



## The Convergence of Ishikawa Iteration for Generalized $\Phi$ -contractive Mappings

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### Abstract

Charles[1] proved the convergence of Picard-type iterative for generalized  $\Phi$ -accretive non-self mappings in a real uniformly smooth Banach space. Based on the theorems of the zeros of strongly  $\Phi$ -quasi-accretive and fixed points of strongly  $\Phi$ -hemi-contractions, we extend the results to Ishikawa iterative and Ishikawa iteration process with errors for generalized  $\Phi$ -hemi-contractive mappings.

*Keywords:* strongly  $\Phi$ -quasi-accretive strongly  $\Phi$ -hemi-contractions Ishikawa iteration process with errors unique solution.

*2010 MSC:* 54H25, 47H10.

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### 1. Introduction

In[2], we can see that the convergence theorems of Ishikawa iterative process with errors for  $\Phi$ -hemi-contractive mappings in uniformly smooth Banach spaces; In[3], we can see that strong convergence of the modified Ishikawa iterative method for infinitely many nonexpansive mappings in Banach spaces; In[4], we can see that Mann and Ishikawa-type iterative schemes for approximating fixed points of multi-valued non-self mappings; In[5], we can see that convergence analysis of the Picard–Ishikawa hybrid iterative process with applications.

In 2009, Charles[1] proved the convergence of Picard-type iterative for generalized  $\Phi$ -accretive non-self maps in a real uniformly smooth Banach space. In this paper, we consider that the Ishikawa iteration process and Ishikawa iteration process with errors will be extended from the results of Charles [1].

In 1974, Ishikawa[6] introduced the Ishikawa iteration process as follows: For a convex subset  $C$  of a Banach space  $E$  and a mapping  $T$  from  $C$  into itself, for any given  $x_0 \in C$ , the sequence  $\{x_n\}$  in  $C$  is

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defined by

$$\begin{aligned}x_{n+1} &= (1 - \alpha_n) x_n + \alpha_n T y_n \\ y_n &= (1 - \beta_n) x_n + \beta_n T x_n, \quad n \geq 0\end{aligned}\tag{0.1}$$

where  $\{\alpha_n\}$  and  $\{\beta_n\}$  are two sequences in  $[0,1]$  satisfying the conditions  $0 \leq \alpha_n, \beta_n \leq 1$  for all  $n$ ,  $\lim_{n \rightarrow \infty} \beta_n = 0$ , and  $\sum_{n=0}^{\infty} \alpha_n \beta_n = \infty$ .

In 1995, Liu[7] introduced what he called the Ishikawa iteration process with errors.

In 1998, Xu[8] introduced the following alternative definitions:

Let  $K$  be a nonempty convex subset of  $E$  and  $T : K \rightarrow K$  be a nonlinear mapping. For any given  $x_0, u_0, v_0 \in K$ , the Ishikawa iterative process with errors  $\{x_n\}_{n=0}^{\infty}$  defined by

$$\begin{aligned}x_{n+1} &= (1 - \beta_n - \gamma_n) x_n + \beta_n T y_n + \gamma_n u_n, \quad n \geq 0, \\ y_n &= (1 - a_n - b_n) x_n + a_n T x_n + b_n v_n \\ &= x_n - a_n (I - T) x_n - b_n (x_n - v_n), \quad n \geq 0,\end{aligned}\tag{0.2}$$

where  $\{u_n\}$ ,  $\{v_n\}$  are any bounded sequences in  $K$ ;  $\{\beta_n\}$ ,  $\{\gamma_n\}$ ,  $\{a_n\}$ ,  $\{b_n\}$  are four real sequences in  $[0,1]$  and satisfy  $\beta_n + \gamma_n \leq 1$ ,  $a_n + b_n \leq 1$ , for all  $n \geq 0$ .

## 2. Preliminaries

**Definition 1.** [1] Given a gauge function  $\varphi$ , the mapping  $J_\varphi : E \rightarrow 2^{E^*}$  defined by

$$J_\varphi x := \{u^* \in E^* : \langle x, u^* \rangle = \|x\| \|u^*\|; \|u^*\| = \varphi(\|x\|)\}$$

is called the duality map with gauge function  $\varphi$  where  $X$  is any normed space.

