begin{aligned} d(\bar{t}, S(\bar{t}, \hbar)) + d(S(\bar{t}, \hbar), \hbar) &= d(1, S(1,1)) + d(S(1,1),1) = d(1,1) + d(1,1) \\ &= 0 \le r[d(\ddot{i}, S(\ddot{i}, \dot{c})) + d(S(\ddot{i}, \dot{c}), \dot{c})]. \end{aligned}$$

Also, let $\bar{t} = 3 \in T2$ and for $\hbar = 1 \in T3$, one writes

$$\begin{aligned} d(\bar{t}, S(\bar{t}, \hbar)) + d(S(\bar{t}, \hbar), \hbar) &= d(3, S(3,1)) + d(S(3,1),1) = d(3,3) + d(3,1) \\ &= 1 \le 2r = r[d(2,1) + d(2,3)] = r[d(2, S(2,3)) + d(S(2,3),3)]. \end{aligned}$$

Case 7. $\ddot{i} = 3, \dot{c} = 1$. Let $\bar{t} = 1 \in T3$ and for $\hbar = 1 \in T1$,

$$\begin{aligned} d(\bar{t}, S(\bar{t}, \hbar)) + d(S(\bar{t}, \hbar), \hbar) &= d(1, S(1,1)) + d(S(1,1),1) = d(1,1) + d(1,1) \\ &= 0 \le r[d(\ddot{i}, S(\ddot{i}, \dot{c})) + d(S(\ddot{i}, \dot{c}), \dot{c})]. \end{aligned}$$

Case 8. $\ddot{i} = 3, \dot{c} = 2$. Let $\bar{t} = 1 \in T3$ and for $\hbar = 1 \in T2$,

$$\begin{aligned} d(\bar{t}, S(\bar{t}, \hbar)) + d(S(\bar{t}, \hbar), \hbar) &= d(1, S(1,1)) + d(S(1,1),1) = d(1,1) + d(1,1) \\ &= 0 \le r[d(\ddot{i}, S(\ddot{i}, \dot{c})) + d(S(\ddot{i}, \dot{c}), \dot{c})]. \end{aligned}$$

Also, let $\bar{t} = 1 \in T3$ and for $\hbar = 3 \in T2$,

$$\begin{aligned} d(z, S(\bar{t}, \hbar)) + d(S(\bar{t}, \hbar), \hbar) &= d(1, S(1,3)) + d(S(1,3),3) = d(1,1) + d(1,3) \\ &= 1 \le 2r = r[d(3,1) + d(1,2)] = r[d(2, S(3,2)) + d(S(3,2),2)]. \end{aligned}$$

Case 9. $\ddot{i} = \dot{c} = 3$. Let $\bar{t} = 1 \in T3$ and for $\hbar = 1$,

$$\begin{aligned} d(\bar{t}, S(\bar{t}, \hbar)) + d(S(\bar{t}, \hbar), \hbar) &= d(1, S(1,1)) + d(S(1,1),1) = d(1,1) + d(1,1) \\ &= 0 \le r[d(\ddot{i}, S(\ddot{i}, \dot{c})) + d(S(\ddot{i}, \dot{c}), \dot{c})]. \end{aligned}$$

By the definition of d, f_T is LSC. Hence, all hypotheses of Corollary 2.5 hold. Here, 1 is the unique FP of T and $1 \in Fix(S)$.

3. The class of (ψ, Γ, α)-contractions

The Gamma function is an analogue of factorial for non-integers. There are many definitions of this function. Here, we take the Euler integral defined by

$$\Gamma(\ddot{i}) = \int_0^\infty t^{\ddot{i}-1} \mathrm{e}^{-t}\, \mathrm{d}t, \quad \ddot{i} > 0.$$

we have the following properties of Gamma. For instance, we may refer to [16].

$$\Gamma(\ddot{i} + 1) = \ddot{i}\Gamma(\ddot{i}), \quad \forall \ddot{i} > 0; \tag{5}$$

- $\Gamma(1) = 1$;
- Γ is logarithmically convex, that is,

$$\Gamma(\lambda\ddot{i} + (1-\lambda)\dot{c}) \le \Gamma^{\lambda}(\ddot{i})\Gamma^{1-\lambda}(\dot{c}), \quad \forall \ddot{i}, \dot{c} > 0, \forall \lambda \in [0,1]. \tag{6}$$

