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On self-similar dendrites in Hilbert space

K. B. Allabergenova[1], M. B. Kadirova[1]

[1]Department of Mechanics and Mathematics, Novosibirsk State University, Novosibirsk, Russia.

Abstract

We consider the relation of the ramification order of self-similar dendrites in $\mathbb{R}^d$ and in the Hilbert space and their Hausdorff dimension and measure.

Key words and phrases: Self-similar set, Open Set Condition, address map, infinitely ramified dendrite, ramification points

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1 Introduction

Let $\mathcal{S} = \{\mathcal{S}_1, \ldots, \mathcal{S}_m\}$ be a system of contracting similarities in a complete metric space X. The unique compact non-empty set K that satisfies equation $K = \bigcup_{i=1}^{m} S_i(K)$ is called *the attractor of the system* $\mathcal{S}$, or a *self-similar set*, defined by the system $\mathcal{S}$ [8]. If the self-similar set K is connected, we call it a *self-similar continuum*. By Hata's Theorem [7], each self-similar continuum is locally connected.

A *dendrite* is a locally connected continuum K that does not contain a simple closed curve [6]. Some authors call dendrites acyclic curves [7]. Thus, each acyclic self-similar continuum is a self-similar dendrite. The present paper considers some properties of self-similar dendrites in $\mathbb{R}^d$ and in the Hilbert space.

Since the inception of the theory of self-similar sets, self-similar dendrites have been addressed by many authors in their papers on the subject. In 1985 M. Hata [7], proved that the set of endpoints of a nontrivial self-similar dendrite has infinite cardinality. In 1995 J. Kigami [9] investigated the shortest path metrics in postcritically finite self-similar dendrites and constructed regular Dirichlet forms

Email Addresses: k.allabergenova@g.nsu.ru (Klara Allabergenova); m.kadirova@g.nsu.ru (Makhliyo Kadirova)

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for such dendrites. In 1991, C. Bandt and K. Keller [3] proposed the intersection graph criterion for the dendrite property of finitely ramified self-similar continua (Proposition 9). The paper [5] contained the idea of the main tree of a PCF self-similar dendrite.

The open set condition (OSC) and the weak separation property (WSP) are essential in the theory of self-similar sets for analyzing their dimension and measure.

In 1992 Bandt and Graf [2] introduced the condition $\mathrm{id} \notin \overline{\mathcal{F}(\mathcal{S})}$ which is equivalent to the positiveness of the Hausdorff measure $H^s(K)$ where s is the similarity dimension of the system $\mathcal{S}$.

In 1994 A. Schief [13] proved that the conditions SOSC, OSC and the positiveness of Hausdorff measure $H^s(K)$ are equivalent for self-similar sets in $\mathbb{R}^d$. In 1996 [14] he extended his results to self-similar sets in complete metric spaces.

The weak separation property (WSP) was defined for self-similar sets by K.-S. Lau and S.-M. Ngai [12] and by M. Zerner[16]. If a self-similar set K has WSP, the measure of K in its dimension is positive. There are many geometric phenomena related to the fulfillment or violation of this property.

As was proved by C. Bandt and H. Rao in [4], any self-similar continuum in the plane that has a finite intersection property satisfies the open set condition, which implies the WSP.

In paper [1], it was proved that all self-similar dendrites in the plane have the Weak Separation Property.

We prove that if a self-similar dendrite in $\mathbb{R}^n$ has the WSP, then the orders of its points are bounded (Theorem 3.1). Here, n is any positive integer.

There exist examples of self-similar dendrites on the plane that satisfy the Weak Separation Property (WSP) but do not satisfy the open set condition (OSC). In this case, some intersections of copies may have positive measures and are attractors of certain graph-directed systems of similarities.

In the case where the system $\mathcal{S}$ satisfies OSC, the intersection of the pieces of the dendrite $K(\mathcal{S})$ is simply connected and has zero measure in K. Therefore, such an intersection is a subdendrite in K. In [1] it was proved that for any n there is a self-similar dendrite K in the plane that contains n pieces $K_1,...,K_n$ and a subdendrite K' such that for any $i,j \in \{1,...,n\}$, $K_i \cap K_j = K'$.