In the particular case  $\varphi(t) = t$ , the duality map  $J = J_\varphi$  is called the normalized duality map.

**Proposition 2.** [9] If a Banach space  $E$  has a uniformly Gateaux differentiable norm, then  $J : E \rightarrow E^*$  is uniformly continuous on bounded subsets of  $E$  from the strong topology of  $E$  to the weak\* topology of  $E^*$ .

**Definition 3.** [10] Let  $E$  be an arbitrary real normed linear space. A mapping  $T : D(T) \subseteq E \rightarrow E$  is called strongly hemi-contractive if  $F(T) \neq \emptyset$ , and there exists  $t > 1$  such that for all  $r > 0$ ,

$$\|x - x^*\| \leq \|(1 + r)(x - x^*) - rt(Tx - x^*)\|\tag{0.1}$$

holds for all  $x \in D(T)$ ,  $x^* \in F(T)$ . If  $t = 1$ , then  $T$  is called hemi-contractive. Finally,  $T$  is called generalized  $\Phi$ -hemi-contractive, if for all  $x \in D(T)$ ,  $x^* \in F(T)$ , there exists  $j(x - x^*) \in J(x - x^*)$  such that

$$\langle (I - T)x - (I - T)x^*, j(x - x^*) \rangle \geq \Phi(\|x - x^*\|).\tag{0.2}$$

It follows from inequality (2.2) that  $T$  is generalized  $\Phi$ -hemi-contractive if and only if

$$\langle Tx - x^*, j(x - x^*) \rangle \leq \|x - x^*\|^2 - \Phi(\|x - x^*\|), \quad \forall n \geq 0.\tag{0.3}$$

**Definition 4.** [1] Let  $N(T) = \{x \in E : Tx = 0\} \neq \emptyset$ . The mapping  $T : D(T) \subseteq E \rightarrow E$  is called generalized  $\Phi$ -quasi-accretive if, for all  $x \in E$ ,  $x^* \in N(T)$ , there exists  $j(x - x^*) \in J(x - x^*)$  such that

$$\langle Tx - Tx^*, j(x - x^*) \rangle \geq \Phi(\|x - x^*\|).\tag{0.4}$$

**Proposition 5.** [1] If  $F(T) = \{x \in E : Tx = x\} \neq \emptyset$ , the mapping  $T : E \rightarrow E$  is strongly hemi-contractive if and only if  $(I - T)$  is strongly quasi-accretive; it is strongly  $\varphi$ -hemi-contractive if and only if  $(I - T)$  is strongly  $\varphi$ -quasi-accretive; and  $T$  is generalized  $\Phi$ -hemi-contractive if and only if  $(I - T)$  is generalized  $\Phi$ -quasi-accretive.

**Proposition 6.** [1] Let  $E$  be a uniformly smooth real Banach space, and let  $J : E \rightarrow 2^{E^*}$  be a normalized duality mapping. Then

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y) \rangle$$

for all  $x, y \in E$ .

**Proposition 7.** [1] Let  $\{\lambda_n\}$  and  $\{\gamma_n\}$  be sequences of nonnegative numbers and  $\{\alpha_n\}$  be a sequence of positive numbers satisfying the conditions  $\sum_{n=1}^{\infty} \alpha_n = \infty$  and  $\frac{\gamma_n}{\alpha_n} \rightarrow 0$ , as  $n \rightarrow \infty$ . Let the recursive inequality

$$\lambda_{n+1} \leq \lambda_n - \alpha_n \psi(\lambda_n) + \gamma_n, n = 1, 2, \dots$$

be given where  $\psi : [0, \infty) \rightarrow [0, \infty)$  is strictly increasing continuous function such that it is positive on  $(0, \infty)$  and  $\psi(0) = 0$ . Then  $\lambda_n \rightarrow 0$ , as  $n \rightarrow \infty$ .

### 3. Main Results

In this section, we will consider to extend the result of Charles[1] to Ishikawa iterative and Ishikawa iteration process with errors under the following assumptions. First, we extend the result of Charles[1] to Ishikawa iterative.