Let Ψ be the collection of functions $\psi : [0,\infty) \to [0,\infty)$ so that

$$\psi(s) \ge cs^{\tau}, \quad \forall s \ge 0, \tag{7}$$

where $c, \tau > 0$ are constants. If $\psi \in \Psi$, then by (7), we have for all $s > 0$,

$$\psi(s) > 0. \tag{8}$$

Definition 3.1 *Let (X,d) be a metric space. A mapping $T : X \to CB(X)$ is named as a multivalued (ψ, Γ, α)-contraction, if there are $\alpha, r \in (0,1)$ and $\psi \in \Psi$ so that:*

for all $\ddot{i}, \dot{c} \in X$ with $\ddot{i} \ne \dot{c}$ and $\bar{t} \in T\ddot{i}$, there is $\hbar \in T\dot{c}$ such that

$$\psi\left(\frac{\Gamma(d(\bar{t}, \hbar) + 1)}{\Gamma(d(\bar{t}, \hbar) + \alpha)}\right) \le r\psi\left(\frac{\Gamma(d(\ddot{i}, \dot{c}) + 1)}{\Gamma(d(\ddot{i}, \dot{c}) + \alpha)}\right). \tag{9}$$

Next FP theorems involve above contractions.

Theorem 3.2: *Let* (X,d) *be a CMS and* $T: X \to CB(X)$ *be a mapping. Assume that:*

- T is a multivalued (ψ,Γ,α)-contraction;
- f_T is LSC on (X,d).

Then, T possesses a FP Υ_* in X.

In addition, assume that $Ta=\{a\}$ if a is a FP of T. Then, Υ_* is unique.

Proof. Letting $x_0, x_1 \in X$ such that $x_1 \in Tx_0$. If $x_1 = x_0$ or $x_1 \in Tx_1$, then x_1 is a FP of T and the proof is finished. Suppose that $x_1 \neq x_0$ and $x_1 \notin Tx_1$. By (9), there exists $x_2 \in Tx_1$ such that

$$\psi\left(\frac{\Gamma(d(x_1,x_2)+1)}{\Gamma(d(x_1,x_2)+\alpha)}\right) \le r\psi\left(\frac{\Gamma(d(x_0,x_1)+1)}{\Gamma(d(x_0,x_1)+\alpha)}\right).$$

Notice that $x_2 \neq x_1$. Again, if $x_2 \in Tx_2$, then x_2 is a FP of T and the proof is finished. Suppose that $x_2 \notin Tx_2$. By (9), there is $x_3 \in Tx_2$ so that

$$\psi\left(\frac{\Gamma(d(x_2,x_3)+1)}{\Gamma(d(x_2,x_3)+\alpha)}\right) \le r\psi\left(\frac{\Gamma(d(x_1,x_2)+1)}{\Gamma(d(x_1,x_2)+\alpha)}\right).$$

By repeating the process above, we construct a sequence $\{x_n\} \subset X$ so that for any $n \ge 1$,

- $x_n \neq x_{n+1}, x_n \notin Tx_n$ and $x_{n+1} \in Tx_n$;

$$\psi\left(\frac{\Gamma(d(x_n,x_{n+1})+1)}{\Gamma(d(x_n,x_{n+1})+\alpha)}\right) \le r\psi\left(\frac{\Gamma(d(x_{n-1},x_n)+1)}{\Gamma(d(x_{n-1},x_n)+\alpha)}\right).$$

By induction, we ge, for all $n = 0,1,\cdots$

$$\psi\left(\frac{\Gamma(d(x_n,x_{n+1})+1)}{\Gamma(d(x_n,x_{n+1})+\alpha)}\right) \le r^n\psi\left(\frac{\Gamma(d(x_0,x_1)+1)}{\Gamma(d(x_0,x_1)+\alpha)}\right), \quad n \in \mathbb{N}. \tag{10}$$

From (7), we have

$$c\left(\frac{\Gamma(d(x_n,x_{n+1})+1)}{\Gamma(d(x_n,x_{n+1})+\alpha)}\right)^{\tau} \le \psi\left(\frac{\Gamma(d(x_n,x_{n+1})+1)}{\Gamma(d(x_n,x_{n+1})+\alpha)}\right). \tag{11}$$