2. Preliminaries

2.1 Self-similar sets.

Definition 2.1: *Let $\mathcal{S} = \{S_1,\ldots,S_m\}$ be a system of contraction similarities in $\mathbb{R}^n$. A compact nonempty set K that satisfies equation $K = \bigcup_{i=1}^{m} S_i(K)$ is called the attractor of the system $\mathcal{S}$ or the set that is self-similar with respect to the system $\mathcal{S}$.*

We denote by $I = \{1,\ldots,m\}$ the set of indices of the system $\mathcal{S}$, then $I^* = \bigcup_{n=1}^{\infty} I^n$ denotes the set of all words $\mathbf{i} = i_1 \ldots i_n$ of finite length in the alphabet I, called *multi-indices*. We use the notation $S_{\mathbf{j}} = S_{j_1 j_2 \ldots j_n} = S_{j_1} S_{j_2} \ldots S_{j_n}$, and denote $S_{\mathbf{j}}(K)$ by $K_{\mathbf{j}}$. The sets $K_{\mathbf{j}}$, where $\mathbf{j} = j_1 j_2 \ldots j_n$, are called *copies of order n* of the set K. For a word $\mathbf{i} = i_1 \ldots i_n$, with $n \geq 1$, we denote $\mathbf{i}^- := i_1 \ldots i_{n-1}$. We denote by $|K|$ the diameter of K.

The set of all infinite strings (or addresses) $I^\infty = \{\beta = \beta_1\beta_2\ldots, \beta_i \in I\}$ is called *the index space* of the system $\mathcal{S}$. The mapping $\pi : I^\infty \to K$ that maps a sequence $\beta \in I^\infty$ to the point $x = \bigcap_{n=1} K_{\beta_1\ldots\beta_n}$ is called the *address map* for the attractor K. Then for each $x \in K$, the set $\pi^{-1}(x)$ is a set of addresses of the point x.

Following M. P. W. Zerner, we define $F := \{S_{\mathbf{i}} : \mathbf{i} \in I^*\}$ to be the semigroup with identity generated by $\mathcal{S}$. We set

$$F_b := \{f \in F : q_f \in (bq_{\min}, b]\}$$

where $q_{\min} := \min\{q_1,...,q_m\}$ and $b \ge 0$.

Given the system $\mathcal{S}$, we define $N(b,x) := \#\{S_{\mathbf{j}} \in F_b : K_{\mathbf{j}} \subset B(x, 2b|K|)\}$ and $N(b) = \max\{N(b,x) : x \in K\}$.

Given a probability vector $\mathbf{p} = \{p_1,..., p_m\}$ we denote by μ_p the Bernoulli measure with the probability vector $\mathbf{p}$ which assigns to each cylinder $i_1...i_n I$ the value $p_{i_1...i_n}$.

Let s be the similarity dimension of the system $\mathcal{S}$, that is, the solution of Moran equation $q_1^s + ... + q_m^s = 1$. We denote by ν the Bernoulli measure μ_p with probability vector $\mathbf{p} = \{q_1^s,..., q_m^s\}$. The natural measure μ^* on K is defined by the equation $\mu^*(A) = \nu(\pi^{-1}(A))$.

Theorem 2.2 ([10], Theorem 1.6.3.): *Let (X,ρ) be a metric space, and μ a Borel probability measure on X. Fix a Borel set $A \subseteq X$. Assume that there exists a constant $c \in (0,\infty]$ such that*

$$\overline{\lim} \frac{\mu(B(x,r))}{r^t} \ge c$$

for all points $x \in A$ except perhaps for countably many.

Then the Hausdorff measure H_t satisfies

$$H_t(E) \le c^{-1} 8^t \mu(E)$$

for every Borel set $E \subseteq A$.

If, conversely,

$$\overline{\lim} \frac{\mu(B(x,r))}{r^t} \le c < +\infty \text{ for all } x \in A,$$

then $\mu(E) \le H_t(E)$ for every Borel set $E \subseteq A$.