**Theorem 8.** Suppose  $D$  is a nonempty closed convex subset of a real uniformly smooth Banach space  $E$ . Suppose  $T : D \rightarrow D$  is a bounded generalized  $\Phi$ -hemi-contractive map with strictly increasing continuous function  $\Phi : [0, \infty) \rightarrow [0, \infty)$  such that  $\Phi(0) = 0$  and  $x^* \in F(T) \neq \emptyset$ . For arbitrary  $x_0 \in D$ ,  $\{x_n\}$  be an Ishikawa iterative sequence defined by (1.1), where  $\{\alpha_n\}, \{\beta_n\} \subseteq [0, 1]$ ,  $\lim_{n \rightarrow \infty} \alpha_n = \lim_{n \rightarrow \infty} \beta_n = 0$  and  $\sum \alpha_n = \infty$ . Then, there exists a constant  $d_0 > 0$  such that if  $0 < \alpha_n, \beta_n \leq d_0$ ,  $\{x_n\}$  converges strongly to the unique fixed point  $x^*$  of  $T$ .

*Proof.* Let  $r$  be sufficiently large such that  $x_1 \in B_r(x^*)$ . Define  $G := \overline{B_r(x^*)} \cap D$ . Then, since  $T$  is bounded we have that  $(I - T)(G)$  is bounded.

Let  $M = \max\{\sup\|(I - T)x_n\|, \sup\|Ty_n - x^*\| : x_n, y_n \in G\}$ . As  $j$  is uniformly continuous on bounded subsets of  $E$ , for  $\varepsilon_0 := \frac{\Phi(\frac{r}{4})}{16M}$ , there exists a  $\delta > 0$  such that  $x, y \in D(T)$ ,  $\|x - y\| < \delta$  implies  $\|j(x) - j(y)\| < \varepsilon_0$ .

$$\text{Set } d_0 = \min\left\{1, \frac{r}{4M}, \frac{r}{2(M+r)}, \frac{\delta}{2M}, \frac{\delta}{2(M+r)}, \frac{\Phi(\frac{r}{4})}{2r^2}\right\}.$$

**Claim1:**  $\{x_n\}$  is bounded.

Suffices to show that  $x_n$  is in  $G$  for all  $n \geq 1$ . The proof is by induction. By our assumption,  $x_1 \in G$ . Suppose  $x_n \in G$ . We prove that  $x_{n+1} \in G$ . Assume for contradiction that  $x_{n+1} \notin G$ . Then, since  $x_{n+1} \in D \forall n \geq 1$ , we have that  $\|x_{n+1} - x^*\| > r$ .

We have the following estimates:

$$\begin{aligned} \|y_n - x^*\| &= \|(1 - \beta_n)x_n + \beta_n T x_n - x^*\| \\ &\leq \|x_n - x^*\| + \beta_n \|(I - T)x_n\| \\ &\leq r + d_0 M \\ &\leq \frac{3}{2} r, \end{aligned}$$

$$\begin{aligned} \|(x_n - x^*) - (y_n - x^*)\| &= \beta_n \|(I - T)x_n\| \\ &\leq d_0 M \\ &\leq \frac{\delta}{2} < \delta, \end{aligned}$$

now

$$\begin{aligned}\|x_n - x^*\| &\geq \|x_{n+1} - x^*\| - \alpha_n \|Ty_n - x_n\| \\ &\geq \|x_{n+1} - x^*\| - \alpha_n [\|Ty_n - x^*\| + \|x_n - x^*\|] \\ &\geq r - d_0(M + r) \\ &\geq r - \frac{r}{2} = \frac{r}{2},\end{aligned}$$

$$\begin{aligned}\|y_n - x^*\| &\geq \|x_n - x^*\| - \beta_n \|(I - T)x_n\| \\ &\geq \frac{r}{2} - d_0M \\ &\geq \frac{r}{4},\end{aligned}$$

and

$$\begin{aligned}\|x_{n+1} - x^*\| &\leq (1 - \alpha_n) \|x_n - x^*\| + \alpha_n \|Ty_n - x^*\| \\ &\leq (1 - \alpha_n)r + \alpha_n M \\ &\leq \frac{3}{2}r,\end{aligned}$$

$$\begin{aligned}\|(x_{n+1} - x^*) - (x_n - x^*)\| &\leq \alpha_n \|Ty_n - x_n\| \\ &\leq \alpha_n [\|Ty_n - x^*\| + \|x_n - x^*\|] \\ &\leq \alpha_n (M + r) \\ &\leq \frac{\delta}{2} < \delta,\end{aligned}$$

therefore,

$$\begin{aligned}\|j(x_n - x^*) - j(y_n - x^*)\| &< \varepsilon_0, \\ \|j(x_{n+1} - x^*) - j(x_n - x^*)\| &< \varepsilon_0.\end{aligned}$$

