Results in Nonlinear Analysis 8 (2025) No. 2, 253–270
https://doi.org/10.31838/rna/2025.08.02.020
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

Higher-order conformable derivatives and their applications

Bouziani Mohammed[a]*, Bahloul Rachid[b]*

[a]Center for Educational Orientation and Planning (COPE), BP 6222, Avenue Zaytoune, Hay Ryad, Rabat, Morocco; [b]Department of Mathematics, Faculty of Applied Sciences, Sultan Moulay Slimane University, Beni Mellal, Morocco

Abstract

After presenting some results and introducing a new definition of the higher-order conformable derivative, we discuss how these findings relate to a novel idea: The higher-order conformable Laplace transform and the higher-order conformable Sumudu transform.

Mathematics Subject Classification (2010): 44A10, 44A20, 45N05

Key words and phrases: Higher-Order Conformable derivative, Higher-Order Conformable Laplace transform, Higher-Order Conformable Sumudu.

1. Introduction

The fractional derivative [13, 15, 10] has attracted the interest of many researchers, each of whom has given the concept of the fractional derivative according to their opinion, for example:

1. Caputo's definition [6]

$$D^{\alpha}\tau(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\tau'(s)}{(t-s)^{\alpha}}\,ds.$$

With $0 < \alpha \leq 1$.

Email addresses: mhdbouziani2010@gmail.com (Bouziani Mohammed); bahloulrachid363@gmail.com (Bahloul Rachid)

Received January 18, 2025; Accepted July 29, 2025; Online August 14, 2025

2. Riemann-Liouville definition [12]

$$D^{\alpha}\tau(t) = \frac{1}{\Gamma(1-\alpha)}\Big(\int_0^t \frac{\tau(s)}{(t-s)^{\alpha}}ds\Big)'.$$

With $0 < \alpha \leq 1$.

3. Conformable derivative [13]

$$D^{\alpha}\tau(t) = \lim_{\mu \to 0}\frac{1}{\mu}[\tau(t+\mu t^{\alpha-1}) - \tau(t)].$$

With $0 < \alpha \leq 1$.

Conformable integral transforms are an important focus of this work, such as:
The conformable Fourier transform [2],

$$\mathcal{F}_{\alpha}[\tau(t)](k) = \frac{\alpha}{p^{\alpha}}\int_0^p e^{-ik\frac{2\pi}{p^{\alpha}}t^{\alpha}}\tau(t)t^{\alpha-1}dt. \tag{1}$$

and in the general case [3]

$$\mathcal{F}_F^{\alpha}[\tau(t)](k) = \frac{1}{G_{\alpha}(p)}\int_0^p e^{-ik\frac{2\pi}{G_{\alpha}(p)}G_{\alpha}(t)}\tau(t)F(t,\alpha)dt. \tag{2}$$

The conformable Laplace transform [1]

$$\mathcal{L}_{\alpha}[\tau(t)](s) = \int_0^{+\infty} e^{-s\frac{t^{\alpha}}{\alpha}}\tau(t)t^{\alpha-1}dt. \tag{3}$$

and in the general case [4].

$$\mathcal{L}_F^{\alpha}[\tau(t)](s) = \int_0^{+\infty} e^{-sG_{\alpha}(t)}\tau(t)F(t,\alpha)dt. \tag{4}$$

And the conformable Sumudo transforms [5].

$$\mathcal{S}_n^{\alpha}[\tau(t)](s) = s\int_0^{+\infty} t^{\alpha(n-1)}e^{-s\frac{t^{\alpha}}{\alpha}}\tau(t)t^{\alpha-1}dt. \tag{5}$$

For more information on this field of research regarding fractional calculus and the conformable derivatives approach, the interested reader can also consult [7, 8, 9, 11, 14, 16, 17].

In this paper, we give the concept of the highest derivative, where α is in the interval $(n-1,n]$ and $n \in \mathbb{N}^*$. Then we introduce the concept of the Higher-Order Conformable Laplace transform and the Higher-Order Conformable Sumudu, similar to the concept of the Higher-Order conformable derivative.

2. Higher-order derivative

Consider $n \in \mathbb{N}^*$ and $\alpha \in (n-1,n]$ throughout the paper.

Definition 2.1 *Given a function $\tau : [0,\infty) \to \mathbb{R}$. Then the Higher-Order conformable derivative of τ of order α is defined by*

$$\mathbb{T}_{\alpha}^{n}\tau(t) = \lim_{\mu \to 0}\frac{1}{\mu}[\tau(t+\mu t^{n-\alpha}) - \tau(t)]$$

If $lim_{t\to 0^+}\mathbb{T}_\alpha^n\tau(t)$ exists, then we define $\mathbb{T}_\alpha^n\tau(t)(0)=lim_{t\to 0^+}\mathbb{T}_\alpha^n\tau(t)$.

1. In case $n=1$, we obtain the derivative

$$\mathbb{T}_\alpha^1\tau(t)=\lim_{\mu\to 0}\frac{\tau(t+\mu t^{1-\alpha})-\tau(t)}{\mu}$$

   defined by Khalil et al. [13].

2. In case $n=\alpha$, we obtain, the classic derivative $\mathbb{T}_\alpha^\alpha\tau(t)=\tau'(t)$.

As a consequence, we obtain the following result, similar to the one in the classical analysis.

Theorem 2.2 *If a function $\tau:[0,\infty)\to\mathbb{R}$ is Higher-Order conformable derivative at $t_0>0$ for $\alpha\in(n-1,n]$, then τ is continuous at t_0.*

Proof. Let t_0 be an arbitrary point greater than zero. Since

$$\lim_{\mu\to 0}\tau(t_0+\mu t_0^{n-\alpha})-\tau(t_0)=\lim_{\mu\to 0}\frac{\tau(t_0+\mu t_0^{n-\alpha})-\tau(t_0)}{t_0^{n-\alpha}\mu}t_0^{n-\alpha}\mu$$

Let $k=t_0^{n-\alpha}\mu$, then $k\to 0$ if $\mu\to 0$, so we have

$$\begin{aligned}\lim_{\mu\to 0}\tau(t_0+\mu t_0^{n-\alpha})-\tau(t_0)&=\left(\lim_{\mu\to 0}\frac{\tau(t_0+\mu t_0^{n-\alpha})-\tau(t_0)}{t_0^{n-\alpha}\mu}\right)\lim_{\mu\to 0}(t_0^{n-\alpha}\mu)\\&=\mathbb{T}_\alpha^n\tau(t)\lim_{\mu\to 0}t_0^{n-\alpha}\mu=0.\end{aligned}$$

From this, we have the continuity of τ at t_0.

Theorem 2.3 *Consider τ,τ_1,τ_2 be $\mathbb{T}_\alpha^n$-differential at $t>0$. Then:*

1. $\mathbb{T}_\alpha^n(a\tau_1+b\tau_2)(t)=a\mathbb{T}_\alpha^n\tau_1(t)+b\mathbb{T}_\alpha^n\tau_2(t)$.
2. $\mathbb{T}_\alpha^n(\lambda)=0,\lambda\in\mathbb{R}$.
3. $\mathbb{T}_\alpha^n(\tau_1\tau_2)(t)=\tau_2(t)\mathbb{T}_\alpha^n\tau_1(t)+\tau_1(t)\mathbb{T}_\alpha^n\tau_2(t)$.
4. $\mathbb{T}_\alpha^n(\frac{\tau_1}{\tau_2})(t)=\frac{\tau_2(t)\mathbb{T}_\alpha^n\tau_1(t)-\tau_1(t)\mathbb{T}_\alpha^n\tau_2(t)}{\tau_2^2(t)}$.
5. For all f, which is differential, we have $\mathbb{T}_\alpha^n\tau(t)=t^{n-\alpha}\tau'(t)$.

