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Utilizing various statistical techniques to estimate the scale parameter for the Rayleigh distribution

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Abstract

The purpose of this paper is to use statistical estimating techniques to estimate the scale parameter of the Rayleigh distribution (maximum likelihood, rank set sampling, Bayes, and robust Bayes estimators) under complete data. A non-Bayes estimator is obtained by rank set sampling and maximum likelihood. The inverted gamma distribution and the inverted exponential distribution based on symmetric unbalanced (squared error) and balanced (quadratic) loss functions are used as Bayesian estimators. Robust Bayes analysis for the Rayleigh distribution depends on (unbalanced and balanced) loss functions based on ML-II-ϵ-contaminated class and derived under the prior contaminant distribution of the Frechet distribution. The estimators' performances are contrasted according to simulation experiments for different cases and sample sizes depending on the value for the mean squared error.

Mathematics Subject Classification: 62F10, 62F15, 62P30.

Key words and phrases: Robust Bayes, Bayes, Rank set, Rayleigh distribution, Frechet distribution.

1. Introduction

The Rayleigh distribution (RD) is credited with naming to John Baron Rayleigh (1842–1919) who introduced it in 1880 about the interference of random phase harmonic oscillations in a communication channel [1]. Furthermore, The Weibull-Rayleigh distribution, also known as the RD, is derived from the amplitude of sound that results from numerous significant sources. Life-testing experiments,

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reliability analysis, applied statistics, and clinical research are just a few of the many uses for the RD with the shape parameter equal to (2), this distribution is a special instance of the two-parameter Weibull distribution [2]. Klivans et al. (2018) developed robust learning algorithms that succeed on a data set contaminated with adversarial corrupted outliers [3]. Al-Ani B. G. et al. (2019) introduced a robust Bayesian method to estimate the reliability function When the shape parameter β is known under the quadratic loss function, they use the initial distribution of the parameter θ with class (ML-II-∈-Contaminated) and discover that the data follows the two-parameter Weibull distribution so that the prior distribution of the scale parameter is a Frechet distribution (FD) [4]. Slob and Burgess (2020) studied Tukey's loss function combined with MM estimation to provide robustness against influential points [5]. Hussein et al. (2023) considered the exponential RD can be extracted mathematically by combining the exponential distribution's cumulative distribution function with the RD's cumulative distribution function [6]. Abraheem S.K. et.al. (2024) used an approximate method to find the reliability function for inverse RD and compared it with two statistical estimation methods. In the approximate method, the reliability function was expanded by using Bernstein polynomials to find the approximate value [7].

This paper focuses on: deriving and estimating the unknown scale parameter of RD on complete data, using two methods maximum likelihood and rank set sampling as the traditional methods, and the Bayes method by assuming informative priors represented by (inverted gamma distribution and inverted exponential distribution) under symmetric loss function unbalanced (squared error) and balanced (quadratic) loss functions. As well as, robust Bayesian estimation using the initial distribution of the scale parameter θ with class ML-II-ϵ-contaminated at the prior contaminant distribution for the FD under unbalanced and balanced loss functions. Finally, finding the best estimator for the scale parameter θ of RD by comparing the performance of those estimators using the Monte-Carlo experiment was performed under a wide range of cases and sample sizes.

2. Rayleigh Distribution

One of the most popular distributions is the RD, which is a continuous probability family with a single scale parameter. The probability density function (p.d.f.) of a continuous random variable that is not negative of RD with scale parameter θ is given by [8]:

$$f(t;\theta)_R = \frac{2}{\theta} t e^{\frac{-t^2}{\theta}} \quad ; t \geq 0, \theta > 0 \tag{1}$$

and otherwise, it equals zero.

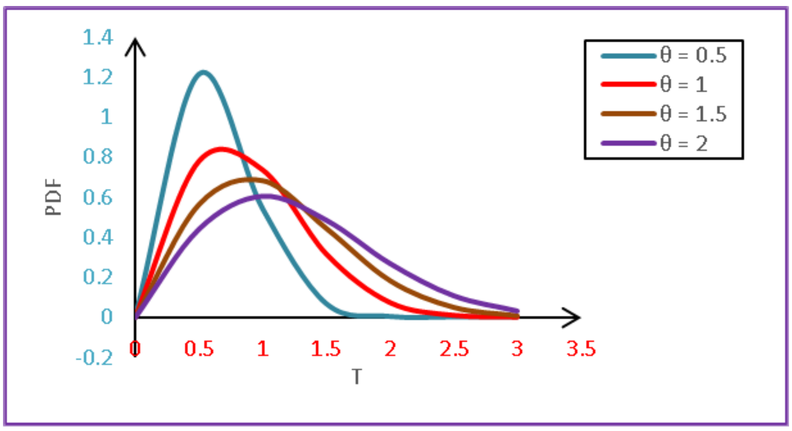

Figure 1: A plot of the p.d.f. for RD using various values θ.

The function of cumulative distribution (CDF) for the RD is:

$$F(t;\theta)_R = 1 - e^{\frac{-t^2}{\theta}} \quad ; t \geq 0, \theta > 0 \tag{2}$$

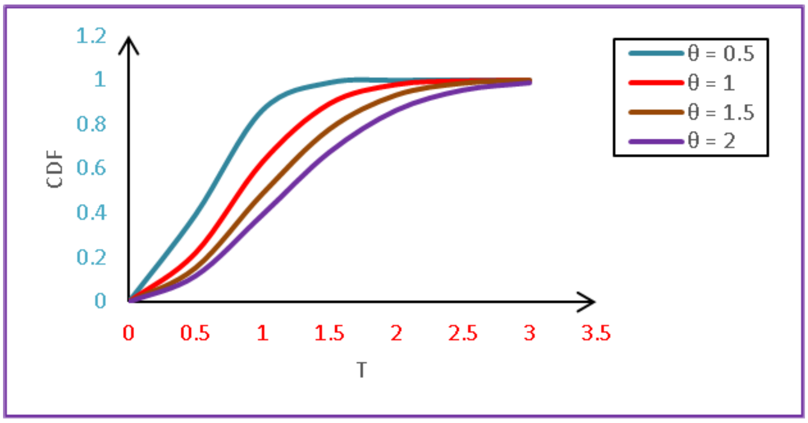

Figure 2: A plot of the CDF for the RD using various values θ.

The reliability function and the failure rate function for the RD as follows:

$$R(t)_R = 1 - F(t;\theta) = e^{\frac{-t^2}{\theta}} \quad ; t \geq 0, \theta > 0 \tag{3}$$

$$h(t) = \frac{2}{\theta} t \; ; t \geq 0, \theta > 0 \tag{4}$$

3. Traditional Estimation Methods

The methods of maximum likelihood (ML) and rank set sampling (RSS) estimations are considered to estimate the parameter of RD with a complete sample.

3.1. Method of Estimation Maximum Likelihood

One such well-known technique is ML estimation, first presented by statistician Fisher in 1992. This approach aims to identify parameter estimators that maximize the likelihood function related to the observed data.

To find the estimate of the parameter, make the likelihood function's derivative for the parameter to be estimated as zero, and what distinguished the ML estimation has the property of non-variability (invariant), as well as the ML estimator have many properties that make it an appealing choice of estimator. It gives estimators that are often unbiased, least variance and sufficient [9].

The ML estimate ($\hat{\theta}$), where the likelihood function under complete samples $(t_1, t_2, \ldots, t_n)$ with size n from the RD as follows [10][11]:

$$L(\theta \mid \underline{t})_{RD}^{MLE} = \prod_{i=1}^{n} f(t_i;\theta)_R = \prod_{i=1}^{n} \left[\frac{2}{\theta} t_i \, e^{\frac{-t_i^2}{\theta}} \right] = \left(\frac{2}{\theta} \right)^n \prod_{i=1}^{n} (t_i) e^{\frac{-\sum_{i=1}^{n} t_i^2}{\theta}} \tag{5}$$

The values for a parameter can be found by [9] as follows:

$$\hat{\theta}_{MLE(RD)} = \frac{\sum_{i=1}^{n} t_i^2}{n} \tag{6}$$

3.2. Estimation Method of Rank Set Sampling

In order to estimate mean pasture yields more efficiently than simple random sampling, McIntyre (1952) proposed a sampling technique. Because only a small percentage of the randomly chosen units from the population are quantified after a preliminary ranking, this sampling technique has become known as RSS [12].

In term of estimation, the RD parameter utilizing RSS method, a procedure will go like this: When increasing ordering random sampling $\left(t_{(1)},t_{(2)},\ldots,t_{(n)}\right)$ is obtained the p.d.f. for the RD as [13]:

$$f\left(t_{(j)}\right) = = \frac{n!}{(j-1)!(n-j)!}\left[F\left(t_{(j)}\right)\right]^{j-1}\left[1-F\left(t_{(j)}\right)\right]^{n-j} f\left(t_{(j)}\right),\ j=1,\ldots,n \tag{7}$$

Put $\frac{n!}{(j-1)!(n-j)!} = B$ where $j=1,\ldots,n$

By substitution of equations (1) and (2) in equation (7) and with simplifying, obtains:

$$f(t_{(j)})_{RD} = B\frac{2}{\theta}t_{(j)}\left[1-e^{\frac{-t^2_{(j)}}{\theta}}\right]^{j-1}\left[e^{\frac{-t^2_{(j)}}{\theta}}\right]^{n-j+1} \quad j=1,\ldots,n \tag{8}$$

The complete data likelihood function $L(\theta \mid \underline{t})^{RSS}_{RD}$ can be expressed by:

$$L(\theta \mid \underline{t})^{RSS}_{RD} = B^n\left(\frac{2}{\theta}\right)^n\prod_{j=1}^{n}t_{(j)}\left[1-e^{\frac{-t^2_{(j)}}{\theta}}\right]^{j-1}\prod_{j=1}^{n}\left[e^{\frac{-t^2_{(j)}}{\theta}}\right]^{n-j+1} \tag{9}$$

Let the natural log for the both sides of equation (9), gets:

$$\ell^{RSS}_{RD} = \ln L\ (\theta \mid \underline{t})^{RSS}_{RD} = n\ln B + n\ln 2 - n\ln\theta + \sum\nolimits_{j=1}^{n}\ln t_{(j)} + \sum\nolimits_{j=1}^{n}(j-1)\left[1-e^{\frac{-t^2_{(j)}}{\theta}}\right] - \sum\nolimits_{j=1}^{n}(n-j+1)\frac{t^2_{(j)}}{\theta}$$

Derived the above equation concerning unknown parameter θ and equal it to zero then simplifying, yields:

$$\frac{-n}{\theta} - \sum\nolimits_{j=1}^{n}\left(\frac{t_{(j)}}{\theta}\right)^2\left[(j-1)\frac{1}{e^{\frac{-t^2_{(j)}}{\theta}}-1} - (n-j+1)\right] = 0 \tag{10}$$

This can be obtained by the solution of equation (10) since this is no closed solution of this equation. Therefore, one iterative method is the Newton-Raphson method, which will be applied to determine the RSS estimator denoted by $\hat{\theta}^{RSS}_{RD}$.