Using Proposition 2.6 and the above formulas, we obtain

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|(1 - \alpha_n)x_n + \alpha_n Ty_n - x^*\|^2 \\ &\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \langle Ty_n - x^*, j(y_n - x^*) \rangle \\ &\quad + 2\alpha_n \langle Ty_n - x^*, j(x_{n+1} - x^*) - j(x_n - x^*) \rangle \\ &\quad + 2\alpha_n \langle Ty_n - x^*, j(x_n - x^*) - j(y_n - x^*) \rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \left[ \|y_n - x^*\|^2 - \Phi(\|y_n - x^*\|) \right] \\ &\quad + 2\alpha_n \|Ty_n - x^*\| \|j(x_{n+1} - x^*) - j(x_n - x^*)\| \\ &\quad + 2\alpha_n \|Ty_n - x^*\| \|j(x_n - x^*) - j(y_n - x^*)\| \\ &\leq (1 - \alpha_n)^2 r^2 + 4\alpha_n \cdot M \cdot \varepsilon_0 \\ &\quad + 2\alpha_n \left[ \|y_n - x^*\|^2 - \Phi(\|y_n - x^*\|) \right],\end{aligned}\tag{0.1}$$

$$\begin{aligned}
 \|y_n - x^*\|^2 &= \|x_n - x^* - \beta_n (I - T)x_n\|^2 \\
 &\leq \|x_n - x^*\|^2 - 2\beta_n \langle (I - T)x_n, j(x_n - x^*) \rangle \\
 &\quad - 2\beta_n \langle (I - T)x_n, j(y_n - x^*) - j(x_n - x^*) \rangle \\
 &\leq \|x_n - x^*\|^2 - 2\beta_n \Phi(\|x_n - x^*\|) \\
 &\quad + 2\beta_n \|(I - T)x_n\| \|j(y_n - x^*) - j(x_n - x^*)\| \\
 &\leq r^2 + 2\beta_n \cdot M \cdot \varepsilon_0.
 \end{aligned} \tag{0.2}$$

Substitute (3.2) into (3.1), since  $0 < \alpha_n, \beta_n \leq d_0$  and  $d_0 = \min \left\{ 1, \frac{r}{4M}, \frac{r}{2(M+r)}, \frac{r}{2M}, \frac{\delta}{2(M+r)}, \frac{\Phi(\frac{r}{4})}{2r^2} \right\}$ , we have

$$\begin{aligned}
 \|x_{n+1} - x^*\|^2 &\leq (1 - \alpha_n)^2 r^2 + 4\alpha_n \cdot M \cdot \varepsilon_0 + 2\alpha_n [r^2 + 2\beta_n \cdot M \cdot \varepsilon_0 - \Phi(\|y_n - x^*\|)] \\
 &\leq r^2 + 2\alpha_n \left[ \frac{\alpha_n}{2} r^2 + 2M\varepsilon_0 + 2\beta_n M\varepsilon_0 \right] - 2\alpha_n \Phi\left(\frac{r}{4}\right) \\
 &\leq r^2 + 2\alpha_n \left[ \frac{\Phi\left(\frac{r}{4}\right)}{2} - \Phi\left(\frac{r}{4}\right) \right] \\
 &\leq r^2
 \end{aligned}$$

i.e.,  $\|x_{n+1} - x^*\| \leq r$ , a contradiction. Therefore  $x_{n+1} \in G$ . Thus by induction  $\{x_n\}$  is bounded. Then,  $\{y_n\}$ ,  $\{Ty_n\}$ ,  $\{Tx_n\}$ ,  $\{(I - T)x_n\}$  are also bounded.