Proof. Let's apply the definition 2.1

1.

$$\begin{aligned}\mathbb{T}_\alpha^n\left(a\tau_1+b\tau_2\right)(t)&=\lim_{\mu\to 0}\frac{1}{\mu}[\left(a\tau_1+b\tau_2\right)(t+\mu t^{n-\alpha})-\left(a\tau_1+b\tau_2\right)(t)]\\&=a\lim_{\mu\to 0}\frac{1}{\mu}[\tau_1(t+\mu t^{n-\alpha})-\tau_1(t)]+b\lim_{\mu\to 0}\frac{1}{\mu}[\tau_2(t+\mu t^{n-\alpha})-\tau_2(t)]\\&=a\mathbb{T}_\alpha^n\tau_1(t)+b\mathbb{T}_\alpha^n\tau_2(t).\end{aligned}$$

2. $\mathbb{T}_\alpha^n(\lambda)=\lim_{\mu\to 0}\frac{1}{\mu}[\lambda-\lambda]=0$.

3.

$$\begin{aligned}\mathbb{T}_\alpha^n(\tau_1\tau_1)(t) &= \lim_{\mu\to 0}\frac{\tau_1(t+\mu t^{n-\alpha})\tau_2(t+\mu t^{n-\alpha})-\tau_1(t)\tau_2(t)}{\mu}\\ &= \lim_{\mu\to 0}\frac{\tau_1(t+\mu t^{n-\alpha})-\tau_1(t)}{\mu}\tau_2(t+\mu t^{n-\alpha})+\tau_1(t)\lim_{\mu\to 0}\frac{\tau_2(t+\mu t^{n-\alpha})-\tau_2(t)}{\mu}\\ &= \mathbb{T}_\alpha^n\tau_1(t)\tau_2(t)+\tau_1(t)\mathbb{T}_\alpha^n\tau_2(t).\end{aligned}$$

4.

$$\begin{aligned}\mathbb{T}_\alpha^n\left(\frac{\tau_1}{\tau_2}\right)(t) &= \lim_{\mu\to 0}\frac{\tau_1(t+\mu t^{n-\alpha})}{\tau_2(t+\mu t^{n-\alpha})}-\frac{\tau_1(t)}{\tau_2(t)}\\ &= \lim_{\mu\to 0}\frac{\tau_2(t)\tau_1(t+\mu t^{n-\alpha})-\tau_1(t)\tau_2(t+\mu t^{n-\alpha})}{\tau_2(t+\mu t^{n-\alpha})\tau_2(t)}\\ &= \lim_{\mu\to 0}\frac{\tau_2(t)[\tau_1(t+\mu t^{n-\alpha})-\tau_1(t)]-\tau_1(t)[\tau_2(t+\mu t^{n-\alpha})-\tau_2(t)]}{\tau_2(t+\mu t^{n-\alpha})\tau_2(t)}\\ &= \frac{\tau_2(t)\mathbb{T}_\alpha^n\tau_1(t)-\tau_1(t)\mathbb{T}_\alpha^n\tau_2(t)}{\tau_2^2(t)}.\end{aligned}$$

5. For $t \in (0,+\infty)$

$$\begin{aligned}\mathbb{T}_\alpha^n\tau(t) &= \lim_{\mu\to 0}\frac{1}{\mu}[\tau(t+\mu t^{n-\alpha})-\tau(t)]\\ &= t^{n-\alpha}\tau'(t).\end{aligned}$$

Example 2.4 *Let* $n=5$ *and* $\alpha=\frac{9}{2}$. *Then for all* $t>0$

1. $\mathbb{T}_\alpha^n[t^2+\sin(t)] = 2t^{\frac{3}{2}}+t^{\frac{1}{2}}\cos(t)$.
2. $\mathbb{T}_\alpha^n(2t^{\frac{1}{2}})=1$.

Higher-Order Conformable Derivatives of certain functions

Theorem 2.5 *Let* $n\in\mathbb{N}^*$ *and* $n-1<\alpha\le n$.

1. $\mathbb{T}_\alpha^n\left(e^{\frac{t^{\alpha+1-n}}{\alpha+1-n}}\right)=e^{\left(\frac{t^{\alpha+1-n}}{\alpha+1-n}\right)}$.
2. $\mathbb{T}_\alpha^n\left(\sin[\frac{t^{\alpha+1-n}}{\alpha+1-n}]\right)=\cos\left(\frac{t^{\alpha+1-n}}{\alpha+1-n}\right)$.
3. $\mathbb{T}_\alpha^n\left(\cos[\frac{t^{\alpha+1-n}}{\alpha+1-n}]\right)=\sin\left(\frac{t^{\alpha+1-n}}{\alpha+1-n}\right)$.
4. $\mathbb{T}_\alpha^n\left(\cosh[\frac{t^{\alpha+1-n}}{\alpha+1-n}]\right)=\sinh\left(\frac{t^{\alpha+1-n}}{\alpha+1-n}\right)$.
5. $\mathbb{T}_\alpha^n\left(\sinh[\frac{t^{\alpha+1-n}}{\alpha+1-n}]\right)=\cosh\left(\frac{t^{\alpha+1-n}}{\alpha+1-n}\right)$.

Proof. Using Theorem 2.3

(1) $\mathbb{T}_\alpha^n\left(e^{\frac{t^{\alpha+1-n}}{\alpha+1-n}}\right) = t^{n-\alpha}t^{\alpha-n}e^{\left(\frac{t^{\alpha+1-n}}{\alpha+1-n}\right)} = e^{\left(\frac{t^{\alpha+1-n}}{\alpha+1-n}\right)}.$

(2) $\mathbb{T}_\alpha^n\left(\sin[\frac{t^{\alpha+1-n}}{\alpha+1-n}]\right) = t^{n-\alpha}t^{\alpha-n}\cos\left(\frac{t^{\alpha+1-n}}{\alpha+1-n}\right) = \cos\left(\frac{t^{\alpha+1-n}}{\alpha+1-n}\right).$

(3) $\mathbb{T}_\alpha^n\left(\cos[\frac{t^{\alpha+1-n}}{\alpha+1-n}]\right) = -t^{n-\alpha}t^{\alpha-n}\cos\left(\frac{t^{\alpha+1-n}}{\alpha+1-n}\right) = -\sin\left(\frac{t^{\alpha+1-n}}{\alpha+1-n}\right).$

(4)
$$\begin{aligned}\mathbb{T}_\alpha^n\left(\cosh[\frac{t^{\alpha+1-n}}{\alpha+1-n}]\right) &= \mathbb{T}_\alpha^n\left(\frac{e^{\frac{t^{\alpha+1-n}}{\alpha+1-n}} + e^{-\frac{t^{\alpha+1-n}}{\alpha+1-n}}}{2}\right)\\ &= \frac{e^{\frac{t^{\alpha+1-n}}{\alpha+1-n}} - e^{-\frac{t^{\alpha+1-n}}{\alpha+1-n}}}{2}\\ &= \sinh\left(\frac{t^{\alpha+1-n}}{\alpha+1-n}\right).\end{aligned}$$

(5)
$$\begin{aligned}\mathbb{T}_\alpha^n\left(\sinh\left[\frac{t^{\alpha+1-n}}{\alpha+1-n}\right]\right) &= \mathbb{T}_\alpha^n\left(\frac{e^{\frac{t^{\alpha+1-n}}{\alpha+1-n}} - e^{-\frac{t^{\alpha+1-n}}{\alpha+1-n}}}{2}\right)\\ &= \frac{e^{\frac{t^{\alpha+1-n}}{\alpha+1-n}} + e^{-\frac{t^{\alpha+1-n}}{\alpha+1-n}}}{2}\\ &= \cosh\left(\frac{t^{\alpha+1-n}}{\alpha+1-n}\right).\end{aligned}$$