4. Bayesian Analysis

In recent years, Bayesian analysis has gained popularity as a framework for practical problems. A crucial component of Bayesian inference is the prior distribution, which represents the information of an uncertain parameter θ and it is combining with the probability distribution of fresh data to produce the posterior distribution. There are two types of prior distribution depending on how much primary information (informative and non-informative priors).

Calculating the Bayes estimators of the RD with a single parameter θ involves multiple steps. Therefore, we need to be aware of the prior and posterior distributions are obtained as:

$$\text{Posterior distribution} = \frac{\text{prior distribution * likelinood function}}{\text{marginal distribution}}$$

The conditional p.d.f. for scale parameter θ, with the random variables $t_1, t_2, \ldots, t_n$ is:

$$\pi(\theta \mid \underline{t}) = \frac{L(\theta \mid \underline{t}) g(\theta)}{\int_\theta L(\theta \mid \underline{t}) g(\theta) d\theta} \tag{11}$$

where

$L(\theta \mid \underline{t})$: likelinood function for sample $t_1, t_2, \ldots, t_n$.

$g(\theta)$:prior distribution.

$\pi(\theta \mid \underline{t})$:posterior distribution.

In this section, two informative prior distributions (inverted gamma and inverted exponentially) have been assumed in order to obtain the posterior distribution for the parameter θ. Two loss functions are taken into consideration: the balanced quadratic loss function (BQLF) and the unbalanced squared error loss function (SELF).

The risk function is a mathematical exception to the loss function, and the standard Bayes estimator $(\hat{\theta}_B)$ for the parameter θ is the value that makes this risk function known as minimal as possible. $\hat{\theta}_B$ Calculated as:

$$\hat{\theta}_B = Risk(\hat{\theta}, \theta) = E\left[L(\hat{\theta}, \theta)\right] = \int_{\forall\theta} L(\hat{\theta}, \theta) \pi(\theta \mid \underline{t}) d\theta \tag{12}$$

4.1. Posterior Density Function Utilizing Invers Gamma Prior

The inverted gamma distribution is the most commonly used prior distribution of the parameter θ, denoted as $I_n \mathrm{Ga}(\alpha, \beta)$ which has the following from of the p.d.f. [14]:

$$g_1(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{1}{\theta}\right)^{\alpha+1} e^{\frac{-\beta}{\theta}} \; ; \qquad \theta > 0, \alpha, \beta > 0 \tag{13}$$

To determine the first posterior density function for the scale parameter θ for the RD, substituting equations (5) and (13) in equation (11), gets:

$$\pi_1(\theta \mid \underline{t}) = \frac{2^n \theta^{-n} \prod_{i=1}^n (t_i) e^{\frac{-\sum_{i=1}^n t_i^2}{\theta}} \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{-\alpha-1} e^{\frac{-\beta}{\theta}}}{\int_0^\infty 2^n \theta^{-n} \prod_{i=1}^n (t_i) e^{\frac{-\sum_{i=1}^n t_i^2}{\theta}} \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{-\alpha-1} e^{\frac{-\beta}{\theta}} d\theta}$$

$$\pi_1(\theta \mid \underline{t}) = \frac{\theta^{-(n+\alpha+1)} e^{\frac{-1}{\theta}(\sum_{i=1}^n t_i^2 + \beta)}}{\int_0^\infty \theta^{-(n+\alpha+1)} e^{\frac{-1}{\theta}(\sum_{i=1}^n t_i^2 + \beta)} d\theta}$$

By using the transformation, $y = \frac{1}{\theta}(\sum_{i=1}^n t_i^2 + \beta)$ which implies that $\theta = \frac{(\sum_{i=1}^n t_i^2 + \beta)}{y}$ and $d\theta = \frac{-(\sum_{i=1}^n t_i^2 + \beta)}{y^2} dy$, the posterior distribution of θ, becomes:

$$\pi_1(\theta \mid \underline{t}) = = \frac{-(\sum_{i=1}^{n} t_i^2 + \beta)^{n+\alpha} \; e^{\frac{-1}{\theta}(\sum_{i=1}^{n} t_i^2 + \beta)}}{\theta^{n+\alpha+1}\,\Gamma(n+\alpha)} \tag{14}$$

Equation (14) is similar to invers gamma distribution IG (α_1, β_1) where $\alpha_1 n + \alpha$ and $\beta_1 = (\sum_{i=1}^{n} t_i^2 + \beta)$

4.2. Posterior Density Function Using Invers Exponential Prior

The second prior distribution is an inverted exponential distribution of hyper-parameter c provided by [15]:

$$g_2 = \frac{c}{\theta^2}\, e^{\frac{-c}{\theta}} \quad ; \qquad \theta > 0 \;,\; c > 0 \tag{15}$$

Substitute equations (5) and (15) in equation (11), yields the following second posterior density function of θ:

$$\pi_2(\theta \mid \underline{t}) = \frac{2^n\, \theta^{-n} \prod_{i=1}^{n} (t_i) e^{\frac{-\sum_{i=1}^{n} t_i^2}{\theta}} \; \frac{c}{\theta^2}\, e^{\frac{-c}{\theta}}}{\int_0^\infty 2^n\, \theta^{-n} \prod_{i=1}^{n} (t_i) e^{\frac{-\sum_{i=1}^{n} t_i^2}{\theta}} \; \frac{c}{\theta^2}\, e^{\frac{-c}{\theta}}\, d\theta}$$

$$\pi_2(\theta \mid \underline{t}) = \frac{\theta^{-n-2}\, e^{\frac{-1}{\theta}\left(\sum_{i=1}^{n} t_i^2 + c\right)}}{\int_0^\infty \theta^{-n-2}\, e^{\frac{-1}{\theta}\left(\sum_{i=1}^{n} t_i^2 + c\right)}\, d\theta}$$

By using the transformation, $y = \frac{1}{\theta}(\sum_{i=1}^{n} t_i^2 + c)$ which implies that $\theta = \frac{(\sum_{i=1}^{n} t_i^2 + c)}{y}$ and $d\theta = \frac{-1}{y^2}(\sum_{i=1}^{n} t_i^2 + c)dy$, the posterior distribution of θ, becomes:

$$\pi_2(\theta \mid \underline{t}) = -\frac{(\sum_{i=1}^{n} t_i^2 + c)^{n+1} \; e^{\frac{-1}{\theta}(\sum_{i=1}^{n} t_i^2 + c)}}{\theta^{n+2}\,\Gamma(n+1)} \tag{16}$$

Equation (16) is similar to invers gamma distribution IG(α_2, β_2) where $\alpha_2 = n+1$ and $\beta_2 = (\sum_{i=1}^{n} t_i^2 + c)$.

5. Loss Functions

The amount of loss resulting from a Bayes decision around an unknown parameter is known as the loss function, and it is one of the metrics used to assess accuracy in the Bayesian estimation process. It measures the difference between this parameter's estimated and actual values ($\hat{\theta} - \theta$. It ought to have a real, non-negative value, and typically symbolized by $L(\hat{\theta}, \theta)$ [16]. It is defined that the loss function $L(\hat{\theta}, \theta)$ is a real-valued function that satisfies [17]:

(i) $L(\hat{\theta}, \theta) \geq 0 \;\; ; \forall \hat{\theta} \neq \theta$

(ii) $L(\hat{\theta}, \theta) = 0 \;\; ; \forall \hat{\theta} = \theta$

Equation (12) can be used to calculate the risk function, which is the mathematical exception to the loss function.

Based on two loss functions: the balanced loss function (BQLF) and the unbalanced loss function (SELF) as symmetric loss functions, Bayesian estimators are obtained [16].

- The SELF for θ is defined as [18]:

$$L\left(\hat{\theta},\theta\right)=(\hat{\theta}-\theta)^2 \tag{17}$$

According to equation (12) and by taking the derivative of equation (17) for $\hat{\theta}$ with setting it to zero and under unbalanced loss function the Bayes estimator of θ symbolized by $\hat{\theta}_{BS}$ then:

$$\hat{\theta}_{BS}=E_{\pi}(\theta|\underline{t}\ ) \tag{18}$$

- The balanced loss function in accordance with Zellner's formula [16]:

$$L_w\left(\hat{\theta},\theta\right)=WL\left(\hat{\theta},\theta_0\right)+\left(1-W\right)L\left(\hat{\theta},\theta\right) \tag{19}$$

where

$L_w\left(\hat{\theta},\theta\right)$: The balanced loss function.

W: weighted coefficient, $w\in\left(0,1\right)$.

θ_0: Primary estimator for the parameter (θ) depends on the observations.

$L\left(\hat{\theta},\theta\right)$: The unbalanced loss function.

$L\left(\hat{\theta},\theta_0\right)$: The unbalanced loss function for the likelihood function.