**Claim2:**  $x_n \rightarrow x^*$ .

Let  $A_n = \|j(x_{n+1} - x^*) - j(x_n - x^*)\|$ ,  $B_n = \|j(x_n - x^*) - j(y_n - x^*)\|$ , Note that  $x_{n+1} - x_n \rightarrow 0, x_n - y_n \rightarrow 0$  as  $n \rightarrow \infty$  and hence by the uniform continuity of  $j$  on bounded subsets of  $E$  we have that

$$A_n \rightarrow 0, B_n \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Let  $M_1 = \max \{ \sup \|x_n - x^*\|, \sup \|y_n - x^*\|, \sup \|Ty_n - x^*\|, \sup \|(I - T)x_n\| \}$ , by (3.1), (3.2), we obtain that

$$\begin{aligned}
 \|x_{n+1} - x^*\|^2 &\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \left[ \|y_n - x^*\|^2 - \Phi(\|y_n - x^*\|) \right] \\
 &\quad + 2\alpha_n \|Ty_n - x^*\| \|j(x_{n+1} - x^*) - j(x_n - x^*)\| \\
 &\quad + 2\alpha_n \|Ty_n - x^*\| \|j(x_n - x^*) - j(y_n - x^*)\| \\
 &\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n M_1 A_n + 2\alpha_n M_1 B_n \\
 &\quad + 2\alpha_n \left[ \|y_n - x^*\|^2 - \Phi(\|y_n - x^*\|) \right]
 \end{aligned} \tag{0.3}$$

and

$$\begin{aligned}
 \|y_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - 2\beta_n \Phi(\|x_n - x^*\|) \\
 &\quad + 2\beta_n \|(I - T)x_n\| \|j(y_n - x^*) - j(x_n - x^*)\| \\
 &\leq \|x_n - x^*\|^2 + 2M_1 B_n.
 \end{aligned} \tag{0.4}$$

Taking (3.4) into (3.3),

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n M_1 A_n + 2\alpha_n M_1 B_n \\ &\quad + 2\alpha_n \left[ \|x_n - x^*\|^2 + 2M_1 B_n - \Phi(\|y_n - x^*\|) \right] \\ &\leq (1 + \alpha_n^2) \|x_n - x^*\|^2 + 2\alpha_n [M_1 A_n + 3M_1 B_n - \Phi(\|y_n - x^*\|)] \\ &\leq \|x_n - x^*\|^2 + 2\alpha_n \left[ \frac{\alpha_n}{2} M_1^2 + M_1 A_n + 3M_1 B_n - \Phi(\|y_n - x^*\|) \right] \\ &\leq \|x_n - x^*\|^2 + 2\alpha_n [Z_n - \Phi(\|y_n - x^*\|)] , \end{aligned}$$

where  $Z_n = \frac{\alpha_n}{2} M_1^2 + M_1 A_n + 3M_1 B_n \rightarrow 0$  as  $n \rightarrow \infty$ .

Set  $\inf \frac{\Phi(\|y_n - x^*\|)}{\Phi(\|x_n - x^*\|) + 1} = L > 0$  since  $\Phi$  is a strictly increasing continuous function, then  $L$  exists.

Thus

$$\Phi(\|y_n - x^*\|) \geq L\Phi(\|x_n - x^*\|) + L \geq L\Phi(\|x_n - x^*\|) .$$

Then

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \|x_n - x^*\|^2 + 2\alpha_n [Z_n - L\Phi(\|x_n - x^*\|)] \\ &\leq \|x_n - x^*\|^2 - 2\alpha_n L\Phi(\|x_n - x^*\|) + 2\alpha_n Z_n . \end{aligned} \tag{0.5}$$

Let  $\lambda_n := \|x_n - x^*\|$  and  $\rho_n = 2\alpha_n Z_n$ , then from inequality (3.5) we obtain that  $\lambda_{n+1} \leq \lambda_n - 2\alpha_n L\Phi(\lambda_n) + \rho_n$ , where  $\frac{\rho_n}{\alpha_n} \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore, the conclusion of the theorem follows from Proposition 2.7. ■

The following corollary follow trivially, since definition 2.3 and definition 2.4.