Theorem 2.6 *Let $\tau:[0,\infty)\to\mathbb{R}$ be a given $\mathbb{T}_\alpha^n$-differential. Then*

$$\mathbb{T}_\alpha^n\tau(t) = t\mathbb{T}_\alpha^{n-1}\tau(t).$$

Proof. By Theorem 2.3

$$\mathbb{T}_\alpha^n\tau(t) = t^{n-\alpha}\tau'(t)$$

and

$$\mathbb{T}_\alpha^{n-1}\tau(t) = t^{n-1-\alpha}\tau'(t)$$

then

$$\mathbb{T}_\alpha^n\tau(t) = t\mathbb{T}_\alpha^{n-1}\tau(t).$$

Example 2.7 *Let $n=5$ and $\alpha=\frac{9}{2}$. then for all $t\in(0,+\infty)$*

$$\mathbb{T}_\alpha^n(t^2) = 2t^{\frac{3}{2}}$$

and

$$\mathbb{T}_{\alpha}^{n-1}(t^2) = 2t^{\frac{1}{2}}$$

then

$$\mathbb{T}_{\alpha}^{n}\tau(t) = t\mathbb{T}_{\alpha}^{n-1}\tau(t).$$

Remark 2.8 *In general*

$$\mathbb{T}_{\alpha+\beta}^{n}\tau(t) \neq \mathbb{T}_{\alpha}^{n}\mathbb{T}_{\beta}^{n}\tau(t).$$

For α,β be such that $n-1<\alpha,\beta \leq n$ and τ be a twice differential.

Example 2.9 *Let* $n=1, \alpha=\beta=\dfrac{1}{2}$ *and* $\tau(t)=t^2$.
Then

$$\mathbb{T}_{\alpha+\beta}^{n}\tau(t) = 2t$$

and

$$\mathbb{T}_{\alpha}^{n}\mathbb{T}_{\beta}^{n}\tau(t) = 3t$$

thus

$$\mathbb{T}_{\alpha+\beta}^{n}\tau(t) \neq \mathbb{T}_{\alpha}^{n}\mathbb{T}_{\beta}^{n}\tau(t).$$

Theorem 2.10 *Let* $\tau:[0,\infty)\to\mathbb{R}$ *be a given twice differential function and* $n-1<\alpha,\beta\leq n$. *Then*

$$\mathbb{T}_{\alpha}^{n}\mathbb{T}_{\beta}^{n}\tau(t) = (n-\beta)\mathbb{T}_{\alpha+\beta}^{2n-1}\tau(t) + \mathbb{T}_{\alpha+\beta}^{2n}\tau'(t), t>0$$

where

$$\mathbb{T}_{\alpha}^{2n}\tau(t) = t^{2n-\alpha}\tau'(t) \text{ and } \mathbb{T}_{\alpha}^{2n-1}\tau(t) = t^{2n-1-\alpha}\tau'(t).$$

Proof. For $t>0$, we have

$$\begin{aligned}\mathbb{T}_{\alpha}^{n}\mathbb{T}_{\beta}^{n}\tau(t) &= t^{n-\alpha}(\mathbb{T}_{\beta}^{n}\tau(t))' \\ &= t^{n-\alpha}[(n-\beta)t^{n-1-\beta}\tau'(t) + t^{n-\beta}\tau''(t)] \\ &= (n-\beta)t^{2n-1-(\alpha+\beta)}\tau'(t) + t^{2n-(\alpha+\beta)}\tau''(t) \\ &= (n-\beta)\mathbb{T}_{\alpha+\beta}^{2n-1}\tau(t) + \mathbb{T}_{\alpha+\beta}^{2n}\tau'(t).\end{aligned}$$

Example 2.11 *Consider the function* $\tau(t)=t^3, n=2, \alpha=\dfrac{5}{4}$ *and* $\beta=\dfrac{3}{2}$. *We have*

$$\mathbb{T}_{\alpha}^{n}\mathbb{T}_{\beta}^{n}\tau(t) = \mathbb{T}_{\frac{5}{4}}^{2}\mathbb{T}_{\frac{3}{2}}^{2}\tau(t) = 3t^{\frac{3}{4}}(\sqrt{t}t^2)' = \frac{15}{2}t^{\frac{9}{4}}$$

and

$$\begin{aligned}(n-\beta)\mathbb{T}_{\alpha+\beta}^{2n-1}\tau(t) + \mathbb{T}_{\alpha+\beta}^{2n}\tau'(t) &= \frac{1}{2}\mathbb{T}_{\frac{11}{4}}^{3}t^3 + \mathbb{T}_{\frac{11}{4}}^{4}(3t^2) \\ &= \frac{3}{2}t^{\frac{9}{4}} + 6t^{\frac{9}{4}} = \frac{15}{2}t^{\frac{9}{4}}\end{aligned}$$

thus

$$\mathbb{T}_{\alpha}^{n}\mathbb{T}_{\beta}^{n}\tau(t) = (n-\beta)\mathbb{T}_{\alpha+\beta}^{2n-1}\tau(t) + \mathbb{T}_{\alpha+\beta}^{2n}\tau'(t).$$

Definition 2.12 *Let $\tau : [0,+\infty) \to \mathbb{R}$ and $\alpha \in (n-1,n]$. The higher-order conformable integral is defined by*

$$\mathbb{I}_{\alpha}^{n}\tau(t) = \int_0^t s^{\alpha-n}\tau(s)ds, t \in [0,+\infty[$$

Example 2.13 *For $n = 7$ and $\alpha = \frac{13}{2}$*

$$\mathbb{I}_{\alpha}^{n}(t^2+3t+2) = \frac{4}{3}t^{\frac{3}{2}} + \frac{2}{7}t^{\frac{7}{2}} + \frac{6}{5}t^{\frac{5}{2}}.$$

Lemma 2.14 *Let $\tau : [0,\infty) \to \mathbb{R}$. for all $t > 0$*

$$\mathbb{T}_{\alpha}^{n}\mathbb{I}_{\alpha}^{n}\tau(t) = \tau(t)$$

Proof. For all $t \in (0,+\infty)$

$$\begin{aligned}\mathbb{T}_{\alpha}^{n}(\mathbb{I}_{\alpha}^{n}\tau)(t) &= t^{n-\alpha}(\mathbb{I}_{\alpha}^{n}\tau(t))' \\ &= t^{n-\alpha}t^{\alpha-n}\tau(t) = \tau(t).\end{aligned}$$

Lemma 2.15 *Let $\tau : [0,+\infty) \to \mathbb{R}$ be a higher-order differential and $\alpha \in (n-1,n]$.* Then

$$\mathbb{I}_{\alpha}^{n}\mathbb{T}_{\alpha}^{n}\tau(t) = \tau(t) - \tau(0).$$

Proof. Let $\alpha \in (n-1,n]$. Then

$$\begin{aligned}\mathbb{I}_{\alpha}^{n}\mathbb{T}_{\alpha}^{n}\tau(t) &= \int_0^t s^{\alpha-n}s^{n-\alpha}\tau'(s)ds \\ &= \tau(t) - \tau(0).\end{aligned}$$

Example 2.16 *Let consider $\tau(t) = 3t^2 + 2t + 1, \alpha = \frac{5}{2}$ and $n = 3$. So*

1. We have

$$\mathbb{I}_{\alpha}^{n}\tau(t) = \frac{6}{5}t^{\frac{5}{2}} + \frac{4}{3}t^{\frac{3}{2}} + 2\sqrt{t}$$

   and

$$\mathbb{T}_{\alpha}^{n}(\mathbb{I}_{\alpha}^{n}\tau)(t) = \sqrt{t}(3t^{\frac{3}{2}} + 2t^{\frac{1}{2}} + t^{-\frac{1}{2}}) = 3t^2 + 2t + 1$$

   then

$$\mathbb{T}_{\alpha}^{n}(\mathbb{I}_{\alpha}^{n}\tau)(t) = \tau(t)$$

2. We have

$$\mathbb{T}_{\alpha}^{n}\tau(t) = \sqrt{t}(6t+2)$$

and

$$\mathbb{I}_\alpha^n(\mathbb{T}_\alpha^n\tau)(t)=\int_0^t(6s+2)ds=3t^2+2t$$

then

$$\mathbb{I}_\alpha^n\mathbb{T}_\alpha^n\tau(t)=\tau(t)-\tau(0).$$

Theorem 2.17 *(Higher Roll's theorem)*
For $a>0$ and $\tau:[a,b]\to\mathbb{R}$ be a given function that satisfies:

1. τ is continuous on $[a,b]$.
2. τ is a higher differential for some $\alpha\in(n-1,n)$.
3. $\tau(a)=\tau(b)$.