Combine (10) and (11) to obtain for all $n = 0,1,\cdots$

$$\frac{\Gamma(d(x_n,x_{n+1})+1)}{\Gamma(d(x_n,x_{n+1})+\alpha)} \le r^{\frac{n}{\tau}}\left[\frac{1}{c}\psi\left(\frac{\Gamma(d(x_0,x_1)+1)}{\Gamma(d(x_0,x_1)+\alpha)}\right)\right]^{\frac{1}{\tau}}. \tag{12}$$

Taking into account the ln-convexity of Γ, for all $n = 0,1,\cdots$ we get

$$\begin{aligned}\Gamma(d(x_n,x_{n+1})+\alpha) &= \Gamma\big((1-\alpha)d(x_n,x_{n+1})+\alpha(d(x_n,x_{n+1})+1)\big)\\ &\le \Gamma^{1-\alpha}(d(x_n,x_{n+1}))\Gamma^{\alpha}(d(x_n,x_{n+1})+1).\end{aligned} \tag{13}$$

Moreover, using (5),

$$\Gamma^{1-\alpha}(d(x_n,x_{n+1})+1) = d^{1-\alpha}(x_n,x_{n+1})\Gamma^{1-\alpha}(d(x_n,x_{n+1})).$$

Hence, from (13),

$$\Gamma(d(x_n,x_{n+1})+\alpha) \le [d(x_n,x_{n+1})]^{\alpha-1}\Gamma(d(x_n,x_{n+1})+1),$$

which implies that for all $n = 0,1,\cdots$

$$d(x_n,x_{n+1}) \le \left(\frac{\Gamma(d(x_n,x_{n+1})+1)}{\Gamma(d(x_n,x_{n+1})+\alpha)}\right)^{\frac{1}{1-\alpha}}. \tag{14}$$

From (12) and (14), we have for all $n = 0,1,\cdots$

$$d^{1-\alpha}(x_n,x_{n+1}) \le r^{\frac{n}{\tau(1-\alpha)}}\left[\frac{1}{c}\psi\left(\frac{\Gamma(d(x_0,x_1)+1)}{\Gamma(d(x_0,x_1)+\alpha)}\right)\right]^{\frac{1}{\tau(1-\alpha)}},$$

that is, for all $n = 0,1,\cdots$

$$d(x_n,x_{n+1}) \le \gamma^n\omega_0, \tag{15}$$

where

$$\gamma = r^{\frac{1}{\tau(1-\alpha)}} < 1 \text{ and } \omega_0 = \left[\frac{1}{c}\psi\left(\frac{\Gamma(d(x_0,x_1)+1)}{\Gamma(d(x_0,x_1)+\alpha)}\right)\right]^{\frac{1}{\tau(1-\alpha)}}.$$

Now, proceeding as the proof of Theorem 2.2, we prove that T has a FP in $\ddot{i}$, that is, there is $\Upsilon_* \in$ so that $\Upsilon_* \in T\Upsilon_*$.

Now, let $\varsigma_* \in Fix(T)$ so that $\Upsilon_* \neq \varsigma_*$. Since $\Upsilon_* \in T\Upsilon_*$, it follows by (9), there is $\dot{\Theta} \in T\varsigma_*$ such that

$$\psi\left(\frac{\Gamma(d(\Upsilon_*,\dot{\Theta})+1)}{\Gamma(d(\Upsilon_*,\dot{\Theta})+\alpha)}\right) \le r\psi\left(\frac{\Gamma(d(\Upsilon_*,\varsigma_*)+1)}{\Gamma(d(\Upsilon_*,\varsigma_*)+\alpha)}\right).$$

As $T\varsigma_* = \{\varsigma_*\}$, we get $\dot{\Theta} = \varsigma_*$. Then

$$\psi\left(\frac{\Gamma(d(\Upsilon_*,\varsigma_*)+1)}{\Gamma(d(\Upsilon_*,\varsigma_*)+\alpha)}\right) \le r\psi\left(\frac{\Gamma(d(\Upsilon_*,\varsigma_*)+1)}{\Gamma(d(\Upsilon_*,\varsigma_*)+\alpha)}\right).$$

In view of $r \in [0,1)$, one gets

$$\psi\left(\frac{\Gamma(d(\Upsilon_*,\varsigma_*)+1)}{\Gamma(d(\Upsilon_*,\varsigma_*)+\alpha)}\right) = 0,$$

which is a contradiction. Finally, $\Upsilon_* = \varsigma_*$.