Definition 2.3: *We denote by J the set of addresses $\alpha = a_1 a_2 ... \in I$ possessing the following property: for any $\mathbf{i} = i_1 i_2 ... i_n \in I^*$ there is $k \in \mathbb{N} \cup \{0\}$ such that $a_{k+1} a_{k+2} ... a_{k+n} = i_1 i_2 ... i_n$.*

Proposition 2.4: *For any Bernoulli measure μ_p on I , $\mu_p(J) = 1$.*

Proof. Consider the set $J^c = I^\infty \setminus J$. If $\alpha = a_1 a_2 ... \in J^c$, then there is $\mathbf{j} = j_1 \dots j_n \in I^*$, such that for any $k \in \mathbb{N} \cup \{0\}$, $a_{k+1} \dots a_{k+n} \ne j_1 \dots j_n$.

Let $W_{\mathbf{j},k} := \{\alpha \in I^\infty : a_{k+1} \dots a_{k+n} \ne j_1 \dots j_n\}$. Notice that $\mu_p(W_{\mathbf{j},k}) = 1 - p_{\mathbf{j}}$ and for any $s \in \mathbb{N}$, $\mu_p(W_{\mathbf{j},k} \cap W_{\mathbf{j},k+n} \cap ... \cap W_{\mathbf{j},k+sn}) = (1 - p_{\mathbf{j}})^{s+1}$.

Since J^c is the intersection of all sets $W_{\mathbf{j},k}$, $J^c \subset (W_{\mathbf{j},k} \cap ... \cap W_{\mathbf{j},k+sn})$ for any $s \in \mathbb{N}$. Hence, $\mu_p(J^c) = 0$ and $\mu_p(J) = 1$.

Throughout this paper, we consider the case where the maps S_i are the similarities and the attractor K is connected. In this case, we say K is a *self-similar continuum*.

We say that the system $\mathcal{S}$ *satisfies the one-point intersection property* if for any nonequal pieces $K_i, K_j, i, j \in I$ of order 1 of the attractor K, $\#(K_i \cap K_j) \le 1$.

2.2 Dendrites

Definition 2.5: *A dendrite is a locally connected continuum that does not contain a simple closed curve. A self-similar dendrite is a self-similar continuum, which is a dendrite.*

We shall use the notion of the order *of a point* in the sense of Menger-Urysohn (see [11, Vol.2, 51, p.274]) and denote by $Ord(p,X)$ the order of the continuum X at a point $p \in X$. If X is a dendrite, then for each point $p \in X$ the number of components of the set $X \setminus \{p\} = Ord(p,X)$ is finite whenever either of these is finite. The points of order 1 in a continuum X are called *end points* of X; A point

p of a continuum X is called a *cut point* of X if $Ord(p,X)=2$. Points of order at least 3 are called *ramification points* of X.

3. Ramification points for self-similar dendrites

First, we recall the theorem we proved in a recent paper [1]

Theorem 3.1: [1]. *Let $\mathcal{S}=\{S_1,...,S_m\}$ be a system of contracting similarities in $\mathbb{R}^d$ and let the attractor K of $\mathcal{S}$ be a dendrite. If $\mathcal{S}$ has the WSP, then there is M such that for any $x\in K$, $Ord(x,K)\le M$.*

Proof. By Zerner's Theorem [16, Th.1,(4a)], for any $a>0$ there is a number M_a such that for any $b>0$ and $x\in\mathbb{R}^d$,

$$\#\{S_{\mathbf{i}}\in F_b : S_{\mathbf{i}}(K)\cap B_{ab}(x)\neq\varnothing\}<M_a \tag{1}$$

Let $Q_1,...,Q_n$ be some finite set of connected components of $K\setminus\{x\}$. Let $\rho<\min\limits_{1\le k\le n}\operatorname{diam}(Q_k)$. For each $1\le k\le n$, take some $z_k\in\partial B(x,\rho)\cap Q_k$. There is $\mathbf{j}_k\in I^*$ such that $z_k\in K_{\mathbf{j}_k}$ and $|K_{\mathbf{j}_k}|\le\rho<|K_{\mathbf{j}_k^-}|$. Then $S_{\mathbf{j}_k}\in F_b$, where $b=\dfrac{\rho}{|K|}$.