We have, there exists $c\in(a,b)$ such that $\mathbb{T}_\alpha^n\tau(c)=0$.

Proof. Since τ is continuous on [a, b] and $\tau(a)=\tau(b)$, $c\in(a,b)$ is a point of local extrema. Assume, for example, that c is a local minimum point. Thus,

$$\mathbb{T}_\alpha^n\tau(c)=\lim_{\varepsilon\to0^+}\frac{\tau(c+\varepsilon t^{n-\alpha})-\tau(c)}{\varepsilon}=\lim_{\varepsilon\to0^-}\frac{\tau(c+\varepsilon t^{n-\alpha})-\tau(c)}{\varepsilon}$$

However, both the first and second limits are non-positive. Therefore,

$$\mathbb{T}_\alpha^n\tau(c)=\mathbb{T}_\alpha^{n,+}\tau(c)=\mathbb{T}_\alpha^{n,-}\tau(c)=0.$$

Theorem 2.18 *(Higher Mean Value Theorem)*
Let $a>0$ and $\tau:[a,b]\to\mathbb{R}$ be a given function that satisfies:

1. τ is continuous on $[a,b]$.
2. τ is a higher differential for some $\alpha\in(n-1,n)$.

Then, there exists $c\in(a,b)$ such that:

$$\mathbb{T}_\alpha^n\tau(c)=(\alpha-n+1)\frac{\tau(b)-\tau(a)}{b^{\alpha-n+1}-a^{\alpha-n+1}}.$$

Proof. Let consider

$$g(x)=\tau(x)-\tau(a)-\frac{\tau(b)-\tau(a)}{b^{\alpha-n+1}-a^{\alpha-n+1}}\left(x^{\alpha-n+1}-a^{\alpha-n+1}\right).$$

Then

$$g(a)=0,\quad g(b)=\tau(b)-\tau(a)-\frac{\tau(b)-\tau(a)}{b^{\alpha-n+1}-a^{\alpha-n+1}}\left(b^{\alpha-n+1}-a^{\alpha-n+1}\right)=0.$$

Higher Roll's Theorem criteria are satisfied by the function g. Consequently, $\exists c\in(a,b)$ such that:

$$\mathbb{T}_\alpha^n g(c)=0$$

Using the knowledge that $\mathbb{T}_\alpha^n(x^{\alpha-n+1})=\alpha-n+1$, we have:

$$\mathbb{T}_\alpha^n\tau(c)-(\alpha-n+1)\frac{\tau(b)-\tau(a)}{b^{\alpha-n+1}-a^{\alpha-n+1}}=0$$

Therefore:

$$\mathbb{T}_\alpha^n \tau(c) = (\alpha - n + 1)\frac{\tau(b) - \tau(a)}{b^{\alpha-n+1} - a^{\alpha-n+1}}.$$

Proposition 2.19 *(Chain Rule). Assume that $\tau, \tau_1 : [0, +\infty) \to \mathbb{R}$ be higher-order differential functions. For $h(t) = \tau \circ \tau_1(t) = \tau(\tau_1(t))$, we have h is higher-order differential with $\tau_1 \neq 0$ and*

$$\mathbb{T}_\alpha^n h(t) = \mathbb{T}_\alpha^n \tau_1(t) \times \mathbb{T}_\alpha^n \tau(\tau_1(t)) \times (\tau_1(t))^{\alpha-n}.$$

Proof. For $t \in (0, +\infty)$

$$\begin{aligned}
\mathbb{T}_\alpha^n h(t) &= t^{n-\alpha} h'(t) \\
&= t^{n-\alpha} \tau_1'(t) \tau'(\tau_1(t)) \\
&= \mathbb{T}_\alpha^n \tau_1(t) \tau'(\tau_1(t)) \\
&= \mathbb{T}_\alpha^n \tau_1(t) (\tau_1(t))^{\alpha-n} [\tau_1(t)]^{n-\alpha} \tau'(\tau_1(t)) \\
&= \mathbb{T}_\alpha^n \tau_1(t) \times \mathbb{T}_\alpha^n \tau(\tau_1(t)) \times (\tau_1(t))^{\alpha-n}.
\end{aligned}$$

Example 2.20 *Let $n = 5, \alpha = \dfrac{9}{2}, \tau(t) = 3t + 1$ and $\tau_1(t) = t^2 + 2t + 1$.*
Then

$$\mathbb{T}_\alpha^n h(t) = \sqrt{t}\ h'(t) = 6\sqrt{t}(t+1), \tag{6}$$

$$\mathbb{T}_\alpha^n \tau_1(t) = \sqrt{t}\ \tau_1'(t) = 2\sqrt{t}(t+1), \tag{7}$$

$$\left(\tau_1(t)\right)^{-\frac{1}{2}} = \frac{1}{t+1}, \tag{8}$$

and

$$\mathbb{T}_\alpha^n \tau(\tau_1(t)) = \mathbb{T}_\alpha^n \tau((t+1)^2) = 3(t+1). \tag{9}$$

Thus, by (6), (7), (8) and (9)

$$\mathbb{T}_\alpha^n h(t) = \mathbb{T}_\alpha^n \tau_1(t) \times \mathbb{T}_\alpha^n \tau(\tau_1(t)) \times (\tau_1(t))^{\alpha-n}.$$

3. Higher laplace transform and higher sumudu transform

This section introduces, with proofs, a set of fundamental properties and rules for the higher-order conformable Laplace and higher-order Sumudu transforms. We also prove the relationship between the two transformations needed to solve the corresponding higher-order conformable differential equations.

3.1 Higher Laplace transform

Definition 3.1 *Let $\tau : [0, \infty) \to \mathbb{R}$ be a given function and $n - 1 < \alpha \leq n$. The higher-order conformable Laplace transform of f is defined by*

$$\mathcal{L}_\alpha^n(\tau(t))(\lambda) = \int_0^{+\infty} e^{-\lambda \frac{t^{\alpha-n+1}}{\alpha-n+1}} t^{\alpha-n} \tau(t) dt, \lambda \in \mathbb{C}$$

Provided the integral exists.