The Bayes estimators can be done by [16]:

$$\hat{\theta}_{BBS}=w\hat{\theta}_{ML}+\left(1-w\right)E_{\pi}(\theta|\underline{t}\ ) \tag{20}$$

- Now determine the Bayes estimators under the unbalanced loss function as the following:

(i) Under the inverse gamma prior distribution using equation (14), the Bayes estimators of θ under the unbalanced loss function according to $\pi_1\ (\theta\mid\underline{t})$ can be found as:

$$E_{\pi_1\ (\theta|\underline{t})}=\int_{\theta}\theta\,\pi_1\ (\theta\mid\underline{t})d\theta\ =\int_0^{\infty}\theta\,\frac{-(\ \sum_{i=1}^{n}t_i^2+\beta)^{n+\alpha}\ \ e^{\frac{-1}{\theta}(\sum_{i=1}^{n}t_i^2+\beta\ )}}{\theta^{n+\alpha+1}\ \Gamma\left(n+\alpha\right)}d\theta$$

Then

$$E_{\pi_1\ (\theta|\underline{t})}=\frac{-(\ \sum_{i=1}^{n}t_i^2+\beta)^{n+\alpha}}{\Gamma\left(n+\alpha\right)}\int_0^{\infty}\theta^{-n-\alpha}\ \ e^{\frac{-1}{\theta}(\sum_{i=1}^{n}t_i^2+\beta\ )}d\theta$$

By using the transformation, $y=\frac{1}{\theta}(\sum_{i=1}^{n}t_i^2+\beta\ )$, which implies that $\theta=\frac{1}{y}\ (\sum_{i=1}^{n}t_i^2+\beta\ )$, and $d\theta=\frac{-1}{y^2}\ (\sum_{i=1}^{n}t_i^2+\beta\ )dy$, gets:

$$E_{\pi_1\ (\theta|\underline{t})}=\frac{\sum_{i=1}^{n}t_i^2+\beta}{n+\alpha-1} \tag{21}$$

The Bayes estimators of θ under the unbalanced loss function based on the assumption of inverse gamma prior information using equation (18) are given as:

$$\hat{\theta}_{Bsig} = \frac{\sum_{i=1}^{n} t_i^2 + \beta}{n + \alpha - 1} \tag{22}$$

(ii)Under the invers exponential prior distribution using equation (16), the Bayes estimators for the θ under the unbalanced loss function according to $\pi_2\ (\theta \mid \underline{t})$ can be found as:

$$E_{\pi_2\ (\theta|\underline{t})} = \int_{\theta} \theta\, \pi_2\ (\theta \mid \underline{t}) d\theta = \int_0^{\infty} \theta \frac{-\left(\sum_{i=1}^{n} t_i^2 + c\right)^{n+1} e^{\frac{-1}{\theta}\left(\sum_{i=1}^{n} t_i^2 + c\right)}}{\theta^{n+2}\, \Gamma(n+1)} d\theta$$

Then

$$E_{\pi_2\ (\theta|\underline{t})} = \frac{-(\sum_{i=1}^{n} t_i^2 + c)^{n+1}}{\Gamma(n+1)} \int_0^{\infty} \theta^{-n-1}\ e^{\frac{-1}{\theta}(\sum_{i=1}^{n} t_i^2 + c)} d\theta$$

By using the transformation, $y = \frac{1}{\theta}(\sum_{i=1}^{n} t_i^2 + c)$ which implies that $\theta = \frac{1}{y}(\sum_{i=1}^{n} t_i^2 + c)$ and $d\theta = \frac{-1}{y^2}(\sum_{i=1}^{n} t_i^2 + c)dy$, gets:

$$E_{\pi_2\ (\theta|\underline{t})} = \frac{\sum_{i=1}^{n} t_i^2 + c}{n} \tag{23}$$

The Bayes estimators for the θ under the unbalanced loss function based on the assumption of inverse exponential prior information using equation (18) are given as:

$$\hat{\theta}_{Bsie} = \frac{\sum_{i=1}^{n} t_i^2 + c}{n} \tag{24}$$

• Bayes estimator under BQLF for RD corresponding to different posterior distribution.

(i) The Bayes estimators for the θ under BQLF based on the assumption of inverse gamma prior information using equations (20) and (21) are given as:

$$\hat{\theta}_{BBsig} = W\hat{\theta}_{ML} + \frac{(1-W)\ (\sum_{i=1}^{n} t_i^2 + \beta\ )}{(n + \alpha - 1)} \tag{25}$$

where $\hat{\theta}_{ML} = \frac{\sum_{i=1}^{n} t_i^2}{n}$

(ii) The Bayes estimators for the θ under the BQLF under the assumption of inverse exponential prior information using equations (20) and (23) are given as:

$$\hat{\theta}_{BBsie} = W\hat{\theta}_{ML} + \frac{(1-W)\ (\sum_{i=1}^{n} t_i^2 + c\ )}{n} \tag{26}$$

where $\hat{\theta}_{ML} = \frac{\sum_{i=1}^{n} t_i^2}{n}$

6. Robust Bayesian Modeling

This section contains: develops a general approach to robust Bayesian modeling, shows how to turn an existing Bayesian model into a robust one, and creates a general computational strategy for it. Our approach uses the initial distribution of the parameter θ with the class (ML-Π-ϵ-contaminated) to study the robust Bayes approach at the prior contaminant distribution are the FD and the main distribution is RD under (unbalanced and balanced) loss functions [4].

A type-II extreme value distribution (Frechet) case is similar to taking the reciprocal of values from a standard Weibull distribution, and the FD is a special case of the generalized extreme value distribution. The p.d.f. and the CDF for FD as [19,20]:

$$f\left(t;\alpha,\theta\right)=\alpha\theta t^{-(\alpha+1)}\exp\left(-\theta t^{-\alpha}\right) \quad ; t>0\ ,\theta>0 \tag{27}$$

$$F\left(t;\alpha,\theta\right)=e^{-\theta t^{-\alpha}} \quad ; \quad t>0 \tag{28}$$

where $\alpha>0$ determine the shape parameter and the scale parameter is $\theta>0$ for the distribution.

Robust Bayesian analysis of the RD based on the ϵ-contamination class of priors of the unknown scale parameter θ and known shape parameter α is considered, and under SELF and BQLF the ML-II Bayes estimators for the parameters were derived. A lot of research has been done on Bayesian analysis under the ϵ-contamination class of priors [21]. The ϵ-contamination class of prior distribution for θ is:

$$\Gamma=\left\{\pi\left(\theta\right):q\left(\theta\right)=\left(1-\in\right)q_0\left(\theta\right)+\in q\left(\theta\right),q\left(\theta\right)\in Q\right\} \tag{29}$$

where $\varepsilon\left(0<\varepsilon<1\right)$ is pre-assigned and denotes the probability of error in the prior $q_0\left(\theta\right)$, elicitation, we take into consideration the base prior, a natural conjugate prior, provided by [21]:

$$q_{F_0}(\theta\mid\sigma_0)=\frac{\sigma_0}{\theta^2}\exp\left(\frac{-\sigma_0}{\theta}\right) \quad ; \ \sigma_0>0\ ,\theta>0 \tag{30}$$

where $\left(\sigma_0,\theta\right)$ represent the vector of hyper-parameters.

The class of all-natural conjugate priors with the vector of hyper-parameter $\left(\sigma,\theta\right)$ is known as the contamination class $q(\theta\mid\sigma)$, which is defined as:

$$q_F\left(\theta\ ,\sigma\right)=\frac{\sigma}{\theta^2}\exp\left(\frac{-\sigma}{\theta}\right) \quad ; \quad \sigma>0\ ,\theta>0 \tag{31}$$

According to the prior prediction $q(\theta\mid\sigma)$, the predictive density is:

$$M_{Frechaet\ R}\left(\underline{t}\mid q\right)=\int_0^\infty L\left(\underline{t}\mid q\right)q(\theta\mid\sigma\ )d\theta \tag{32}$$

Using equations (5) and (31), then from above equation, obtains:

$$\begin{aligned}M_{Frechaet\ R}\left(\underline{t}\mid q\right)&=\int_0^\infty\frac{2^n}{\theta^n}\left(\prod_{i=1}^n t_i\right)e^{\frac{-\sum_{i=1}^n t_i^2}{\theta}}\ \frac{\sigma}{\theta^2}\ e^{\frac{-\sigma}{\theta}}\,d\theta\\&=2^n\left(\prod_{i=1}^n t_i\right)\sigma\int_0^\infty\theta^{-n-2}e^{\frac{-\left(\sum_{i=1}^n t_i^2+\sigma\right)}{\theta}}\,d\theta\end{aligned}$$

Using the transformation, $y=\frac{\sum_{i=1}^{n}t_i^2+\sigma}{\theta}$ which implies that $\theta=\frac{\sum_{i=1}^{n}t_i^2+\sigma}{y}$ and $d\theta=\frac{-\left(\sum_{i=1}^{n}t_i^2+\sigma\right)}{y^2}dy$, gets:

$$\Rightarrow\quad 2^n\left(\prod_{i=1}^{n}t_i\right)\sigma\int_0^\infty\frac{-(\sum_{i=1}^{n}t_i^2+\sigma\ )^{-n-2}}{y^{-n-2}}e^{-y}\left(\frac{-\left(\sum_{i=1}^{n}t_i^2+\sigma\right)}{y^2}\right)dy$$

$$\Rightarrow\quad -2^n\left(\prod_{i=1}^{n}t_i\right)\sigma\left(\sum_{i=1}^{n}t_i^2+\sigma\right)^{-n-1}\int_0^\infty y^n\,e^{-y}dy$$

Then

$$M_{Frechaet\,R}\left(\underline{t}\mid q\right)=-2^n\left(\prod_{i=1}^{n}t_i\right)\Gamma\left(n+1\right)\sigma\left(\sum_{i=1}^{n}t_i^2+\sigma\right)^{-n-1} \tag{33}$$

Now maximized $M_{Frechaet\,R}\left(\underline{t}\mid q\right)$, substitute's σ at its ML estimator, which is provided by:

$$\begin{aligned}\frac{dM_{Frechaet\,R}\left(\underline{t}\mid q\right)}{d\sigma}&=-2^n\left(\prod_{i=1}^{n}t_i\right)\Gamma\left(n+1\right)\left[\sigma\left(-n-1\right)(\sum_{i=1}^{n}t_i^2+\sigma\ )^{-n-2}+(\sum_{i=1}^{n}t_i^2+\sigma)^{-n-1}\right]\\&=-2^n\left(\prod_{i=1}^{n}t_i\right)\Gamma\left(n+1\right)\left(\sum_{i=1}^{n}t_i^2+\sigma\right)^{-(n+2)}\left[\sigma\left(-n-1\right)+\left(\sum_{i=1}^{n}t_i^2+\sigma\right)\right]\end{aligned} \tag{34}$$