Since K is a dendrite and $x\notin K_{\mathbf{j}_k}$, the sets $K_{\mathbf{j}_k}$ lie in Q_k and therefore are disjoint. All they have non-empty intersection with the ball B_ρ. Taking $a=|K|$ we have $ab=\rho$.

So, by (1), the number of components $n\le M_a$ for any $x\in K$.

Note that $\operatorname{Ord}(x,K)$ allows us to evaluate the upper density of the measure μ^* at the point x.

Lemma 3.2: *Let $\mathcal{S}=\{S_1,...,S_m\}$ be a system of contractive similarities in the Hilbert space X and let the attractor K of $\mathcal{S}$ be a dendrite. If $\operatorname{Ord}(x_0,K)\ge M$ for some $x_0\in K$, then $\overline{\lim\limits_{b\to0}}\dfrac{\mu^*(B(x_0,b))}{b^s}\ge M\left(\dfrac{q_{min}}{3}\right)^s$.*

Proof. Choose M components $Q_1,\dots,Q_M$ of $K\setminus\{x_0\}$. Let $\delta=\min\{|Q_k|, 1\le k\le M\}$ and take any $b<\delta$. For each k, $Q_k\cap S\left(x_0,\frac{2}{3}b\right)\neq\varnothing$. Take some $x_k\in Q_k\cap S\left(x_0,\frac{2}{3}b\right)$. There is a copy $K_{\mathbf{j}_k}$ of K that contains x_k and satisfies the inequality $\frac{b}{3}q_{min}<|K_{\mathbf{j}_k}|\le\frac{b}{3}$, which implies $\mu^*(K_{\mathbf{j}_k})>\left(\frac{b}{3}q_{min}\right)^s$.

Each of the copies $K_{\mathbf{j}_k}$ is contained in $Q_k\cap B(x_0,b)$, therefore they are disjoint. Consequently, for any $b<\delta$,

$$\frac{\mu^*(B(x_0,b))}{b^s}\ge\sum_{k=1}^{M}\frac{\mu^*(K_{\mathbf{j}_k})}{b^s}\ge M\left(\frac{q_{min}}{3}\right)^s.$$

As a direct consequence, we see that if the order of a point x_0 is infinite, then the upper density of μ^* at this point is also infinite.

Lemma 3.3: *Let $\mathcal{S}=\{S_1,...,S_m\}$ be a system of contractive similarities in the Hilbert space X and let the attractor K of $\mathcal{S}$ be a dendrite. If $Ord(x_0,K)=\infty$ for some $x_0\in K$, then $\overline{\lim\limits_{b\to0}}\dfrac{\mu^*(B(x_0,b))}{b^s}=\infty$.*

Surprisingly, the point x_0 is not a unique point that has an infinite density, because this density property is inherited by all points in the set $\pi(J)$.

Proposition 3.4: *Let $\mathcal{S}=\{S_1,...,S_m\}$ be a system of contractive similarities in the Hilbert space X and let the attractor K of $\mathcal{S}$ be a dendrite. If $Ord(x_0,K)=\infty$ for some $x_0\in K$, then for any $y\in\pi(J)$,*

$$\overline{\lim_{b\to 0}}\frac{\mu^*(B(y,b))}{b^s}=\infty.$$

Proof. For any $M>0$ there is a $\delta>0$ such that for any $b\in(0,\delta)$ the inequality $\frac{\mu^*(B(x_0,b))}{b^s}\ge M\left(\frac{q_{min}}{3}\right)^s$ holds. Fix such b and take $\mathbf{j}\in I^*$ such that $x_0\in K_{\mathbf{j}}$ and $bq_{min}<|K_{\mathbf{j}}|\le b$.

If $z\in K_{\mathbf{j}}$, then $\frac{\mu^*(B(z,2b))}{(2b)^s}\ge M\left(\frac{q_{min}}{6}\right)^s$.