Example 3.2 *Let* $n=8$ *and* $\alpha=\frac{15}{2}$. *Then for* $\lambda>\alpha$,

$$\mathcal{L}^{8}_{\frac{15}{2}}(e^{2\alpha\sqrt{t}})(\lambda)=\frac{1}{\lambda-\alpha}.$$

Theorem 3.3 *Let* $\tau:[0,\infty)\to\mathbb{R}$ *be a given function and* $n-1<\alpha\le n$. *Then*

$$\mathcal{L}^{n}_{\alpha}(\tau(t))(\lambda)=\mathcal{L}\left(\tau[\sqrt[\alpha-n+1]{(\alpha-n+1)u}\ ]\right)(\lambda),\ \lambda>0.$$

Proof. Applying Definition 3.1 and letting $u=\dfrac{t^{\alpha-n+1}}{\alpha-n+1}$

$$\begin{aligned}\mathcal{L}^{n}_{\alpha}[\tau(t)](\lambda)&=\int_0^{+\infty}e^{-\lambda\frac{t^{\alpha-n+1}}{\alpha-n+1}}t^{\alpha-n}\tau(t)dt\\&=\int_0^{\infty}e^{-su}\tau[\sqrt[\alpha-n+1]{(\alpha-n+1)u}\ ]du\\&=\mathcal{L}\left(\tau[\sqrt[\alpha-n+1]{(\alpha-n+1)u}\ ]\right)(\lambda).\end{aligned}$$

Theorem 3.4 *Let* $\tau:[0,\infty)\to\mathbb{R}$ *be a continuously higher-order differential function and* $n-1<\alpha\le n$. *Then*

$$\mathcal{L}^{n}_{\alpha}[\mathbb{T}^{n}_{\alpha}\tau(t)](\lambda)=\lambda\mathcal{L}^{n}_{\alpha}[\tau(t)](\lambda)-\tau(0),\ \lambda>0.$$

Proof. Applying Definition 3.1 and Theorem 2.3

$$\begin{aligned}\mathcal{L}^{n}_{\alpha}(\mathbb{T}^{n}_{\alpha}f(t))(\lambda)&=\int_0^{+\infty}e^{-\lambda\frac{t^{\alpha-n+1}}{\alpha-n+1}}\tau'(t)dt\\&=-\tau(0)+\lambda\int_0^{+\infty}e^{-\lambda\frac{t^{\alpha-n+1}}{\alpha-n+1}}\tau(t)t^{\alpha-n}dt\\&=\lambda\mathcal{L}^{n}_{\alpha}[\tau(t)](\lambda)-\tau(0).\end{aligned}$$

Theorem 3.5 *Assume that* $\tau:[0,\infty)\to\mathbb{R}$ *is a continuously higher-order differential function and* $n-1<\alpha\le n$. *Then*

$$\mathcal{L}^{n}_{\alpha}({}^{2}\mathbb{T}^{n}_{\alpha}\tau(t))(\lambda)=\lambda^{2}\mathcal{L}^{n}_{\alpha}(\tau(t))(\lambda)-\lambda\tau(0)-\mathbb{T}^{n}_{\alpha}\tau(0),\lambda>0.$$

Where ${}^{2}\mathbb{T}^{n}_{\alpha}\tau(t)=\mathbb{T}^{n}_{\alpha}\mathbb{T}^{n}_{\alpha}\tau(t)$.

Proof. Applying Theorem 3.4

$$\begin{aligned}\mathcal{L}^{n}_{\alpha}({}^{2}\mathbb{T}^{n}_{\alpha}\tau(t))(\lambda)&=\lambda\mathcal{L}^{n}_{\alpha}(\mathbb{T}^{n}_{\alpha}\tau(t))(\lambda)-\mathbb{T}^{n}_{\alpha}\tau(0)\\&=\lambda[\lambda\mathcal{L}^{n}_{\alpha}(\tau(t))(\lambda)-\tau(0)]-\mathbb{T}^{n}_{\alpha}\tau(0)\\&=\lambda^{2}\mathcal{L}^{n}_{\alpha}(\tau(t))(\lambda)-\lambda\tau(0)-\mathbb{T}^{n}_{\alpha}\tau(0).\end{aligned}$$

Theorem 3.6 *Let* f *be a real function taking its values in* $[0,t]$ *and*

$$n-1<\alpha\le n$$

$$\mathcal{L}^{n}_{\alpha}[\mathbb{I}^{n}_{\alpha}\tau(t)](\lambda)=\frac{1}{\lambda}\mathcal{L}^{n}_{\alpha}\tau(t)(\lambda),\lambda>0.$$

Proof. We have

$$\mathcal{L}_{\alpha}^{n}(\mathbb{T}_{\alpha}^{n}\mathbb{I}_{\alpha}^{n}\tau(t))(\lambda)=\mathcal{L}_{\alpha}^{n}(\tau(t))(\lambda)$$

and according to the theorem 3.4

$$\mathcal{L}_{\alpha}^{n}(\mathbb{T}_{\alpha}^{n}\mathbb{I}_{\alpha}^{n}\tau(t))(\lambda)=\lambda\mathcal{L}_{\alpha}^{n}(\mathbb{I}_{\alpha}^{n}\tau(t))(\lambda)-\mathbb{I}_{\alpha}^{n}\tau(0)=\lambda\mathcal{L}_{\alpha}^{n}(\mathbb{I}_{\alpha}^{n}\tau(t))(\lambda)$$

so

$$\mathcal{L}_{\alpha}^{n}(\mathbb{I}_{\alpha}^{n}\tau(t))(\lambda)=\frac{1}{\lambda}\mathcal{L}_{\alpha}^{n}\tau(t)(\lambda),\forall\lambda>0.$$

Theorem 3.7 *Let* $a\in\mathbb{R}$ *and* $n-1<\alpha,\beta\leq n$.

1. $\mathcal{L}_{\alpha}^{n}\left(e^{a\frac{t^{\alpha+1-n}}{\alpha+1-n}}\right)(\lambda)=\dfrac{1}{\lambda-a},\lambda>a.$
2. $\mathcal{L}_{\alpha}^{n}\left(\sin[a\dfrac{t^{\alpha+1-n}}{\alpha+1-n}]\right)(\lambda)=\dfrac{a}{\lambda^2+a^2},\lambda>0.$
3. $\mathcal{L}_{\alpha}^{n}\left(\cos[a\dfrac{t^{\alpha+1-n}}{\alpha+1-n}]\right)(\lambda)=\dfrac{\lambda}{\lambda^2+a^2},\lambda>0.$
4. $\mathcal{L}_{\alpha}^{n}\left(\sinh[a\dfrac{t^{\alpha+1-n}}{\alpha+1-n}]\right)(\lambda)=\dfrac{a}{\lambda^2-a^2},\lambda>|a|.$
5. $\mathcal{L}_{\alpha}^{n}\left(\cosh[a\dfrac{t^{\alpha+1-n}}{\alpha+1-n}]\right)(\lambda)=\dfrac{\lambda}{\lambda^2-a^2},\lambda>|a|.$

Proof. Using definition 3.1

(1)

$$\begin{aligned}\mathcal{L}_{\alpha}^{n}\left(e^{a\frac{t^{\alpha+1-n}}{\alpha+1-n}}\right)(\lambda)&=\int_0^{+\infty}e^{-(\lambda-a)\frac{t^{\alpha-n+1}}{\alpha-n+1}}t^{\alpha-n}dt\\&=\int_0^{+\infty}e^{-(\lambda-a)u}du\\&=\frac{1}{\lambda-a}.\end{aligned}$$

(2)

$$\begin{aligned}\mathcal{L}_{\alpha}^{n}\left(\sin[a\frac{t^{\alpha+1-n}}{\alpha+1-n}]\right)(\lambda)&=\int_0^{+\infty}e^{-\lambda\frac{t^{\alpha-n+1}}{\alpha-n+1}}\sin[a\frac{t^{\alpha+1-n}}{\alpha+1-n}]t^{\alpha-n}dt\\&=\int_0^{+\infty}e^{-\lambda u}\sin(au)du\\&=\frac{a}{\lambda}\int_0^{+\infty}e^{-\lambda u}\cos(au)du\\&=\frac{a}{\lambda^2}-\frac{a^2}{\lambda^2}\int_0^{+\infty}e^{-\lambda u}\sin(au)du.\end{aligned}$$

Then

$$\mathcal{L}_{\alpha}^{n}\left(\sin[a\frac{t^{\alpha+1-n}}{\alpha+1-n}]\right)(\lambda)=\frac{a}{\lambda^2+a^2}.$$