**Corollary 9.** *Suppose  $E$  is a real uniformly smooth Banach space. Suppose  $T : E \rightarrow E$  is a bounded generalized  $\Phi$ -accretive map with strictly increasing continuous function  $\Phi : [0, \infty) \rightarrow [0, \infty)$  such that  $\Phi(0) = 0$  and the solution  $x^*$  of the equation  $Tx = 0$  exists. For arbitrary  $x_0 \in E$ , the sequence  $\{x_n\}$  in  $E$  is defined by*

$$\begin{cases} x_{n+1} = (1 - \alpha_n) x_n - \alpha_n T y_n \\ y_n = (1 - \beta_n) x_n + \beta_n T x_n, \quad n \geq 0 \end{cases}$$

where  $\{\alpha_n\}, \{\beta_n\} \subseteq [0, 1]$ ,  $\lim_{n \rightarrow \infty} \alpha_n = \lim_{n \rightarrow \infty} \beta_n = 0$  and  $\sum \alpha_n = \infty$ . Then, there exists a constant  $d_0 > 0$  such that if  $0 < \alpha_n, \beta_n \leq d_0$ ,  $\{x_n\}$  converges strongly to the unique solution of  $Tx = 0$ .

Now, we consider to generalize to a more general case, we extend the result of Charles[1] to Ishikawa iteration process with errors as follows.

**Theorem 10.** *Suppose  $D$  is a nonempty closed convex subset of a real uniformly smooth Banach space  $E$ . Suppose  $T : D \rightarrow D$  is a bounded generalized  $\Phi$ -hemi-contractive map with strictly increasing continuous function  $\Phi : [0, \infty) \rightarrow [0, \infty)$  such that  $\Phi(0) = 0$  and  $x^* \in F(T) \neq \emptyset$ . For arbitrary  $x_0 \in D$ ,  $\{x_n\}$  be an Ishikawa iteration process with errors defined by (1.2), where  $\{\beta_n\}, \{\gamma_n\}, \{a_n\}, \{b_n\} \subseteq [0, 1]$ ,  $\lim_{n \rightarrow \infty} \beta_n = \lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} a_n = 0$ ,  $\gamma_n = o(\beta_n)$  and  $\sum \beta_n = \infty$ . Then, there exists a constant  $d_0 > 0$  such that if  $0 < \beta_n, \gamma_n, a_n, b_n \leq d_0$ ,  $\{x_n\}$  converges strongly to the unique fixed point  $x^*$  of  $T$ .*

*Proof.* Let  $r$  be sufficiently large such that  $x_1 \in B_r(x^*)$ . Define  $G := \overline{B_r(x^*)} \cap D$ . Then, since  $T$  is bounded we have that  $(I - T)(G)$  is bounded.

$$\begin{aligned} \text{Let } M &= \sup \{ \|(I - T)x_n\| : x_n \in G \} + \sup \{ \|x_n - v_n\| : x_n \in G \} \\ &\quad + \sup \{ \|Ty_n - x^*\| + 1 : y_n \in G \} + \sup \{ \|u_n - x^*\| \}. \end{aligned}$$

As  $j$  is uniformly continuous on bounded subsets of  $E$ , for  $\varepsilon_0 := \frac{\Phi(\frac{r}{32M})}{32M}$ , there exists a  $\delta > 0$  such that  $x, y \in D(T), \|x - y\| < \delta$  implies  $\|j(x) - j(y)\| < \varepsilon_0$ .

Set  $d_0 = \min \left\{ 1, \frac{r}{8M}, \frac{\delta}{2(2M+r)}, \frac{r}{2(2M+r)}, \frac{\delta}{4M}, \frac{\Phi(\frac{r}{4})}{4r^2}, \frac{\Phi(\frac{r}{4})}{20Mr}, \frac{\Phi^2(\frac{r}{4})}{48Mr^3} \right\}$ .

**Claim1:**  $\{x_n\}$  is bounded.

Suffices to show that  $x_n$  is in  $G$  for all  $n \geq 1$ . The proof is by induction. By our assumption,  $x_1 \in G$ . Suppose  $x_n \in G$ . We prove that  $x_{n+1} \in G$ . Assume for contradiction that  $x_{n+1} \notin G$ . Then, since  $x_{n+1} \in D$ ,  $\forall n \geq 1$ , we have that  $\|x_{n+1} - x^*\| > r$ .