If $y\in\pi(J)$, then for any $\mathbf{j}$ there is $\mathbf{i}$ such that $y\in S_{\mathbf{i}}(K_{\mathbf{j}})$ and therefore $B(y,2bq_{\mathbf{i}})\supset S_{\mathbf{i}}(B(x_0,b))$. Therefore,

$$\frac{\mu^*(B(y,2bq_{\mathbf{i}}))}{(2bq_{\mathbf{i}})^s}>M\left(\frac{q_{min}}{6}\right)^s. \tag{2}$$

We see that for any $y\in\pi(J)$ and any M there is $\delta>0$ such that inequality (2) holds for any $b<\delta$, which completes the proof.

Theorem 3.5: *Let $\mathcal{S}=\{S_1,...,S_m\}$ be a system of contractive similarities in the Hilbert space X and let the attractor K of $\mathcal{S}$ be a dendrite. If $Ord(x,K)=\infty$ for some $x\in K$, then $H^s(K)=0$.*

Proof. By Proposition 2.4, $\nu(J^c)=0$. Therefore, by the definition of μ^*, $\mu^*\pi(J)=\mu^*(K)$. Consequently, for any $y\in\pi(J)$, $\mu^*(B(y,r)\cap\pi(J))=\mu^*(B(y,r))$. It follows from Theorem 2.2, that $H^s(K)=0$.

4. An example of infinitely ramified dendrite in the Hilbert space

Let X be the Hilbert space l_2 with orthonormal base $\{e_1,e_2,\cdots\}$. A point of X is denoted by $\mathbf{x}=(x_1,x_2,...)=\sum x_k e_k$ and $\|\mathbf{x}\|=\sqrt{\sum x_k^2}$.

Denote by σ the following permutation of the set $\mathbb{N}$:

$$\sigma(n)=\begin{cases}2k+1 & \text{if} \quad n=2k-1, \quad k\in\mathbb{N}\\ 2k & \text{if} \quad n=2k+2, k\in\mathbb{N}\\ 1 & \text{if} \quad n=2\end{cases}$$

We define an orthonormal linear map O_σ by formula

$$O_\sigma(\mathbf{x})=\sum_{k=1}x_k e_{\sigma(k)}=(x_2,x_4,x_1,x_6,x_3,x_8,x_5...).$$

Let X_1 be the subspace of X which is the linear hull of the set $\{e_{2k-1},k\in\mathbb{N}\}$ and set inductively $X_{n+1}=O_\sigma(X_n)$. Thus, X_n is the linear hull of the set $\{e_{2(n+k)-3},k\in\mathbb{N}\}$. By definition, $X_k\supset X_{k+1}$ for any $k\in\mathbb{N}$.

Define the system of similarities $\mathcal{S}=\{S_0,S_1,S_2\}$ by equations

$$S_0(\mathbf{x})=1/2(\mathbf{x}+e_1),\quad S_1(\mathbf{x})=1/2(e_1-\mathbf{x}),\quad S_2(\mathbf{x})=O_\sigma(\mathbf{x}/2).$$

Since for $i=0,1,2$, $\mathrm{L}ip\, S_i=1/2$, the similarity dimension s of the system $\mathcal{S}$ is equal to $\log_2 3$.

Let K be the attractor of the system $\mathcal{S}$. The attractor K contains the unit segment $I_0=[0,e_1]$ and all its images $I_k=S_2^k(I_0)=[0,2^{-k}e_{2k+1}]$, which form an infinite countable set of pairwise orthogonal segments with common endpoint 0.

We state the following properties of the system $\mathcal{S}$ and its attractor K.

Lemma 4.1: *The system $\mathcal{S}$ satisfies the open set condition.*

Proof. We construct the open set W for the system $\mathcal{S}$ in the following.

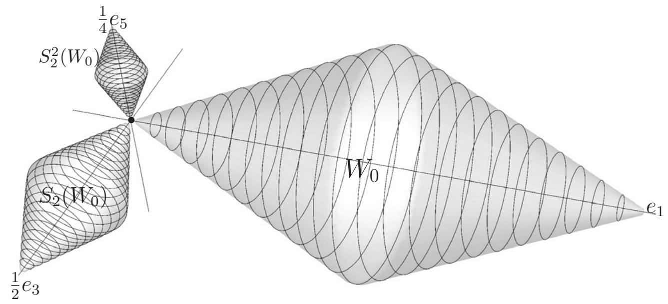

Figure 1: The projection of the set W to the subspace $Span(e_1, e_3, e_5)$.