The following formula is obtained by equating the partial derivative to zero:

$$\hat{\sigma}=\frac{\sum_{i=1}^{n}t_i^{\,2}}{n}=\hat{\theta}_{ML} \tag{35}$$

Put equation (35) in equation (31). Then we have:

$$q_{FR}\left(\theta\vert\,\hat{\sigma}\right)=\begin{cases}\frac{\sum_{i=1}^{n}t_i^{\,2}}{n\theta^2}\exp\left(\frac{-\sum_{i=1}^{n}t_i^{\,2}}{n\theta}\right) & if\,\sigma_0<\hat{\sigma}\\ q_{FR_0}(\theta\vert\,\sigma_0) & if\,\sigma_0\geq\hat{\sigma}\end{cases}$$

Thus, ML-II posterior density is found by:

$$\hat{\pi}_{FR}\left(\theta\right)=\left(1-\varepsilon\right)q_{FR_0}(\theta\vert\,\sigma_0)+\varepsilon\,q_{FR}\left(\theta\vert\,\hat{\sigma}\right) \tag{36}$$

From [21], the ML-II posterior of θ is obtained as:

$$\hat{\pi}^*_{FR}\left(\theta\right)=\hat{\lambda}q^*{}_{FR_0}\left(\theta\vert\,\sigma_0\right)+\left(1-\hat{\lambda}\right)q^*{}_{FR}\left(\theta\vert\,\hat{\sigma}\right)\quad;\quad 0<\theta<\infty$$

and the ML-II posterior mean of θ is given:

$$E\left(\hat{\pi}^*_{FR}\left(\theta\right)\right)=\hat{\lambda}E\left(q^*{}_{FR_0}\left(\theta\vert\,\sigma_0\right)\right)+\left(1-\hat{\lambda}\right)E(q^*{}_{FR}\left(\theta\vert\,\hat{\sigma}\right)\quad;\quad 0<\theta<\infty \tag{37}$$

where

$$\hat{\lambda}=\frac{\left(1-\varepsilon\right)M_{Frechaet\,R}\left(\underline{t}\vert\,q_0\left(\theta\right)\right)}{\left(1-\varepsilon\right)M_{Frechaet\,R}\left(\underline{t}\vert\,q_0\left(\theta\right)\right)+\varepsilon\,M_{Frechaet\,R}\left(\underline{t}\vert\,\hat{q}\left(\theta\right)\right)}\ \text{and}\ q^*_{FR}(\theta\vert\,\sigma)=\frac{L\left(\theta\right)q\left(\theta\vert\sigma\right)}{M(t\vert q\left(\theta\right))}$$

$$\hat{\lambda} = \begin{cases} \left[1 + \dfrac{\varepsilon M_{Frechaet\,R}\left(\underline{t} | \hat{q}(\theta)\right)}{(1-\varepsilon) M_{Frechaet\,R}\left(\underline{t} | q_0(\theta)\right)}\right]^{-1} & if\, \sigma_0 < \hat{\sigma} \\ (1-\varepsilon) & if\, \sigma_0 \geq \hat{\sigma} \end{cases}$$

$$= \begin{cases} \left[1 + \dfrac{\varepsilon(-2^n)\left(\prod_{i=1}^{n} t_i\right)\Gamma(n+1)\hat{\sigma}\ (\sum_{i=1}^{n} t_i^2 + \hat{\sigma}\ )^{-(n+1)}}{(1-\varepsilon)(-2^n)\left(\prod_{i=1}^{n} t_i\right)\Gamma(n+1)\sigma_0\ (\sum_{i=1}^{n} t_i^2 + \sigma_0)^{-(n+1)}}\right]^{-1} & if\, \sigma_0 < \hat{\sigma} \\ (1-\varepsilon) & if\, \sigma_0 \geq \hat{\sigma} \end{cases}$$

$$= \begin{cases} \left[1 + \dfrac{\varepsilon}{(1-\varepsilon)} \dfrac{\hat{\sigma}}{\sigma_0} \left(\dfrac{n\hat{\sigma} + \sigma_0}{n\hat{\sigma} + \hat{\sigma}}\ \right)^{(n+1)}\right]^{-1} & if\, \sigma_0 < \hat{\sigma} \\ (1-\varepsilon) & if\, \sigma_0 \geq \hat{\sigma} \end{cases}$$

Then

$$\hat{\lambda} = \begin{cases} \left[1 + \dfrac{K\varepsilon}{(1-\varepsilon)} \quad \left(\dfrac{Kn+1}{K(n+1)}\ \right)^{(n+1)}\right]^{-1} & if\, \sigma_0 < \widehat{\sigma} \\ (1-\varepsilon) & if\, \sigma_0 \geq \hat{\sigma} \end{cases} \tag{38}$$

where $K = \dfrac{\hat{\sigma}}{\sigma_0}$ and $\hat{\sigma} = \dfrac{\sum_{i=1}^{n} t_i^2}{n}$. Now, find

$$q^*_{FR_0}(\theta | \sigma_0) = \frac{L(\theta) q_0(\theta | \sigma_0)}{M(\underline{t} | q_0\ (\theta\ ))}$$

$$= \frac{\dfrac{2^n}{\theta^n} \prod_{i=1}^{n} t_i\, e^{-\frac{\sum_{i=1}^{n} t_i^2}{\theta}}\ \dfrac{\sigma_0}{\theta^2}\ e^{-\frac{\sigma_0}{\theta}}}{-2^n\ (\prod_{i=1}^{n} t_i)\Gamma(n+1)\sigma_0 (\sum_{i=1}^{n} t_i^2 + \sigma_0\ )^{-(n+1)}}$$

$$= \frac{\theta^{-(n+2)}\ \ e^{-\frac{\left(\sum_{i=1}^{n} t_i^2 + \sigma_0\right)}{\theta}}}{-\Gamma(n+1)\ (\sum_{i=1}^{n} t_i^2 + \sigma_0\ )^{-(n+1)}}$$

$$q^*_{FR_0}\ (\theta \mid \sigma_0) = \frac{(\sum_{i=1}^{n} t_i^2 + \sigma_0\ )^{(n+1)} \theta^{-(n+2)}\ \ e^{-\frac{\left(\sum_{i=1}^{n} t_i^2 + \sigma_0\right)}{\theta}}}{-\Gamma(n+1)} \tag{39}$$

The probability function in equation (39) is similar to invers gamma distribution IG (α_1 ,β_1), where $\alpha_1 = (n+1)$ and $\beta_1 = \left(\sum_{i=1}^{n} t_i^2 + \sigma_0\right)$. Similarly, we get:

$$q^*_{FR}(\theta|\hat{\sigma}) = \frac{(\sum_{i=1}^{n} t_i^2 + \hat{\sigma})^{n+1} \theta^{-(n+2)} e^{-\frac{\left(\sum_{i=1}^{n} t_i^2 + \hat{\sigma}\right)}{\theta}}}{-\Gamma(n+1)} \tag{40}$$

The probability function in equation (40) is similar to invers gamma distribution IG (α_2 ,β_2), where $\alpha_2 = (n+1)$ and $\beta_2 = \left(\sum_{i=1}^{n} t_i^2 + \hat{\sigma}\right)$.

Under SELF using equation (17), the ML-II estimator θ is given by:

$$E\left(q^*_{FR_0}\ (\theta \mid \sigma_0)\right) = \int_0^\infty \theta \frac{(\sum_{i=1}^{n} t_i^2 + \sigma_0)^{(n+1)}\ \theta^{-(n+2)}\ e^{-\frac{\left(\sum_{i=1}^{n} t_i^2 + \sigma_0\right)}{\theta}}}{-\Gamma(n+1)} d\theta$$

$$= \frac{(\sum_{i=1}^{n} t_i^2 + \sigma_0)^{(n+1)}}{-\Gamma(n+1)} \int_0^\infty \theta^{-(n+1)}\ e^{-\frac{\left(\sum_{i=1}^{n} t_i^2 + \sigma_0\right)}{\theta}} d\theta$$

Using the transformation $y = \frac{\sum_{i=1}^{n} t_i^2 + \sigma_0}{\theta}$ which implies that $\theta = \frac{\sum t_i^2 + \sigma_0}{y}$ and $d\theta = \frac{-\left(\sum t_i^2 + \sigma_0\right)}{y^2} dy$, gets:

$$\Rightarrow \quad \frac{(\sum_{i=1}^{n} t_i^2 + \sigma_0)^{(n+1)}}{-\Gamma(n+1)} \int_0^\infty \frac{(\sum_{i=1}^{n} t_i^2 + \sigma_0)^{-(n+1)}}{y^{-n-1}} e^{-y}\ \frac{-\left(\sum_{i=1}^{n} t_i^2 + \sigma_0\right)}{y^2}\ dy$$

$$\Rightarrow \quad \frac{\left(\sum_{i=1}^{n} t_i^2 + \sigma_0\right)}{\Gamma(n+1)} \int_0^\infty y^{n-1}\ e^{-y}\ dy = \frac{\left(\sum_{i=1}^{n} t_i^2 + \sigma_0\right)}{\Gamma(n+1)} \Gamma(n)$$

That is,

$$E(q^*_{FR_0}\ (\theta \mid \sigma_0)) = \frac{\sum_{i=1}^{n} t_i^2 + \sigma_0}{n} \tag{41}$$

Similarly,

$$E(q^*_{FR_0}\ (\theta \mid \hat{\sigma})) = \frac{\sum_{i=1}^{n} t_i^2 + \frac{\sum_{i=1}^{n} t_i^2}{n}}{n}$$

$$= \frac{n\sum_{i=1}^{n} t_i^2 + \sum_{i=1}^{n} t_i^2}{n^2}$$

Then

$$E(q^*_{FR_0}\ (\theta|\hat{\sigma})) = \frac{(n+1)\sum_{i=1}^{n} t_i^2}{n^2} \tag{42}$$

Put equations (41) and (42) in (37), the robust Bayes estimators of θ for the mixed posterior distribution under SELF is given by:

$$\hat{\theta}_{RBSE} = E\left(\hat{\pi}^*_{RBSE}(\theta)\right) = \begin{cases} \frac{\hat{\lambda}}{n}\left(\sum_{i=1}^{n} t_i^2 + \sigma_0\right) + \frac{\left(1-\hat{\lambda}\right)\left(n+1\right)\sum_{i=1}^{n} t_i^2}{n^2} & if\, \sigma_0 < \hat{\sigma} \\ \frac{1}{n}\left(\sum_{i=1}^{n} t_i^2 + \sigma_0\right) & if\, \sigma_0 \geq \hat{\sigma} \end{cases}$$

$$= \begin{cases} \hat{\lambda}\left(\hat{\sigma} + \frac{\sigma_0}{n}\right) + \frac{\left(n+1\right)\left(1-\hat{\lambda}\right)\hat{\sigma}}{n} & if\, \sigma_0 < \hat{\sigma} \\ \left(\hat{\sigma} + \frac{\sigma_0}{n}\right) & if\, \sigma_0 \geq \hat{\sigma} \end{cases}$$

Then

$$\hat{\theta}_{\text{RBSE}} = \begin{cases} \hat{\lambda}\frac{\sigma_0}{n} + \left(n+1-\hat{\lambda}\right)\frac{\hat{\sigma}}{n} & if\, \sigma_0 < \hat{\sigma} \\ \left(\hat{\sigma} + \frac{\sigma_0}{n}\right) & if\, \sigma_0 \geq \hat{\sigma} \end{cases} \tag{43}$$

where $\hat{\lambda}$ yield equation (38).