We provide the following result in the case when T is WPC on (X,d).

Corollary 3.3: *Let (X,d) be a CMS and $T: X \to CB(X)$ be a mapping. Assume that:*

- T is a multivalued (ψ,Γ,α)-contraction;
- T is WPC on (X,d).

Then, T possesses a FP Υ_* in X.

In addition, assume that $Ta = \{a\}$ if a is a FP of T. Then, Υ_* is unique.

Notice that Theorem 3.2 of Huang and Samet [14] is a consequence of Theorem 3.2. Namely, we have the following result.

Corollary 3.4: (See [14]) Let (X,d) be a CMS, $\psi \in \Psi$ and $\alpha, r \in (0,1)$. Let $T : X \to X$ be a mapping so that

$$\psi\left(\frac{\Gamma(d(T\ddot{i},T\dot{c})+1)}{\Gamma(d(T\ddot{i},T\dot{c})+\alpha)}\right) \le r\psi\left(\frac{\Gamma(d(\ddot{i},\dot{c})+1)}{\Gamma(d(\ddot{i},\dot{c})+\alpha)}\right)$$

for all $\ddot{i},\dot{c} \in X$ with $T\ddot{i} \ne T\dot{c}$;

- T is WPC on (X,d).

Then:

- For every $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to a FP of T;
- T has a unique FP in X.

Next example is an illustration of Theorem 3.2.

Example 3.5 Let $X = \{1,2,3\}$ and d be the metric given as

$$d(\ddot{i},\dot{c}) = d(\dot{c},\ddot{i}), d(\ddot{i},\ddot{i}) = 0 \text{ for all } \ddot{i},\dot{c} \in X$$

and

$$d(1,2) = 1, d(1,3) = 2 \text{ and } d(2,3) = 3.$$

Notice that (X,d) is a CMS. Consider the mapping $T : X \to CB(X)$ defined by

$$T1 = T3 = \{1\} \text{ and } T2 = \{1,3\}.$$

Remark that T is not a contraction in the sense of Nadler [5]. Indeed,

$$H(T1,T2) = 1 = \max\{d(1,1), d(1,3)\} = d(1,3) = 2 > 1 = d(1,2).$$

Take

$$\psi(t) = \begin{cases} \frac{\sqrt{\pi}}{2}t, & if \quad 0 \le t \le \frac{2}{\sqrt{\pi}}, \\ \frac{3\sqrt{\pi}}{8}t, & if \quad \frac{2}{\sqrt{\pi}} < t \le \frac{8}{3\sqrt{\pi}}, \\ \frac{5\sqrt{\pi}}{16}t + 2, & if \quad t > \frac{8}{3\sqrt{\pi}}. \end{cases}$$

Clearly,

$$\psi(t) \ge \frac{\sqrt{\pi}}{4}t, \quad t \ge 0,$$

which shows that $\psi \in \Psi$ with $c = \frac{\sqrt{\pi}}{4}$ and $\tau = 1$.

We shall prove that all conditions of Theorem 3.2 hold for $\alpha = \frac{1}{2}$ and $r \in [\frac{1}{2}, 1)$. To check this, we have the following cases:

Case 1. $\ddot{i} = 1, \dot{c} = 2$. Let $\bar{t} = 1 \in T1$ and for $\hbar = 1 \in T2$,

$$\frac{\psi\left(\frac{\Gamma(d(\bar{t},\hbar)+1)}{\Gamma(d(\bar{t},\hbar)+\alpha)}\right)}{\psi\left(\frac{\Gamma(d(\ddot{i},\dot{c})+1)}{\Gamma(d(\ddot{i},\dot{c})+\alpha)}\right)}=\frac{\psi\left(\frac{\Gamma(d(1,1)+1)}{\Gamma(d(1,1)+\alpha)}\right)}{\psi\left(\frac{\Gamma(d(1,2)+1)}{\Gamma(d(1,2)+\alpha)}\right)}=\frac{\psi\left(\frac{\Gamma(1)}{\Gamma(\frac{1}{2})}\right)}{\psi\left(\frac{\Gamma(2)}{\Gamma(\frac{1}{2})}\right)}=\frac{\psi\left(\frac{1}{\sqrt{\pi}}\right)}{\psi\left(\frac{2}{\sqrt{\pi}}\right)}=\frac{1}{2}.$$