Consider an open bicone-shaped set (see Fig.1)

$$W_0 = \{\mathbf{x} \in X : 0 < x_1 < 1, \| \mathbf{x} - x_1 e_1 \| < 1/2 - |1/2 - x_1|\}$$

and note that for any $k,l \in \mathbb{N}$, $S_2^k(\overline{W_0}) \cap S_2^l(-\overline{W_0}) = \{0\}$ and for any $k \neq l$ in $\mathbb{N}$, $S_2^k(\overline{W}_0) \cap S_2^l(\overline{W}_0) = \{0\}$. Put $W = \bigcup_{k=0}^{\infty} S_2^k(W_0)$.

It is clear that $S_2(W) \subset W$ and that $S_0(W) \cup S_1(W) \subset W_0 \subset W$.

Since $\overline{W} \cap -\overline{W} = \{0\}$, $S_0(\overline{W}) \cap S_1(\overline{W}) = \{\frac{1}{2}e_1\}$. As a consequence, $S_0(W) \cap S_2(W) = \varnothing$.

From $S_2(\overline{W}) \cap \overline{W}_0 = \{0\}$ one sees that $S_2(\overline{W}) \cap S_1(\overline{W}) = \{0\}$ and $S_2(\overline{W}) \cap S_0(\overline{W}) = \varnothing$. Hence, for any nonequal $i, j \in \{0,1,2\}$, $S_i(W) \cap S_j(W) = \varnothing$.

Although the system $\mathcal{S}$ satisfies the Open Set Condition, this does not guarantee that the Hausdorff dimension of K is equal to s because the space X is not finite dimensional.

Lemma 4.2: *The self-similar boundary of K is $\partial K = \{0, e_1\}$. K has the single intersection property, and K is a dendrite.*

Proof. From $[0, e_1] \subset K \subset \overline{W}$ it follows that $K_1 \cap K_2 = \{0\}$, $K_1 \cap K_0 = \left\{\frac{1}{2}e_1\right\}$, and $K_0 \cap K_2 = \varnothing$.

Therefore, K is connected and has the single intersection property. Moreover, the intersection graph of the system $\{K_0, K_1, K_2\}$ is a tree. By [15, Theorem 2.6], K is a dendrite.

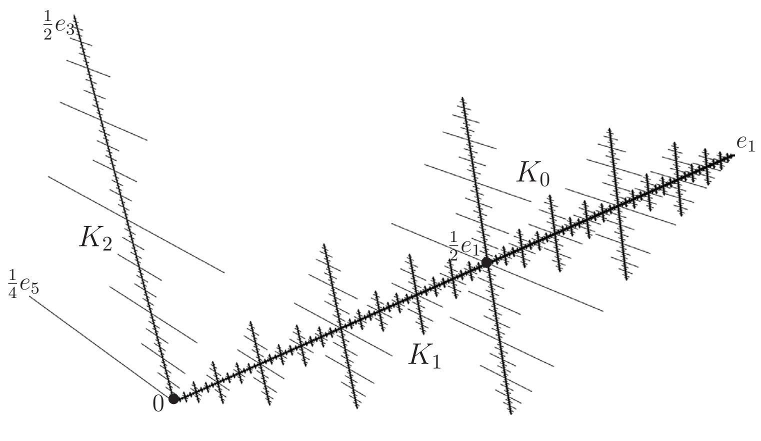

Figure 2: The projection of the set K to the subspace $Span(e_1, e_3, e_5)$.

Lemma 4.3: *The attractor K contains a ramification point of infinite order. Consequently, $H^s(K)=0$.*

Proof. Since $K_0 \cup K_1 \subset W_0$ is connected, for any $k \in \mathbb{N}$, $S_2^k(K_0 \cup K_1)$ lies in $S_2^k(W_0)$ and is connected. Therefore, the set $K \setminus \{0\}$ is a disjoint union $\coprod_{k=0}^{\infty} S_2^k(K_0 \cup K_1 \setminus \{0\})$ of connected components. Since $\mathrm{Ord}(0,K)=\infty$, by Theorem 3.5, $H^s(K)=0$.