Similarly, the ML-II estimator of θ under BQLF, given as:

$$E\left(q^*(\theta|\sigma_0)\right) = W\hat{\theta}_{ML} + \frac{(1-w)}{n}\left(\sum_{i=1}^{n} t_i^2 + \sigma_0\right) \tag{44}$$

And,

$$E\left(q^*(\theta|\hat{\sigma})\right) = W\hat{\theta}_{ML} + \frac{(1-w)\,(n+1)}{n^2}\sum_{i=1}^{n} t_i^2 \tag{45}$$

where $\hat{\theta}_{ML} = \frac{\sum_{i=1}^{n} t_i^2}{n}$.

Put equations (44) and (45) in (37), robust Bayes estimators for θ of a mixed posterior distribution under BQLF are given by:

$$\hat{\theta}_{RBBQ} = E\left(\hat{\pi}^*_{RBBQ}(\theta)\right) = \begin{cases} \hat{\lambda}\left[W\hat{\theta}_{ML} + \frac{(1-W)}{n}\left(\sum_{i=1}^{n} t_i^2 + \sigma_0\right)\right] & \\ +\left(1-\hat{\lambda}\right)\left[W\hat{\theta}_{ML} + \frac{(1-W)\,(n+1)}{n^2}\sum_{i=1}^{n} t_i^2\right] & if\, \sigma_0 < \hat{\sigma} \\ W\hat{\theta}_{ML} + \frac{(1-W)}{n}\left(\sum_{i=1}^{n} t_i^2 + \sigma_0\right) & if\, \sigma_0 \geq \hat{\sigma} \end{cases} \tag{46}$$

7. Simulation Study

This section focuses on the simulation experiments that will be directed to look at estimates values of the unknown scale parameter θ for RD. The simulation experiments can be summarized as the following stages:

Stage 1: The constant and parameter values imposed in simulation experiments are defined in Table 1.

Table 1: Values assumed for simulation experiments' constants and parameters

10 , 15, 25, 50, 100	n	**Samples size**
1, 1.5, 2.5	θ	**Scale parameter**
1	β	**Hyper-parameter of the inverted gamma distribution**
α < β, α = β, α > β 0.5, 1 , 2	α	
0.9, 2	c	**Hyper-parameter of the inverted exponential distribution**
(0, 1)	w	**Weighted coefficient**
(0, 1)	ϵ	**Probability of error**
0.05, 0.1, 0.5	σ_0	**Hyper-parameter of natural conjugate prior Frechet distribution**
1000	L	**The number of sample replicates**

- A scale parameter for RD was chosen in three different cases: $\theta = 1$, 1.5, and 2.5.
- Hyper-parameters α, and β of an inverted gamma distribution to observe their effect on the estimates are taken as β =1 and α = 0.5, 1, 2 (α < β, α = β, α > β).
- Hyper-parameters (c) of an inverted exponential distribution are used in two different values (c = 0.9, and 2).
- Selected the two different values of weighted coefficient (w) on the interval (0, 1) is (w = 0.2, and 0.6), and two different values of probability of error (ϵ) on the interval (0, 1) is ($\epsilon = 0.00001$, and 0.9).
- Hyper-parameters (σ_0) of the natural conjugate prior FD are taken as ($\sigma_0 = 0.05$, 0.1, and 0.5).
- 1000 independent samples of size n are obtained by repeating the process 1000 times.

Stage 2: Generating n observations from the random number U_k where ($k = 1, 2, \ldots, n$) by a continuous uniform distribution on the unit interval (0, 1) and utilize the inverse transformation method, which is predicated on determining the inverse of CDF as follows::

$$U = F(t;\theta) \tag{47}$$

$$t = F^{-1}(U) \tag{48}$$

Substituting equation (2) in equation (47), gets:

$$U_k = \left(1 - e^{\frac{-t_k^2}{\theta}}\right) ;\ t \ge 0,\ \theta > 0\ \ k = 1, 2, \ldots, n \tag{49}$$

Simplifying equation (49), have the following:

$$t_k = \left[\ln\left(\frac{1}{1-U_k}\right)^{\theta}\right]^{\frac{1}{2}} ;\ t \ge 0,\ \theta > 0\ \ k = 1, 2, \ldots, n \tag{50}$$

Stage 3: Calculate the ML, RSS, Bayesian, and robust Bayesian estimators for an unknown scale parameter θ for RD and compare these estimators by using MSE.

Stage 4: The estimator with the lowest MSE value is the best; MSE is defined as [7]:

$$\mathrm{MSE}\left(\hat{\theta}\right)=\frac{1}{\mathrm{L}}\sum_{k=1}^{\mathrm{L}}\left(\hat{\theta}_k-\ \theta\right)^2 \tag{51}$$

where
L: Number of sample replicates.
$\hat{\theta}_k$: θ estimate at a kth-replicate.
The simulation study is illustrated by the following algorithm:

Algorithm: Methodology Estimation
To find the estimate value of ($\hat{\theta}$**)** using three estimation methods:

- Using traditional methods (MLE and RSS).

Let samples size (n), scale parameter (θ**)**, and number of sample replicates (L).
Evaluate random sample (t) using equation (50).

Find the ML estimator denoted by $\hat{\theta}_{MLE(RD)}=\frac{\sum_{i=1}^{n}t_i^2}{n}$

Find the RSS estimator denoted by $\hat{\theta}_{RD}^{RSS}$ using the Newton-Raphson method, which will be applied to equation (10).

- Using Bayes estimation method (inverted gamma and inverted exponentially under (BQLF) and (SELF)).

Let hyper-parameter of the inverted gamma (*β* and *α*).
Let hyper-parameter of the inverted exponential (*c*).
Let weighted coefficient (*w*).

Set $\hat{\theta}_{Bsig}=\frac{(\sum_{i=1}^{n}t_i^2+\beta\ )}{\left(n+\alpha-1\right)}$

Set $\hat{\theta}_{Bsie}=\frac{(\sum_{i=1}^{n}t_i^2+c)}{n}$

Set $\hat{\theta}_{BBsig}=W\hat{\theta}_{ML}+\frac{\left(1-W\right)\ (\sum_{i=1}^{n}t_i^2+\beta\ )}{\left(n+\alpha-1\right)}$

Set $\hat{\theta}_{BBsie}=W\hat{\theta}_{ML}+\frac{\left(1-W\right)\ (\sum_{i=1}^{n}t_i^2+c\ )}{n}$

- Using robust Bayes estimation method (FD) under (BQLF) and (SELF).

Let probability of error (ϵ).
Let hyper-parameter of natural conjugate prior FD (σ_0 **).**

Set $\hat{\sigma}=\frac{\sum_{i=1}^{n}t_i^2}{n}$, and $K=\frac{\hat{\sigma}}{\sigma_0}$

Set $\hat{\lambda}=\begin{cases}\left[1+\frac{K\varepsilon}{\left(1-\varepsilon\right)}\ \ (\frac{Kn+1}{K\left(n+1\right)})^{(n+1)}\right]^{-1} & if\ \sigma_0<\widehat{\sigma}\\ \left(1-\varepsilon\right) & if\ \sigma_0\geq\hat{\sigma}\end{cases}$

$$\textbf{Set } \hat{\theta}_{\text{RBSE}} = \begin{cases} \hat{\lambda}\dfrac{\sigma_0}{n} + \left(n+1-\hat{\lambda}\right)\dfrac{\hat{\sigma}}{n} & \text{if } \sigma_0 < \hat{\sigma} \\ \left(\hat{\sigma} + \dfrac{\sigma_0}{n}\right) & \text{if } \sigma_0 \geq \hat{\sigma} \end{cases}$$

$$\textbf{Set } \hat{\theta}_{RBBQ} = \begin{cases} \hat{\lambda}\left[W\hat{\theta}_{ML} + \dfrac{(1-W)}{n}\left(\sum_{i=1}^{n} t_i^2 + \sigma_0\right)\right] + \left(1-\hat{\lambda}\right)\left[W\hat{\theta}_{ML} + \dfrac{(1-W)\,(n+1)}{n^2}\sum_{i=1}^{n} t_i^2\right] & \text{if } \sigma_0 < \hat{\sigma} \\ W\hat{\theta}_{ML} + \dfrac{(1-W)}{n}\left(\sum_{i=1}^{n} t_i^2 + \sigma_0\right) & \text{if } \sigma_0 \geq \hat{\sigma} \end{cases}$$

Find MSE by using equation (51) for all the estimate value of ($\hat{\theta}$)
end

8. Results of the Simulation for Estimating the Scale Parameter

The simulation results depend on the Monte Carlo simulation to estimate the unknown scale parameter of RD and the MSE values are presented in Tables (2-7) using MLE, RSS, Bayes, and robust Bayes methods with different values of weighted coefficient (w = 0.2, 0.6), hyper-parameter of natural conjugate prior FD (σ_0= 0.05, 0.1, 0.5), and different simple sizes (n =10, 15, 25, 50, 100) are shown below.