Case 2. $\ddot{i}=1,\dot{c}=3$. Let $\bar{t}=1\in T1$ and for $\hbar=1\in T3$,

$$\frac{\psi\left(\frac{\Gamma(d(\bar{t},\hbar)+1)}{\Gamma(d(\bar{t},\hbar)+\alpha)}\right)}{\psi\left(\frac{\Gamma(d(\ddot{i},\dot{c})+1)}{\Gamma(d(\ddot{i},\dot{c})+\alpha)}\right)}=\frac{\psi\left(\frac{\Gamma(d(1,1)+1)}{\Gamma(d(1,1)+\alpha)}\right)}{\psi\left(\frac{\Gamma(d(1,3)+1)}{\Gamma(d(1,3)+\alpha)}\right)}=\frac{\psi\left(\frac{\Gamma(1)}{\Gamma(\frac{1}{2})}\right)}{\psi\left(\frac{\Gamma(3)}{\Gamma(2+\frac{1}{2})}\right)}=\frac{\psi\left(\frac{1}{\sqrt{\pi}}\right)}{\psi\left(\frac{8}{\sqrt{3\pi}}\right)}=\frac{1}{2}.$$

Case 3. $\ddot{i}=2,\dot{c}=1$. Let $\bar{t}=1\in T2$ and for $\hbar=1\in T3$,

$$\frac{\psi\left(\frac{\Gamma(d(\bar{t},\hbar)+1)}{\Gamma(d(\bar{t},\hbar)+\alpha)}\right)}{\psi\left(\frac{\Gamma(d(\ddot{i},\dot{c})+1)}{\Gamma(d(\ddot{i},\dot{c})+\alpha)}\right)}=\frac{\psi\left(\frac{\Gamma(d(1,1)+1)}{\Gamma(d(1,1)+\alpha)}\right)}{\psi\left(\frac{\Gamma(d(2,1)+1)}{\Gamma(d(2,1)+\alpha)}\right)}=\frac{\psi\left(\frac{\Gamma(1)}{\Gamma(\frac{1}{2})}\right)}{\psi\left(\frac{\Gamma(1)}{\Gamma(1+\frac{1}{2})}\right)}=\frac{\psi\left(\frac{1}{\sqrt{\pi}}\right)}{\psi\left(\frac{2}{\sqrt{\pi}}\right)}=\frac{1}{2}.$$

Also, let $\bar{t}=3\in T1$ and for $\hbar=1\in T3$,

$$\frac{\psi\left(\frac{\Gamma(d(\bar{t},\hbar)+1)}{\Gamma(d(\bar{t},\hbar)+\alpha)}\right)}{\psi\left(\frac{\Gamma(d(\ddot{i},\dot{c})+1)}{\Gamma(d(\ddot{i},\dot{c})+\alpha)}\right)}=\frac{\psi\left(\frac{\Gamma(d(1,1)+1)}{\Gamma(d(1,1)+\alpha)}\right)}{\psi\left(\frac{\Gamma(d(3,1)+1)}{\Gamma(d(3,1)+\alpha)}\right)}=\frac{1}{2}.$$