Lemma 4.4: *For any copy $K_{i_1 \dots i_n}$ of order n the number of its neighbors of equal size is not greater than $2n$.*

Proof. The self-similar boundary of K consists of 2 points, 0 and e_1, thus, for any copy $K_{i_1 \dots i_n}$, its boundary points are $S_{i_1 \dots i_n}(0)$ and $S_{i_1 \dots i_n}(e_1)$.

Note that if $i_n = 2$, then $S_{i_1 \dots i_n}(0) = S_{i_1 \dots i_{n-1}}(0)$. If $i_n \neq 2$, $S_{i_1 \dots i_n}(0)$ is equal to $S_{i_1 \dots i_{n-1}}(e_1/2)$. Consequently, all the ramification points of K except the point 0 can be represented as $S_{i_1 \dots i_n}(e_1/2)$ for some $i_1,..,i_n$, where $i_n \neq 2$.

In the same way, if $i_n = 1$, then $S_{i_1 \dots i_n}(e_1) = S_{i_1 \dots i_{n-1}}(0)$ and if $i_n \neq 1$, then $S_{i_1 \dots i_n}(e_1)$ is an end point of K. Therefore, the only neighbor of $K_{i_1 \dots i_{n-1} 1}$ of the same size at the point $S_{i_1 \dots i_n}(0)$ is $K_{i_1 \dots i_{n-1} 0}$.

If $\partial K_{i_1 \dots i_n}$ contains the origin 0, then $S_{i_1 \dots i_n} = S_2^k S_1 S_0^l$, where $k+l+1=n$ and $k \geq 0, l \geq 0$. There are n possible choices for k and l, so $K_{i_1 \dots i_n}$ has $n-1$ neighbors of the same size at this point. The total number of neighbors of $K_{i_1 \dots i_n}$ in this case will be n.

If $\partial K_{i_1 \dots i_n}$ contains the point $e_1/2$, then $S_{i_1 \dots i_n} = S_0 S_2^k S_1 S_0^l$ or $S_1 S_2^k S_1 S_0^l$, where $k+l+2=n$ and $k \geq 0, l \geq 0$. There are $n-1$ possible choices of k and l, so $K_{i_1 \dots i_n}$ has $2n-1$ neighbors of the same size at the point $e_1/2$. The total number of neighbors of $K_{i_1 \dots i_n}$ in this case will be $2n$.

Consider a copy $S_{\mathbf{j}}(K_{\mathbf{i}})$, where $\mathbf{i} = i_1 \dots i_{n-k}$, $\mathbf{j} = j_1 \dots j_k$. Its order is n. Its self-similar boundary is $\{S_{\mathbf{j}}(0), S_{\mathbf{j}}(e_1/2)\}$, and the number of its neighbors is equal to $2(n-k)$ which is less or equal than $2n$.

Lemma 4.5: *If $K_{i_1 \dots i_n} \cap K_{j_1 \dots j_n} = \varnothing$, then*

$$\delta(K_{i_1 \dots i_n}, K_{j_1 \dots j_n}) := \inf\{d(x,y) : x \in K_{i_1 \dots i_n}, y \in K_{j_1 \dots j_n}\} \geq 2^{-n}. \tag{3}$$

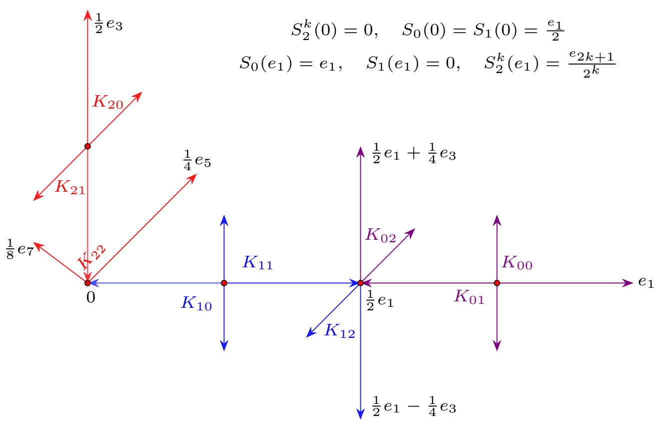

Figure 3: The copies K_{ij} of order 2 of the set K.