- Table 2 with w = 0.2, and σ_0= 0.05 we observe:
  When (θ = 1, 1.5, 2.5) with (c = 0.9, 2), (α = 0.5, 1), and (ϵ = 0.00001) the MSE values associated with robust Bayes (unbalanced, and balanced loss functions) estimates are less than MLE, RSS, and Bayes while values of the MSE for robust Bayes under an unbalanced loss function are less than values of the MSE for robust Bayes based on the balanced loss function for all sample sizes except (θ = 2.5, and n =10, 15).
- Table 3 with w = 0.2, and σ_0= 0.1 we observe:
  When (θ = 1, 1.5, 2.5) with (c = 0.9, 2), (α = 0.5, 1), and (ϵ = 0.00001) the MSE values associated with robust Bayes (unbalanced, and balanced loss functions) estimates are less than MLE, RSS, and Bayes while values of the MSE for robust Bayes under an unbalanced loss function are less than the MSE values of robust Bayes under the balanced loss function for all sizes of sample excepting (n = 10, 15).
- Table 4 with w = 0.2, and σ_0= 0.5 we observe:
  I. When (θ = 1, 1.5, 2.5) with (c = 0.9, 2), (α = 0.5, 1), and (ϵ = 0.00001) the MSE values associated with robust Bayes (unbalanced, and balanced loss functions) estimates are less than RSS, and Bayes while values of the MSE for robust Bayes under the balanced loss function are less than values of the MSE for robust Bayes under the unbalanced loss function for all sample sizes.
  II. When (θ = 1, 1.5, 2.5), and (ϵ = 0.00001), values of the MSE for MLE are less than values of the MSE for robust Bayes (unbalanced and balanced loss functions) for all sample sizes except (θ = 1, and n = 50, 100), (θ = 1.5, and n = 25, 100), and (θ = 2.5, and n = 25, 50, 100).
- Table 5 with w = 0.6, and σ_0= 0.05 we observe:
  I. When (θ = 1, 1.5, 2.5) with (c = 0.9,2), (α = 0.5, 1), and (ϵ = 0.00001) the MSE values associated with robust Bayes (unbalanced, and balanced loss functions) estimates are less than MLE, RSS, and Bayes under an unbalanced loss function while values of the MSE for robust Bayes under an unbalanced loss function are less than values of the MSE for robust Bayes under the balanced loss function for all sample sizes except (θ = 2.5, and n = 15).

II. When (α = 1), and (ϵ = 0.00001) the MSE values associated with Bayes estimators of inverted gamma prior under balanced loss function less than robust Bayes (unbalanced, and balanced loss functions) estimates when (θ = 1, and n = 50), (θ = 1.5, and n = 25, 100), and (θ = 2.5, and n = 25, 50, 100), but they are greater than robust Bayes (unbalanced loss function) and less than robust Bayes (balanced loss function) when (θ = 1, and n = 100).

III. When (c = 0.9), and (ϵ = 0.00001) the MSE values associated with Bayes estimators of inverted exponential prior under balanced loss function less than robust Bayes (unbalanced, and balanced loss functions) estimates when (θ = 1, and n = 50, 100), (θ = 1.5, and n = 25, 100), and (θ = 2.5, and n = 25, 50, 100).

- Table 6 with w = 0.6, and σ_0 = 0.1 we observe:

I. When (θ = 1, 1.5, 2.5) with (c = 0.9, 2), (α = 0.5, 1), and (ϵ = 0.00001) the MSE values associated with robust Bayes (unbalanced, and balanced loss functions) estimates are less than MLE, RSS, and Bayes under an unbalanced loss function while values of the MSE for robust Bayes under an unbalanced loss function are less than values of the MSE for robust Bayes under the balanced loss function for all sample sizes except (θ = 1, 1.5, and n = 10), and (θ = 2.5, and n = 10, 15).

II. When (α = 1), and (ϵ = 0.00001) the MSE values associated with Bayes estimators of inverted gamma prior under balanced loss function less than robust Bayes (unbalanced, and balanced loss functions) estimates when (θ = 1.5, and n = 25, 100), and (θ = 2.5, and n = 25, 50, 100), but they are greater than robust Bayes (unbalanced loss function) and less than robust Bayes (balanced loss function) when (θ = 1, and n = 50).

III. When (c = 0.9), and (ϵ = 0.00001) the MSE values associated with Bayes estimators of inverted exponential prior under balanced loss function less than robust Bayes (unbalanced, and balanced loss functions) estimates when (θ = 1, and n = 50), (θ = 1.5, and n = 25, 100), and (θ = 2.5, and n = 25, 50, 100), but they are greater than robust Bayes (unbalanced loss function) and less than robust Bayes (balanced loss function) when (θ = 1, and n = 100).

- Table 7 with w = 0.6, and σ_0 = 0.5 we observe:

I. When (θ = 1, 1.5, 2.5) with (c =0.9, 2), (α = 0.5, 1), and (ϵ = 0.00001) the MSE values associated with robust Bayes (unbalanced, and balanced loss functions) estimates are less than RSS, and Bayes based on an unbalanced loss function while values of the MSE for robust Bayes under an balanced loss function are less than values of the MSE for robust Bayes under the unbalanced loss function for all sample sizes except (θ = 1.5, and n = 25).

II. When (α = 1), and (ϵ = 0.00001) the MSE values associated with Bayes estimators of inverted gamma prior under balanced loss functions are less than robust Bayes estimates (unbalanced, and balanced loss functions) for some states: (θ = 1, and n = 10), (θ = 1.5, and n = 25), and (θ = 2.5, and n = 25, 50), but they are greater than robust Bayes estimates under (balanced loss functions) and less than robust Bayes (unbalanced loss function) for the other cases.

III. When (c = 0.9), and (ϵ = 0.00001) the MSE values associated with Bayes estimators of inverted exponential prior under balanced loss functions are less than robust Bayes estimates (unbalanced, and balanced loss functions) for some states: (θ = 1.5, and n = 25), and (θ = 2.5, and n = 25, 50), but they are greater than robust Bayes estimates under (balanced loss functions) and less than robust Bayes (unbalanced loss function)for the other cases.

- From tables (2-7), it appears that when (α = 2), the MSE values of Bayes estimators with inverted gamma prior under (unbalanced and balanced loss functions) are less than of all the estimation methods for all sample sizes.

Table 2: Values of MSE to the scale parameter estimators θ using non-Bayes, Bayes and robust Bayes methods for RD with w = 0.2, σ_0 = 0.05.

θ			1					1.5					2.5				
$\hat{\theta}$ \ n			10	15	25	50	100	10	15	25	50	100	10	15	25	50	100
$\hat{\theta}_{MLE}$			0.09803	0.05661	0.03913	0.01979	0.00954	0.22540	0.14199	0.09218	0.04109	0.023049	0.55762	0.40516	0.24601	0.12350	0.058836
$\hat{\theta}_{RSS}$			0.09990	0.05841	0.03984	0.01993	0.00955	0.22738	0.14283	0.092196	0.041136	0.023053	0.56151	0.40656	0.24607	0.12353	0.058839
$\hat{\theta}_{Bsie}$	**c**	0.9	0.10494	0.05965	0.04010	0.01994	0.00958	0.23242	0.14497	0.09246	0.04130	0.02308	0.56431	0.40818	0.24641	0.12361	0.05887
		2	0.13539	0.07313	0.04481	0.02101	0.00985	0.26299	0.15838	0.09632	0.04243	0.02335	0.59448	0.42166	0.25042	0.12462	0.05913
$\hat{\theta}_{Bsig}$	**α**	0.5	0.13136	0.07028	0.04393	0.02082	0.00979	0.28135	0.16522	0.09902	0.04294	0.02350	0.67004	0.45613	0.26226	0.12752	0.05982
		1	0.10671	0.06043	0.04037	0.01999	0.00959	0.23420	0.14574	0.09265	0.04136	0.02310	0.56605	0.40897	0.24662	0.12366	0.05888
		2	0.08101	0.04976	0.03618	0.01902	0.00935	0.18885	0.12608	0.08612	0.03966	0.02264	0.48138	0.36572	0.23216	0.11992	0.05797
$\hat{\theta}_{BBsig}$	**α**	0.5	0.12068	0.06583	0.04237	0.02047	0.009706	0.26465	0.15822	0.09684	0.04237	0.02336	0.63837	0.44201	0.25765	0.12638	0.05954
		1	0.10337	0.05895	0.03987	0.01989	0.00957	0.23084	0.14428	0.09230	0.04124	0.02307	0.56276	0.40749	0.24624	0.12356	0.05886
		2	0.08429	0.05109	0.03676	0.01917	0.00939	0.19554	0.12902	0.08725	0.03992	0.02272	0.49295	0.37196	0.23435	0.12049	0.05811
$\hat{\theta}_{BBsie}$	**c**	0.9	0.10226	0.05846	0.03970	0.01986	0.00956	0.22972	0.14379	0.092197	0.04120	0.02306	0.56167	0.40700	0.24612	0.123534	0.05885
		2	0.12152	0.06698	0.04265	0.02051	0.009723	0.24907	0.15226	0.09447	0.04191	0.02322	0.58071	0.41552	0.24851	0.12414	0.05901
$\hat{\theta}_{RBSE}$	ϵ	10^{-5}	0.097989	0.056594	0.039117	0.019780	0.009534	0.22537	0.141971	0.09213	0.041086	0.023047	0.557605	0.405155	0.24597	0.123491	0.058833
		0.9	0.12687	0.06806	0.04351	0.02079	0.009782	0.29279	0.17025	0.10148	0.04343	0.02366	0.73214	0.48673	0.27339	0.13035	0.06050
$\hat{\theta}_{RBBQ}$	ϵ	10^{-5}	0.097993	0.056596	0.039119	0.019782	0.0095353	0.22538	0.141973	0.09214	0.041087	0.023048	0.557603	0.405153	0.24598	0.123493	0.058834
		0.9	0.11941	0.06504	0.04237	0.02052	0.009717	0.27545	0.16294	0.09904	0.04282	0.02349	0.68653	0.46575	0.26628	0.12858	0.06007

Table 3: Values of MSE to the scale parameter estimators θ using non-Bayes, Bayes and robust Bayes methods for RD with w = 0.2, σ_0 = 0.1.