Case 4. $\ddot{i}=2,\dot{c}=3$. Let $\bar{t}=1\in T2$ and for $\hbar=1\in T3$,

$$\frac{\psi\left(\frac{\Gamma(d(\bar{t},\hbar)+1)}{\Gamma(d(\bar{t},\hbar)+\alpha)}\right)}{\psi\left(\frac{\Gamma(d(\ddot{i},\dot{c})+1)}{\Gamma(d(\ddot{i},\dot{c})+\alpha)}\right)}=\frac{\psi\left(\frac{\Gamma(d(1,1)+1)}{\Gamma(d(1,1)+\alpha)}\right)}{\psi\left(\frac{\Gamma(d(2,3)+1)}{\Gamma(d(2,3)+\alpha)}\right)}=\frac{\psi\left(\frac{1}{\sqrt{\pi}}\right)}{\psi\left(\frac{\Gamma(4)}{\Gamma(3+\frac{1}{2})}\right)}=\frac{\psi\left(\frac{1}{\sqrt{\pi}}\right)}{\psi\left(\frac{16}{5\sqrt{\pi}}\right)}=\frac{1}{6}.$$

Also, let $\bar{t}=3\in T2$ and for $\hbar=1\in T3$,

$$\frac{\psi\left(\frac{\Gamma(d(\bar{t},\hbar)+1)}{\Gamma(d(\bar{t},\hbar)+\alpha)}\right)}{\psi\left(\frac{\Gamma(d(\ddot{i},\dot{c})+1)}{\Gamma(d(\ddot{i},\dot{c})+\alpha)}\right)}=\frac{\psi\left(\frac{\Gamma(d(3,1)+1)}{\Gamma(d(3,1)+\alpha)}\right)}{\psi\left(\frac{\Gamma(d(2,3)+1)}{\Gamma(d(2,3)+\alpha)}\right)}=\frac{1}{3}.$$

Case 5. $\ddot{i}=3, \dot{c}=1$. Let $\bar{t}=1\in T3$ and for $\hbar=1\in T1$,

$$\frac{\psi\left(\frac{\Gamma(d(\bar{t},\hbar)+1)}{\Gamma(d(\bar{t},\hbar)+\alpha)}\right)}{\psi\left(\frac{\Gamma(d(\ddot{i},\dot{c})+1)}{\Gamma(d(\ddot{i},\dot{c})+\alpha)}\right)}=\frac{\psi\left(\frac{\Gamma(d(1,1)+1)}{\Gamma(d(1,1)+\alpha)}\right)}{\psi\left(\frac{\Gamma(d(3,1)+1)}{\Gamma(d(3,1)+\alpha)}\right)}=\frac{1}{2}.$$

Case 6. $\ddot{i}=3, \dot{c}=2$. Let $\bar{t}=1\in T3$ and for $\hbar=1\in T2$,

$$\frac{\psi\left(\frac{\Gamma(d(\bar{t},\hbar)+1)}{\Gamma(d(\bar{t},\hbar)+\alpha)}\right)}{\psi\left(\frac{\Gamma(d(\ddot{i},\dot{c})+1)}{\Gamma(d(\ddot{i},\dot{c})+\alpha)}\right)}=\frac{\psi\left(\frac{\Gamma(d(1,1)+1)}{\Gamma(d(1,1)+\alpha)}\right)}{\psi\left(\frac{\Gamma(d(3,2)+1)}{\Gamma(d(3,2)+\alpha)}\right)}=\frac{1}{3}.$$

Also, let $\bar{t}=1\in T3$ and for $\hbar=3\in T2$, we have

$$\frac{\psi\left(\frac{\Gamma(d(\bar{t},\hbar)+1)}{\Gamma(d(\bar{t},\hbar)+\alpha)}\right)}{\psi\left(\frac{\Gamma(d(\ddot{i},\dot{c})+1)}{\Gamma(d(\ddot{i},\dot{c})+\alpha)}\right)}=\frac{\psi\left(\frac{\Gamma(d(1,3)+1)}{\Gamma(d(1,3)+\alpha)}\right)}{\psi\left(\frac{\Gamma(d(3,2)+1)}{\Gamma(d(3,2)+\alpha)}\right)}=\frac{1}{3}.$$

Notice that f_T is LSC. Hence, all hypotheses of Theorem 3.2 hold, and 1 is the unique FP of T..

4. Conclusions

In this paper, we studied the case of multivalued extension of the recent work of Huang and Samet. We also presented some illustrated examples where the main result of Nadler is not applicable. As future works, we may suggest to extend the obtained results in the direction of Patle et al. [4] and Mudhesh et al. [10, 11], or for more generalized MSs, like partial MSs [17], Branciari MSs [18], b-MSs [19], G-MSs [20].

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