Proof. This is clear for $n=1$, because $\delta(K_0, K_2) = 1/2$.

In the case where $i_1 \dots i_{n-1} = j_1 \dots j_{n-1}$, $K_{i_1 \dots i_{n-1} i_n} \cap K_{i_1 \dots i_{n-1} j_n} = \varnothing$ implies $\{i_n, j_n\} = \{0, 2\}$. Therefore, $\delta(K_{i_1 \dots i_{n-1} 0}, K_{i_1 \dots i_{n-1} 2}) = 2^{-n}$.

If $K_{i_1 \dots i_{n-1}} \cap K_{j_1 \dots j_{n-1}} = \varnothing$, then $\delta(K_{i_1 \dots i_n}, K_{j_1 \dots j_n}) \geq 2^{1-n}$.

If $K_{i_1 \dots i_{n-1}}$ and $K_{j_1 \dots j_{n-1}}$ have a common point $y=0$ or $y = S_k\left(\frac{e_1}{2}\right)$, then one of the copies, say $K_{\mathbf{i}}$ does not contain y, therefore $\delta(K_{\mathbf{i}}, y) = 2^{-n}$, which implies (3).

Lemma 4.6: *For any* $x \in K$, $\mu^*(B(x, 2^{-n})) \leq \frac{2n+1}{3^n}$.

Proof. There is a copy $K_{i_1 \dots i_n}$ that contains x. If such a copy is unique, then the ball $B(x, 2^{-n})$ intersects at most $2n+1$ copies of order n, which implies $\mu^*(B(x, 2^{-n})) \leq \frac{2n+1}{3^n}$. If x is a common point of several copies of order n then the open ball $B(x, 2^{-n})$ intersects at most $2n$ copies of order n, and $\mu^*(B(x, 2^{-n})) \leq \frac{2n}{3^n}$.

Lemma 4.7: $\dim_H(K) = s$.

Proof. If $2^{-n-1} < r < 2^{-n}$, then $\mu^*(B(x,r)) < \mu^*(B(x, 2^{-n})) \leq \frac{2n+1}{3^n}$.

Combining these two inequalities, we obtain $\frac{\mu^*(B(x,r))}{r^s} < 3(1 - 2\log_2 r)$. Therefore, for any $t < s$, $\lim\limits_{r \to 0} \frac{\mu^*(B(x,r))}{r^t} \leq \lim\limits_{r \to 0} 3(1 - 2\log_2 r) r^{s-t} = 0$.

By Theorem 2.2, this means that for any $t < s$, $H_t(K) > \mu^*(K) = 1$, we therefore have $H_t(K) = \infty$ for any $t < s$. Consequently, $\dim_H(K) = s$.

5 Conclusion

This study is devoted to self-similar dendrites in the $\mathbb{R}^d$ and Hilbert space. It has already been proved that for any self-similar dendrite K in $\mathbb{R}^d$ that satisfies the weak separation property, the ramification order of K is finite. This paper presents a new theorem for self-similar dendrites in a Hilbert space. This theorem is as follows: if a self-similar dendrite K has a ramification point of infinite order, then the s-dimensional Hausdorff measure K is zero, where s is the similarity dimension.

Currently, no examples of infinitely ramified self-similar dendrites are known. We have constructed an infinitely ramified self-similar dendrite K in a Hilbert space that satisfies the Open Set Condition. In this case, as proved in the article, its Hausdorff measure $H^s(K) = 0$. Moreover, the Hausdorff dimension is equal to the similarity dimension. This result is obtained by applying the inverse Frostman type theorem for Hausdorff Measures in metric spaces.

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Conflicts of interest

The authors declare no conflicts of interest.

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