θ			1					1.5					2.5				
$\hat{\theta}$ \ n			10	15	25	50	100	10	15	25	50	100	10	15	25	50	100
$\hat{\theta}_{MLE}$			0.09803	0.05661	0.03913	0.01979	0.00954	0.22540	0.14199	0.09218	0.04109	0.023049	0.55762	0.40516	0.24601	0.12350	0.058836
$\hat{\theta}_{RSS}$			0.09990	0.05841	0.03984	0.01993	0.00955	0.22738	0.14283	0.092196	0.041136	0.023053	0.56151	0.40656	0.24607	0.12353	0.058839
$\hat{\theta}_{Bsie}$	**c**	0.9	0.10494	0.05965	0.04010	0.01994	0.00958	0.23242	0.14497	0.09246	0.04130	0.02308	0.56431	0.40818	0.24641	0.12361	0.05887
		2	0.13539	0.07313	0.04481	0.02101	0.00985	0.26299	0.15838	0.09632	0.04243	0.02335	0.59448	0.42166	0.25042	0.12462	0.05913
$\hat{\theta}_{Bsig}$	**α**	0.5	0.13136	0.07028	0.04393	0.02082	0.00979	0.28135	0.16522	0.09902	0.04294	0.02350	0.67004	0.45613	0.26226	0.12752	0.05982
		1	0.10671	0.06043	0.04037	0.01999	0.00959	0.23420	0.14574	0.09265	0.04136	0.02310	0.56605	0.40897	0.24662	0.12366	0.05888
		2	0.08101	0.04976	0.03618	0.01902	0.00935	0.18885	0.12608	0.08612	0.03966	0.02264	0.48138	0.36572	0.23216	0.11992	0.05797
$\hat{\theta}_{BBsig}$	**α**	0.5	0.12068	0.06583	0.04237	0.02047	0.009706	0.26465	0.15822	0.09684	0.04237	0.02336	0.63837	0.44201	0.25765	0.12638	0.05954
		1	0.10337	0.05895	0.03987	0.01989	0.00957	0.23084	0.14428	0.09230	0.04124	0.02307	0.56276	0.40749	0.24624	0.12356	0.05886
		2	0.08429	0.05109	0.03676	0.01917	0.00939	0.19554	0.12902	0.08725	0.03992	0.02272	0.49295	0.37196	0.23435	0.12049	0.05811
$\hat{\theta}_{BBsie}$	**c**	0.9	0.10226	0.05846	0.03970	0.01986	0.00956	0.22972	0.14379	0.092197	0.04120	0.02306	0.56167	0.40700	0.24612	0.123534	0.05885
		2	0.12152	0.06698	0.04265	0.02050	0.009723	0.24907	0.15226	0.09447	0.04191	0.02322	0.58071	0.41552	0.24851	0.12414	0.05901
$\hat{\theta}_{RBSE}$	ϵ	10^{-5}	0.09799	0.056595	0.039111	0.019774	0.009533	0.22539	0.141968	0.09209	0.041083	0.023045	0.55758	0.40515	0.24593	0.123481	0.058831
		0.9	0.12658	0.06794	0.04346	0.02078	0.009779	0.29234	0.17005	0.10142	0.04342	0.02365	0.73138	0.48639	0.27328	0.13032	0.06049
$\hat{\theta}_{RBBQ}$	ϵ	10^{-5}	0.09798	0.056591	0.039113	0.019776	0.009534	0.22538	0.141965	0.09210	0.041084	0.023046	0.55757	0.40514	0.24595	0.123484	0.058832
		0.9	0.11921	0.06496	0.04235	0.02051	0.009716	0.27515	0.16281	0.09901	0.04281	0.023497	0.68603	0.46553	0.26621	0.12856	0.06007

Table 4: Values of MSE to the scale parameter estimators θ using non-Bayes, Bayes and robust Bayes methods for RD with w = 0.2, σ_0 = 0.5.

θ			1					1.5					2.5				
$\hat{\theta}$ \ n			10	15	25	50	100	10	15	25	50	100	10	15	25	50	100
$\hat{\theta}_{MLE}$			0.09803	0.05661	0.03913	0.01978	0.009537	0.22540	0.14199	0.09218	0.04109	0.023049	0.55762	0.40516	0.24601	0.12350	0.058836
$\hat{\theta}_{RSS}$			0.09990	0.05841	0.03984	0.01993	0.00955	0.22738	0.14283	0.092196	0.041136	0.023053	0.56151	0.40656	0.24607	0.12353	0.058839
$\hat{\theta}_{Bsie}$	**c**	0.9	0.10494	0.05965	0.04010	0.01994	0.00958	0.23242	0.14497	0.09246	0.04130	0.02308	0.56431	0.40818	0.24641	0.12361	0.05887
		2	0.13539	0.07313	0.04481	0.02101	0.00985	0.26299	0.15838	0.09632	0.04243	0.02335	0.59448	0.42166	0.25042	0.12462	0.05913
$\hat{\theta}_{Bsig}$	**α**	0.5	0.13136	0.07028	0.04393	0.02082	0.00979	0.28135	0.16522	0.09902	0.04294	0.02350	0.67004	0.45613	0.26226	0.12752	0.05982
		1	0.10671	0.06043	0.04037	0.01999	0.00959	0.23420	0.14574	0.09265	0.04136	0.02310	0.56605	0.40897	0.24662	0.12366	0.05888
		2	0.08101	0.04976	0.03618	0.01902	0.00935	0.18885	0.12608	0.08612	0.03966	0.022645	0.48138	0.36572	0.23216	0.11992	0.05797
$\hat{\theta}_{BBsig}$	**α**	0.5	0.12068	0.06583	0.04237	0.02046	0.009706	0.26465	0.15822	0.09684	0.04237	0.02336	0.63837	0.44201	0.25765	0.12638	0.05954
		1	0.10337	0.05895	0.03987	0.01989	0.00957	0.23084	0.14428	0.09230	0.04124	0.02307	0.56276	0.40749	0.24624	0.12356	0.05886
		2	0.08429	0.05109	0.03676	0.01917	0.00939	0.19554	0.12902	0.08725	0.03992	0.022721	0.49295	0.37196	0.23435	0.12049	0.05811
$\hat{\theta}_{BBsie}$	**c**	0.9	0.10226	0.05846	0.03970	0.01986	0.00956	0.22972	0.14379	0.092197	0.04120	0.02306	0.56167	0.40700	0.24612	0.123534	0.05885
		2	0.12152	0.06698	0.04265	0.02050	0.009723	0.24907	0.15226	0.09447	0.04191	0.023224	0.58071	0.41552	0.24851	0.12414	0.05901
$\hat{\theta}_{RBSE}$	ϵ	10^{-5}	0.09987	0.05741	0.03935	0.01979	0.009540	0.22730	0.14275	0.09202	0.041126	0.023048	0.55934	0.40595	0.24591	0.12348	0.058834
		0.9	0.12487	0.06729	0.04323	0.02072	0.00977	0.28941	0.16882	0.10099	0.04331	0.02363	0.72598	0.48394	0.27247	0.13012	0.06045
$\hat{\theta}_{RBBQ}$	ϵ	10^{-5}	0.09910	0.05707	0.03924	0.01977	0.009535	0.22652	0.14242	0.09199	0.041103	0.023045	0.55859	0.40561	0.24587	0.12347	0.058831
		0.9	0.11790	0.06447	0.04216	0.02047	0.009705	0.27301	0.16191	0.09869	0.04274	0.02348	0.68224	0.46381	0.26565	0.12842	0.06003

Table 5: Values of MSE to the scale parameter estimators θ using non-Bayes, Bayes and robust Bayes methods for RD with w = 0.6, σ_0 = 0.05.

θ			1					1.5					2.5				
$\hat{\theta}$		n	10	15	25	50	100	10	15	25	50	100	10	15	25	50	100
$\hat{\theta}_{MLE}$			0.09803	0.05661	0.03913	0.01978	0.009537	0.22540	0.14199	0.09218	0.04109	0.023049	0.55762	0.40516	0.24601	0.12350	0.058836
$\hat{\theta}_{RSS}$			0.09990	0.05841	0.03984	0.01993	0.00955	0.22738	0.14283	0.092196	0.041136	0.023053	0.56151	0.40656	0.24607	0.12353	0.058839
$\hat{}_{Bsie}$	**c**	0.9	0.10494	0.05965	0.04010	0.01994	0.00958	0.23242	0.14497	0.09246	0.04130	0.02308	0.56431	0.40818	0.24641	0.12361	0.05887
		2	0.13539	0.07313	0.04481	0.02101	0.00985	0.26299	0.15838	0.09632	0.04243	0.02335	0.59448	0.42166	0.25042	0.12462	0.05913
$\hat{\theta}_{Bsig}$	**α**	0.5	0.13136	0.07028	0.04393	0.02082	0.00979	0.28135	0.16522	0.09902	0.04294	0.02350	0.67004	0.45613	0.26226	0.12752	0.05982
		1	0.10671	0.06043	0.04037	0.01999	0.00959	0.23420	0.14574	0.09265	0.04136	0.02310	0.56605	0.40897	0.24662	0.12366	0.05888
		2	0.08101	0.04976	0.03618	0.01902	0.00935	0.18885	0.12608	0.08612	0.03966	0.022645	0.48138	0.36572	0.23216	0.11992	0.05797
$\hat{\theta}_{BBsig}$	**α**	0.5	0.10534	0.05950	0.04015	0.01998	0.009585	0.23952	0.14776	0.09369	0.04153	0.02316	0.58880	0.41966	0.25047	0.12461	0.05911
		1	0.09910	0.05707	0.03924	0.019777	0.0095354	0.22652	0.14242	0.091985	0.04110	0.023045	0.55859	0.40561	0.24587	0.123467	0.058831
		2	0.09103	0.05382	0.03794	0.019479	0.00946	0.20985	0.13526	0.08964	0.04049	0.02288	0.52159	0.38691	0.23959	0.12185	0.05844
$\hat{\theta}_{BBsie}$	**c**	0.9	0.09885	0.05696	0.03921	0.019772	0.0095341	0.22627	0.14232	0.09198	0.041096	0.023044	0.55835	0.40550	0.24586	0.123465	0.058829
		2	0.10337	0.05895	0.03987	0.01989	0.00957	0.23084	0.14428	0.09230	0.04124	0.02307	0.56276	0.40749	0.24624	0.12356	0.05886
$\hat{\theta}_{RBSE}$	ϵ	10^{-5}	0.09799	0.05659	0.03911	0.019780	0.0095349	0.22538	0.14197	0.09213	0.041086	0.023047	0.557605	0.405155	0.24597	0.123491	0.058834
		0.9	0.12687	0.06806	0.04351	0.02079	0.00978	0.29279	0.17025	0.10148	0.04343	0.02366	0.73214	0.48673	0.27339	0.13035	0.06050
$\hat{\theta}_{RBBQ}$	ϵ	10^{-5}	0.09801	0.05660	0.03912	0.019785	0.009536	0.22539	0.14198	0.09216	0.041089	0.023048	0.557606	0.405153	0.24599	0.123496	0.058835
		0.9	0.10703	0.06010	0.04049	0.02009	0.00961	0.24657	0.15080	0.09504	0.04181	0.02324	0.61136	0.43079	0.25452	0.12564	0.05935

Table 6: Values of MSE to the scale parameter estimators θ using non-Bayes, Bayes and robust Bayes methods for RD with w = 0.6, σ_0 = 0.1.