(3)

$$\begin{aligned}\mathcal{L}_{\alpha}^{n}\left(\cos[a\frac{t^{\alpha+1-n}}{\alpha+1-n}]\right)(\lambda)&=\int_0^{+\infty}e^{-\lambda\frac{t^{\alpha-n+1}}{\alpha-n+1}}\cos[a\frac{t^{\alpha+1-n}}{\alpha+1-n}]t^{\alpha-n}dt\\&=\int_0^{+\infty}e^{-\lambda u}\cos(au)du\\&=\frac{1}{\lambda}-\frac{a}{\lambda}\int_0^{+\infty}e^{-\lambda u}\sin(au)du\\&=\frac{a}{\lambda}-\frac{a^2}{\lambda^2}\int_0^{+\infty}e^{-\lambda u}\cos(au)du.\end{aligned}$$

Then

$$\mathcal{L}_{\alpha}^{n}\left(\cos[a\frac{t^{\alpha+1-n}}{\alpha+1-n}]\right)(\lambda)=\frac{\lambda}{\lambda^2+a^2}.$$

(4)

$$\begin{aligned}\mathcal{L}_{\alpha}^{n}\left(\sinh[a\frac{t^{\alpha+1-n}}{\alpha+1-n}]\right)(\lambda)&=\mathcal{L}_{\alpha}^{n}\left(\frac{e^{a\frac{t^{\alpha+1-n}}{\alpha+1-n}}-e^{-a\frac{t^{\alpha+1-n}}{\alpha+1-n}}}{2}\right)(\lambda)\\&=\frac{1}{2}\left(\frac{1}{\lambda-a}-\frac{1}{\lambda+a}\right)\\&=\frac{a}{\lambda^2-a^2}.\end{aligned}$$

(5)

$$\begin{aligned}\mathcal{L}_{\alpha}^{n}\left(\cosh[a\frac{t^{\alpha+1-n}}{\alpha+1-n}]\right)(\lambda)&=\mathcal{L}_{\alpha}^{n}\left(\frac{e^{a\frac{t^{\alpha+1-n}}{\alpha+1-n}}+e^{-a\frac{t^{\alpha+1-n}}{\alpha+1-n}}}{2}\right)(\lambda)\\&=\frac{1}{2}\left(\frac{1}{\lambda-a}+\frac{1}{\lambda+a}\right)\\&=\frac{\lambda}{\lambda^2-a^2}.\end{aligned}$$

Theorem 3.8 *Let $\tau_1,\tau_2:[0,\infty)\to\mathbb{R}$ be two given functions and $n-1<\alpha\leq n$.*

$$\mathcal{L}_{\alpha}^{n}[(\tau_1*\tau_2)(t)](\lambda)=\mathcal{L}_{\alpha}^{n}[\tau_1(t)](\lambda)\mathcal{L}_{\alpha}^{n}[\tau_2(t)](\lambda)$$

where

$$(\tau_1*\tau_2)(t)=\int_0^t\tau_1(s)\tau_2\left(\sqrt[\alpha-n+1]{t^{\alpha-n+1}-s^{\alpha-n+1}}\right)s^{\alpha-n}ds.$$

Proof. By Definition 3.1

$$\begin{aligned}\mathcal{L}_\alpha^n[(\tau_1 * \tau_2)(t)](\lambda) &= \int_0^{+\infty} e^{-\lambda\frac{t^{\alpha-n+1}}{\alpha-n+1}} t^{\alpha-n}(\tau_1 * \tau_2)(t)dt \\ &= \int_0^{+\infty} e^{-\lambda\frac{t^{\alpha-n+1}}{\alpha-n+1}} t^{\alpha-n}\left(\int_0^t \tau_1(s)\tau_2\left(\sqrt[\alpha-n+1]{t^{\alpha-n+1}-s^{\alpha-n+1}}\right)s^{\alpha-n}ds\right)dt \\ &= \int_0^{+\infty} \tau_1(s)s^{\alpha-n}\left(\int_s^{+\infty} e^{-\lambda\frac{t^{\alpha-n+1}}{\alpha-n+1}} \tau_2\left(\sqrt[\alpha-n+1]{t^{\alpha-n+1}-s^{\alpha-n+1}}\right)t^{\alpha-n}dt\right)ds.\end{aligned}$$

By changing the variable $v = \sqrt[\alpha-n+1]{t^{\alpha-n+1}-s^{\alpha-n+1}}$.

$$\begin{aligned}\mathcal{L}_\alpha^n[(\tau_1 * \tau_2)(t)](\lambda) &= \int_0^{+\infty} \tau_1(s)s^{\alpha-n}\left(\int_0^{+\infty} e^{-\lambda\frac{v^{\alpha-n+1}+s^{\alpha-n+1}}{\alpha-n+1}} \tau_2(v)v^{\alpha-n}dv\right)ds \\ &= \left(\int_0^{+\infty} \tau_1(s)s^{\alpha-n}e^{-\lambda\frac{s^{\alpha-n+1}}{\alpha-n+1}}ds\right)\left(\int_0^{+\infty} \tau_1(v)v^{\alpha-n}e^{-\lambda\frac{v^{\alpha-n+1}}{\alpha-n+1}}dv\right) \\ &= \mathcal{L}_\alpha^n[\tau_1(t)](\lambda)\mathcal{L}_\alpha^n[\tau_2(t)](\lambda).\end{aligned}$$

3.2 Higher Sumudu transform

Definition 3.9 *Over the following set of functions:*

$$A_\alpha(\tau) = \{\tau(x) : \exists M, \tau_1, \tau_2 > 0, |\tau(x)| < Me^{\frac{|x^{\alpha-n+1}|}{(\alpha-n+1)\tau_j}}, \text{ if } x^{\alpha-n+1} \in (-1)^j \times [0,+\infty), j = 1,2\}.$$

The Higher conformable Sumudu transform of f can be generalized by:

$$\mathcal{S}_\alpha^n(\tau(t))(u) = \frac{1}{u}\int_0^{+\infty} e^{-\frac{1}{u}\frac{t^{\alpha-n+1}}{\alpha-n+1}} t^{\alpha-n}\tau(t)dt, u \in \mathbb{C}$$

The next theorem gives a relationship between the higher-order conformable Sumudu transform and the higher-order conformable Laplace transform.

Theorem 3.10 *Let* $\tau : [0,\infty) \to \mathbb{R}$ *be a given function and* $n-1 < \alpha \le n$.Then

$$\mathcal{S}_\alpha^n(\tau(t))(u) = \frac{1}{u}\mathcal{L}_\alpha^n(\tau(t))(\frac{1}{u}).$$

Proof. Applying Definition 3.9, we get:

$$\begin{aligned}\mathcal{S}_\alpha^n(\tau(t))(u) &= \frac{1}{u}\int_0^{+\infty} e^{-\frac{1}{u}\frac{t^{\alpha-n+1}}{\alpha-n+1}} t^{\alpha-n}\tau(t)dt, u \in \mathbb{C} \\ &= v\int_0^{+\infty} e^{-v\frac{t^{\alpha-n+1}}{\alpha-n+1}} t^{\alpha-n}\tau(t)dt, u \in \mathbb{C}(v = \frac{1}{u}) \\ &= v\mathcal{L}_\alpha^n(\tau(t))(v) \\ &= \frac{1}{u}\mathcal{L}_\alpha^n(\tau(t))(\frac{1}{u}).\end{aligned}$$

which completes the proof of the Theorem 3.10.

Theorem 3.11 *Let* $\tau:[0,\infty)\to\mathbb{R}$ *be a given function and* $n-1<\alpha\le n$*.Then*

$$\mathcal{S}_{\alpha}^{n}(\tau(t))(u)=\mathcal{S}\left(\tau[\sqrt[\alpha-n+1]{(\alpha-n+1)s}\,]\right)(u),u>0$$

Proof. Using Theorems 3.10 and Theorem 3.3, we get:

$$\begin{aligned}\mathcal{S}_{\alpha}^{n}(\tau(t))(u)&=\frac{1}{u}\mathcal{L}_{\alpha}^{n}(\tau(t))(\frac{1}{u})\\&=\frac{1}{u}\mathcal{L}\left(\tau[\sqrt[\alpha-n+1]{(\alpha-n+1)s}\,]\right)(\frac{1}{u})\\&=\mathcal{S}\left(\tau[\sqrt[\alpha-n+1]{(\alpha-n+1)s}\,]\right)(u).\end{aligned}$$

Theorem 3.12 *Let* $\tau:[0,\infty)\to\mathbb{R}$ *be a given continuously higher-order differential function and* $n-1<\alpha\le n$. *Then*

$$\mathcal{S}_{\alpha}^{n}[\mathbb{T}_{\alpha}^{n}\tau(t)](u)=\frac{\mathcal{S}_{\alpha}^{n}[\tau(t)](u)-\tau(0)}{u},u>0.$$

Proof. By Theorem 3.10 and Theorem 3.4

$$\begin{aligned}\mathcal{S}_{\alpha}^{n}[\mathbb{T}_{\alpha}^{n}\tau(t)](u)&=\frac{1}{u}\mathcal{L}_{\alpha}^{n}(\tau(t))(\frac{1}{u})\\&=\frac{\frac{1}{u}\mathcal{L}_{\alpha}^{n}[\tau(t)](\frac{1}{u})-\tau(0)}{u}\\&=\frac{\mathcal{S}_{\alpha}^{n}[\tau(t)](u)-\tau(0)}{u}.\end{aligned}$$

Theorem 3.13 *Let* $\tau:[0,\infty)\to\mathbb{R}$ *be a given continuously higher-order differential function and* $n-1<\alpha\le n$. *Then*

$$\mathcal{S}_{\alpha}^{n}({}^{2}\mathbb{T}_{\alpha}^{n}\tau(t))(u)=\frac{1}{u^{2}}\mathcal{S}_{\alpha}^{n}(\tau(t))(u)-\frac{1}{u^{2}}\tau(0)-\frac{1}{u}\mathbb{T}_{\alpha}^{n}\tau(0),u>0.$$

Where ${}^{2}\mathbb{T}_{\alpha}^{n}\tau(t)=\mathbb{T}_{\alpha}^{n}\mathbb{T}_{\alpha}^{n}\tau(t)$.

Proof. By Theorem 3.10 and Theorem 3.5

$$\begin{aligned}\mathcal{S}_{\alpha}^{n}({}^{2}\mathbb{T}_{\alpha}^{n}\tau(t))(u)&=\frac{1}{u}\mathcal{L}_{\alpha}^{n}({}^{2}\mathbb{T}_{\alpha}^{n}\tau(t))(u)\\&=\frac{1}{u}[\frac{1}{u^{2}}\mathcal{L}_{\alpha}^{n}(\tau(t))(\frac{1}{u})-\frac{1}{u}\tau(0)-\mathbb{T}_{\alpha}^{n}\tau(0)]\\&=\frac{1}{u^{2}}\mathcal{S}_{\alpha}^{n}(\tau(t))(u)-\frac{1}{u^{2}}\tau(0)-\frac{1}{u}\mathbb{T}_{\alpha}^{n}\tau(0).\end{aligned}$$

Theorem 3.14 *For a continuous function* $\tau:[0,+\infty)\to\mathbb{R}$.

$$\mathcal{S}_{\alpha}^{n}[\mathbb{I}_{\alpha}^{n}\tau(t)](u)=u\mathcal{S}_{\alpha}^{n}\tau(t)(u),u>0.$$

Proof. Using Theorems 3.10 and Theorem 3.6, we get:

$$\begin{aligned}\mathcal{S}_\alpha^n[\mathbb{I}_\alpha^n \tau(t)](u) &= \frac{1}{u}\mathcal{S}_\alpha^n[\mathbb{I}_\alpha^n \tau(t)](\frac{1}{u}) \\ &= \frac{1}{u} u \mathcal{S}_\alpha^n[\tau(t)](\frac{1}{u}) \\ &= u\mathcal{S}_\alpha^n \tau(t)(u).\end{aligned}$$

Which completes the proof of Theorem 3.14.

Theorem 3.15 *Let* $a \in \mathbb{R}$ *and* $n-1 < \alpha, \beta \leq n$.

1. $\mathcal{S}_\alpha^n\left(e^{a\frac{t^{\alpha+1-n}}{\alpha+1-n}}\right)(u) = \frac{1}{1-au}, u > \frac{1}{a}$.
2. $\mathcal{S}_\alpha^n\left(\sin[a\frac{t^{\alpha+1-n}}{\alpha+1-n}]\right)(u) = \frac{au}{1+a^2u^2}, u > \frac{1}{|a|}$.
3. $\mathcal{S}_\alpha^n\left(\cos[a\frac{t^{\alpha+1-n}}{\alpha+1-n}]\right)(u) = \frac{1}{1+a^2u^2}, u > \frac{1}{|a|}$.
4. $\mathcal{S}_\alpha^n\left(\sinh[a\frac{t^{\alpha+1-n}}{\alpha+1-n}]\right)(u) = \frac{au}{1-a^2u^2}, u > \frac{1}{|a|}$.
5. $\mathcal{S}_\alpha^n\left(\cosh[a\frac{t^{\alpha+1-n}}{\alpha+1-n}]\right)(u) = \frac{1}{1-a^2u^2}, u > \frac{1}{|a|}$.

Proof. Using Theorem 3.10 and Theorem 3.7

(1)

$$\begin{aligned}\mathcal{S}_\alpha^n\left(e^{a\frac{t^{\alpha+1-n}}{\alpha+1-n}}\right)(u) &= \frac{1}{u}\mathcal{L}_\alpha^n\left(e^{a\frac{t^{\alpha+1-n}}{\alpha+1-n}}\right)(\frac{1}{u}) \\ &= \frac{1}{u}\left(\frac{1}{\frac{1}{u}-a}\right) \\ &= \frac{1}{1-au}.\end{aligned}$$

(2)

$$\begin{aligned}\mathcal{S}_\alpha^n\left(\sin[a\frac{t^{\alpha+1-n}}{\alpha+1-n}]\right)(u) &= \frac{1}{u}\mathcal{L}_\alpha^n\left(\sin[a\frac{t^{\alpha+1-n}}{\alpha+1-n}]\right)(\frac{1}{u}) \\ &= \frac{1}{u}\frac{au^2}{1+a^2u^2} \\ &= \frac{au}{1+a^2u^2}.\end{aligned}$$

(3)

$$\begin{aligned}\mathcal{S}_\alpha^n\left(\cos[a\frac{t^{\alpha+1-n}}{\alpha+1-n}]\right)(u) &= \frac{1}{u}\mathcal{L}_\alpha^n\left(\cos[a\frac{t^{\alpha+1-n}}{\alpha+1-n}]\right)(\frac{1}{u})\\ &= \frac{1}{u^2}\frac{u^2}{1+a^2u^2}\\ &= \frac{1}{1+a^2u^2}.\end{aligned}$$

(4)

$$\begin{aligned}\mathcal{S}_\alpha^n\left(\sinh[a\frac{t^{\alpha+1-n}}{\alpha+1-n}]\right)(u) &= \frac{1}{u}\mathcal{L}_\alpha^n\left(\sinh[a\frac{t^{\alpha+1-n}}{\alpha+1-n}]\right)(\frac{1}{u})\\ &= \frac{1}{u}\frac{au^2}{1-a^2u^2}\\ &= \frac{au}{1-a^2u^2}.\end{aligned}$$