θ			1					1.5					2.5				
$\hat{\theta}$		n	10	15	25	50	100	10	15	25	50	100	10	15	25	50	100
$\hat{\theta}_{MLE}$			0.09803	0.05661	0.03913	0.01978	0.009537	0.22540	0.14199	0.09218	0.04109	0.023049	0.55762	0.40516	0.24601	0.12350	0.058836
$\hat{\theta}_{RSS}$			0.09990	0.05841	0.03984	0.01993	0.00955	0.22738	0.14283	0.092196	0.041136	0.023053	0.56151	0.40656	0.24607	0.12353	0.058839
$\hat{\theta}_{Bsie}$	**c**	0.9	0.10494	0.05965	0.04010	0.01994	0.00958	0.23242	0.14497	0.09246	0.04130	0.02308	0.56431	0.40818	0.24641	0.12361	0.05887
		2	0.13539	0.07313	0.04481	0.02101	0.00985	0.26299	0.15838	0.09632	0.04243	0.02335	0.59448	0.42166	0.25042	0.12462	0.05913
$\hat{\theta}_{Bsig}$	**α**	0.5	0.13136	0.07028	0.04393	0.02082	0.00979	0.28135	0.16522	0.09902	0.04294	0.02350	0.67004	0.45613	0.26226	0.12752	0.05982
		1	0.10671	0.06043	0.04037	0.01999	0.00959	0.23420	0.14574	0.09265	0.04136	0.02310	0.56605	0.40897	0.24662	0.12366	0.05888
		2	0.08101	0.04976	0.03618	0.01902	0.00935	0.18885	0.12608	0.08612	0.03966	0.022645	0.48138	0.36572	0.23216	0.11992	0.05797
$\hat{\theta}_{BBsig}$	**α**	0.5	0.10534	0.05950	0.04015	0.01998	0.009585	0.23952	0.14776	0.09369	0.04153	0.02316	0.58880	0.41966	0.25047	0.12461	0.05911
		1	0.09910	0.05707	0.03924	0.019777	0.0095354	0.22652	0.14242	0.091985	0.04110	0.0230450	0.55859	0.40561	0.24587	0.123467	0.058831
		2	0.09103	0.05382	0.03794	0.01948	0.00946	0.20985	0.13526	0.08964	0.04049	0.02288	0.52159	0.38691	0.23959	0.12185	0.05844
$\hat{\theta}_{BBsie}$	**c**	0.9	0.09885	0.05696	0.03921	0.019772	0.0095341	0.22627	0.14232	0.09198	0.041096	0.023044	0.55835	0.40550	0.24586	0.123465	0.058829
		2	0.10337	0.05895	0.03987	0.01989	0.00957	0.23084	0.14428	0.09230	0.04124	0.02307	0.56276	0.40749	0.24624	0.12356	0.05886
$\hat{\theta}_{RBSE}$	ϵ	10^{-5}	0.097997	0.0565948	0.03911	0.019774	0.009533	0.22539	0.14196	0.09209	0.041083	0.0230453	0.55758	0.405147	0.24593	0.12348	0.058832
		0.9	0.12658	0.06794	0.04346	0.02078	0.00978	0.29234	0.17005	0.10142	0.04342	0.02365	0.73138	0.48639	0.27328	0.13032	0.06049
$\hat{\theta}_{RBBQ}$	ϵ	10^{-5}	0.097992	0.0565949	0.03912	0.019782	0.0095353	0.22537	0.14197	0.09214	0.041087	0.023048	0.55757	0.405141	0.24598	0.12349	0.058834
		0.9	0.10696	0.06008	0.04349	0.02009	0.00961	0.24647	0.15076	0.09503	0.04181	0.02324	0.61122	0.43073	0.25451	0.12563	0.05935

Table 7: Values of MSE to the scale parameter estimators θ using non-Bayes, Bayes and robust Bayes methods for RD with w = 0.6, σ_0 = 0.5.

θ			1					1.5					2.5				
$\hat{\theta}$ \ n			10	15	25	50	100	10	15	25	50	100	10	15	25	50	100
$\hat{\theta}_{MLE}$			0.09803	0.05661	0.03913	0.01978	0.009537	0.22540	0.14199	0.09218	0.04109	0.023049	0.55762	0.40516	0.24601	0.12350	0.058836
$\hat{\theta}_{RSS}$			0.09990	0.05841	0.03984	0.01993	0.00955	0.22738	0.14283	0.092196	0.041136	0.023053	0.56151	0.40656	0.24607	0.12353	0.058839
$\hat{\theta}_{Bsie}$	**c**	0.9	0.10494	0.05965	0.04010	0.01994	0.00958	0.23242	0.14497	0.09246	0.04130	0.02308	0.56431	0.40818	0.24641	0.12361	0.05887
		2	0.13539	0.07313	0.04481	0.02101	0.00985	0.26299	0.15838	0.09632	0.04243	0.02335	0.59448	0.42166	0.25042	0.12462	0.05913
$\hat{\theta}_{Bsig}$	**α**	0.5	0.13136	0.07028	0.04393	0.02082	0.00979	0.28135	0.16522	0.09902	0.04294	0.02350	0.67004	0.45613	0.26226	0.12752	0.05982
		1	0.10671	0.06043	0.04037	0.01999	0.00959	0.23420	0.14574	0.09265	0.04136	0.02310	0.56605	0.40897	0.24662	0.12366	0.05888
		2	0.08101	0.04976	0.03618	0.01902	0.00935	0.18885	0.12608	0.08612	0.03966	0.022645	0.48138	0.36572	0.23216	0.11992	0.05797
$\hat{\theta}_{BBsig}$	**α**	0.5	0.10534	0.05950	0.04015	0.01998	0.009585	0.23952	0.14776	0.09369	0.04153	0.02316	0.58880	**0.41966**	**0.25047**	**0.12461**	**0.05911**
		1	0.09910	0.05707	0.03924	0.019777	0.009535	0.22652	0.14242	0.091985	0.04110	0.023045	0.55859	**0.40561**	**0.24587**	**0.123467**	**0.058831**
		2	0.09103	0.05382	0.03794	0.01989	0.00946	0.20985	0.13526	0.08964	0.04049	0.02288	0.52159	**0.38691**	**0.23959**	**0.12185**	**0.05844**
$\hat{\theta}_{BBsie}$	**c**	0.9	0.09885	0.05696	0.03921	0.019772	0.009534	0.22627	0.14232	0.091982	0.041096	0.023044	0.55835	**0.40550**	**0.24586**	**0.123465**	**0.0588297**
		2	0.10337	0.05895	0.03987	0.01989	0.009566	0.23084	0.14428	0.09230	0.04124	0.023073	0.56276	**0.40749**	**0.24624**	**0.12356**	**0.058858**
$\hat{\theta}_{RBSE}$	ϵ	10^{-5}	0.09987	0.05741	0.03935	0.01979	0.009540	0.22730	0.14275	0.092016	0.04113	0.023048	0.55934	**0.40595**	**0.27247**	**0.12348**	**0.058834**
		0.9	0.12487	0.06729	0.04323	0.02072	0.009766	0.28941	0.16882	0.10099	0.04331	0.02363	0.72598	**0.48394**	**0.24591**	**0.13012**	**0.06045**
$\hat{\theta}_{RBBQ}$	ϵ	10^{-5}	0.09816	0.05666	0.03912	0.019766	0.009532	0.22556	0.14203	0.092019	0.041081	0.023043	0.55771	**0.40521**	**0.25430**	**0.123468**	**0.0588292**
		0.9	0.10637	0.05986	0.04041	0.02007	0.009606	0.24560	0.15040	0.09490	0.04178	0.02323	0.60987	**0.43010**	**0.24588**	**0.12558**	**0.05934**

9. Conclusion

Some statistical estimations methods were used to find estimates values of the scale parameter θ for RD. These methods are compared depending on MSE to show which is best and all the computations were performed in (MATLAB 2015). The perfect results are presented and a comparison is done as follows:

- The smaller ϵ and approximate to the zero, values of the MSE for robust Bayes (unbalanced and balanced loss functions) are the best from values of the MSE for robust Bayes (unbalanced and balanced loss functions) when ϵ approximate to the one.
- When ($w = 10^{-5}$), means that it is approximately zero, become $\hat{\theta}_{Bsig}$ equal to $\hat{\theta}_{BBsig}$ and $\hat{\theta}_{Bsie}$ equal to $\hat{\theta}_{BBsie}$. But when w is approximately one, become $\hat{\theta}_{BBsig}$ and $\hat{\theta}_{BBsie}$ equal to $\hat{\theta}_{MLE}$ according to equations (25) and (26).
- From the tables (2-7) and for all sample sizes, we notice that when ($\alpha = 2$), the MSE values associated with Bayes estimators of inverted gamma prior under unbalanced and balanced loss function less than the other statistical estimation methods. This means that when (α) grows up the best values of MSE get.
- From the tables (2-7) and for all sample sizes, when (c) values are small, the MSE values associated with Bayes estimators of inverted exponential prior under balanced and unbalanced loss functions become smaller.
- From the tables (2-7), when the value of sample sizes increases, the MSE values of all the statistical methods used decrease and approximate each other.

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