(5)

$$\begin{aligned}\mathcal{S}_\alpha^n\left(\cosh[a\frac{t^{\alpha+1-n}}{\alpha+1-n}]\right)(u) &= \frac{1}{u}\mathcal{L}_\alpha^n\left(\cosh[a\frac{t^{\alpha+1-n}}{\alpha+1-n}]\right)(\frac{1}{u})\\ &= \frac{1}{u^2}\frac{u^2}{1-a^2u^2}\\ &= \frac{1}{1-a^2u^2}.\end{aligned}$$

Theorem 3.16 *Let* $\tau_1,\tau_2:[0,\infty)\to\mathbb{R}$ *be two given functions and* $n-1<\alpha\le n$.

$$\mathcal{S}_\alpha^n[(\tau_1*\tau_2)(t)](u) = u\mathcal{S}_\alpha^n[\tau_1(t)](u)\mathcal{S}_\alpha^n[\tau_2(t)](u).$$

Proof. Using Theorem 3.8 and Theorem 3.10

$$\begin{aligned}\mathcal{S}_\alpha^n[(\tau_1*\tau_2)(t)](u) &= \frac{1}{u}\mathcal{L}_\alpha^n[(\tau_1*\tau_2)(t)](\frac{1}{u})\\ &= \frac{1}{u}\mathcal{L}_\alpha^n[(\tau_1(t)](\frac{1}{u})\mathcal{L}_\alpha^n[(\tau_2(t)](\frac{1}{u})\\ &= u\mathcal{S}_\alpha^n[(\tau_1(t)](u)\mathcal{S}_\alpha^n[(\tau_2(t)](u).\end{aligned}$$

4. Applications

Example 4.1 *Let's study the following differential equation:*

$$u'(t)+3t^{-\frac{1}{2}}u(t) = t^{-\frac{1}{2}}e^{2t^{\frac{1}{2}}} \tag{10}$$

where $u(0)=0, t>0$.

So according to theorem 2.3, We replace the equation (10) with

$$\mathbb{T}^{8}_{\frac{15}{2}}u(t)+3y(t)=e^{2\sqrt{t}}$$

By utilizing the $\mathcal{L}^{n}_{\alpha}$-Laplace transform and Theorem 3.4, we have for $\lambda>1$

$$\mathcal{L}^{n}_{\alpha}(u(t))=\frac{3}{4}\frac{1}{\lambda+3}+\frac{1}{4}\frac{1}{\lambda-1}$$

And by Theorem 3.7, we find

$$u(t)=\frac{3}{4}e^{-6\sqrt{t}}+\frac{1}{4}e^{2\sqrt{t}}.$$

Example 4.2 *We want to study the following higher-order harmonic oscillator equation:*

$$^{2}\mathbb{T}^{n}_{\alpha}\tau(t)+9\tau(t)=f(t),\tau(0)=a,\mathbb{T}^{n}_{\alpha}\tau(0)=b \tag{11}$$

By applying the higher-order Laplace transform to this equation and using the fact that it is a linear operator

$$\mathcal{L}^{n}_{\alpha}\left({}^{2}\mathbb{T}^{n}_{\alpha}\tau(t)\right)(\lambda)+9\mathcal{L}^{n}_{\alpha}[\tau(t)](\lambda)=\mathcal{L}^{n}_{\alpha}[f(t)](\lambda).$$

By Theorem 3.5

$$\lambda^{2}\mathcal{L}^{n}_{\alpha}(\tau(t))(\lambda)-\lambda\tau(0)-\mathbb{T}^{n}_{\alpha}\tau(0)+9\mathcal{L}^{n}_{\alpha}(\tau(t))(\lambda)=\mathcal{L}^{n}_{\alpha}(f(t))(\lambda)$$

then

$$\mathcal{L}^{n}_{\alpha}[\tau(t)](\lambda)=a\frac{\lambda}{\lambda^{2}+3^{2}}+b\frac{1}{\lambda^{2}+3^{2}}+\frac{1}{\lambda^{2}+3^{2}}\mathcal{L}^{n}_{\alpha}[f(t)](\lambda)$$

And using Theorem 3.8 and Theorem 3.7

$$\tau(t)=a\cos[3\frac{t^{\alpha+1-n}}{\alpha+1-n}]+\frac{b}{3}\sin[3\frac{t^{\alpha+1-n}}{\alpha+1-n}]+\frac{1}{3}\int_{0}^{t}\sin[3\frac{s^{\alpha+1-n}}{\alpha+1-n}]f\left(\sqrt[\alpha-n+1]{t^{\alpha-n+1}-s^{\alpha-n+1}}\right)s^{\alpha-n}ds.$$

References

[1] F. Alrawajeh Z. Al-Zhouri, N. Al-Mutai, and R. Alkhasawnh. New results on conformable fractional sumudo transform. Theories and applications, International Journal of Analysis and Applications, 17(6):1019–1033, 2019.

[2] R. Bahloul, R.M. Soufiane, T. Abdeljawad, and B. Abdalla: Some Results of Conformable Fourier Transform. European Journal of Pure and Applied Mathematics, Vol. 17, No. 4, (2024), 2405–2430.

[3] R. Bahloul and M. Sbabheh, Some results of the new definition of $\mathcal{N}^{\alpha}_{F}$-Fourier transform and their applications. Gulf Journal of Mathematics Vol 19, Issue 2 (2025) 228–246.

[4] R. Bahloul, R. Houssame and A. Thabet, New Definition of $\mathcal{N}^{\alpha}_{F}$-Laplace Conformable Transform and Their Applications. Journal of Computational Analysis and Applications Vol. 34, No. 4 (2025), 730–746.

[5] M. Bouziani, R. Houssame and R. Bahloul, Conformable ARA Transform Function and its Properties. Asia Pac. J. Math. 2025 12:58.

[6] M. Caputo, Linear models of dissipation whose q is almost frequency independent-ii. Geophysical J. of the Royal Astronomical Society. (1967); 13(5):529–539.

[7] I. Kadria, M. Horanib, and R. Khali, Solution of a fractional Laplace type equation in a conformable sense using fractional Fourier series with separation of variables technique, Results in Nonlinear Analysis 6 (2023) No. 2, 53–59.

[8] A. Kilic Man, H. Eltayeb, and Kamatan, A Note on the Comparison Between Laplace and Sumudu Transforms. Bulletin of the Iranian Mathematical Society Vol. 37 No. 1 (2011), pp. 131–141.

[9] KS. Killer, B. Ross. An Introduction to the Fractional Calculus and Fractional Differential Equations. New York: John Wiley and Sons (1993).

[10] JT. Machado, V. Kiryakova, F. Mainardi. Recent history of fractional calculus. Communications in Nonlinear Science and Numerical Simulation. (2011); 16(3):1140–1153.

[11] V.T. Nguyen, Note on the convergence of fractional conformable diffusion equation with linear source term, Results in Nonlinear Analysis 5(2022) No. 3, 387–392.

[12] M. Riesz, L'intégrale de Riemann-Liouville et le problème de Cauchy. Acta Mathematica. (1949) ; 81(1) : 1–222.

[13] KH. Roshdi, M. Al Horani, A. Yousef, M. Sababheh. A new definition of fractional derivative, J. Comput. Appl. Math. 264 (2014) 65–70.

[14] F.S. Silva, Davidson M. Moreira, and Marcelo A. Moret, Conformable Laplace Transform of Fractional Differential Equations; Axioms 2018, 7, 55.

[15] V. Stojiljkovic, A New Conformable Fractional Derivative and Applications. Sel.Mat. (2022), 9, 370–380.

[16] A. Thabet, On conformable fractional calculus, J. Comput. Appl. Math. 279 (2015) 57–66.

[17] H. Zhou, S. Yang, S. Zhang, Conformable derivative approach to anomalous diffusion, Physica A: Stat. Mech. Appl. 491 (2018) 1001–1013.