We have the following estimates:

$$\begin{aligned} \|y_n - x^*\| &= \|x_n - x^* - a_n(I - T)x_n - b_n(x_n - v_n)\| \\ &\leq \|x_n - x^*\| + a_n\|(I - T)x_n\| + b_n\|x_n - v_n\| \\ &\leq r + d_0(M + M) \\ &\leq \frac{5}{4}r, \end{aligned}$$

$$\begin{aligned} \|(x_n - x^*) - (y_n - x^*)\| &\leq a_n\|(I - T)x_n\| + b_n\|x_n - v_n\| \\ &\leq d_0(M + M) \\ &\leq \frac{\delta}{2} < \delta, \end{aligned}$$

now

$$\begin{aligned} \|x_n - x^*\| &\geq \|x_{n+1} - x^*\| - \beta_n\|Ty_n - x_n\| - \gamma_n\|u_n - x_n\| \\ &\geq \|x_{n+1} - x^*\| - \beta_n(\|Ty_n - x^*\| + \|x_n - x^*\|) - \gamma_n\|u_n - x_n\| \\ &\geq r - d_0(M + r + M) \\ &\geq \frac{r}{2}, \end{aligned}$$

$$\begin{aligned} \|y_n - x^*\| &\geq \|x_n - x^*\| - a_n\|(I - T)x_n\| - b_n\|x_n - v_n\| \\ &\geq \|x_n - x^*\| - d_0(M + M) \\ &\geq \frac{r}{2} - \frac{r}{4} = \frac{r}{4}, \end{aligned}$$

and

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq \|x_n - x^*\| + \beta_n\|Ty_n - x_n\| + \gamma_n\|u_n - x_n\| \\ &\leq \|x_n - x^*\| + \beta_n(\|Ty_n - x^*\| + \|x_n - x^*\|) + \gamma_n\|u_n - x_n\| \\ &\leq r + d_0(M + r + M) \\ &\leq \frac{3}{2}r, \end{aligned}$$

$$\begin{aligned} \|(x_{n+1} - x^*) - (x_n - x^*)\| &\leq \beta_n\|Ty_n - x_n\| + \gamma_n\|u_n - x_n\| \\ &\leq \beta_n(\|Ty_n - x^*\| + \|x_n - x^*\|) + \gamma_n\|u_n - x_n\| \\ &\leq d_0(M + r + M) \\ &\leq \frac{\delta}{2} < \delta, \end{aligned}$$

therefore,

$$\|j(x_n - x^*) - j(y_n - x^*)\| < \varepsilon_0,$$

$$\|j(x_{n+1} - x^*) - j(x_n - x^*)\| < \varepsilon_0 .$$

Using Proposition 2.6 and the above formulas, we obtain

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|(1 - \beta_n - \gamma_n)x_n + \beta_n T y_n + \gamma_n u_n - x^*\|^2 \\ &\leq (1 - \beta_n - \gamma_n)^2 \|x_n - x^*\|^2 + 2\beta_n \langle T y_n - x^*, j(x_{n+1} - x^*) \rangle \\ &\quad + 2\gamma_n \langle u_n - x^*, j(x_{n+1} - x^*) \rangle \\ &\leq (1 - \beta_n)^2 \|x_n - x^*\|^2 + 2\beta_n \langle T y_n - x^*, j(y_n - x^*) \rangle \\ &\quad + 2\beta_n \langle T y_n - x^*, j(x_{n+1} - x^*) - j(x_n - x^*) \rangle \\ &\quad + 2\beta_n \langle T y_n - x^*, j(x_n - x^*) - j(y_n - x^*) \rangle \\ &\quad + 2\gamma_n \langle u_n - x^*, j(x_{n+1} - x^*) \rangle \\ &\leq (1 - \beta_n)^2 \|x_n - x^*\|^2 + 2\beta_n \left[ \|y_n - x^*\|^2 - \Phi(\|y_n - x^*\|) \right] \\ &\quad + 2\beta_n \|T y_n - x^*\| \|j(x_{n+1} - x^*) - j(x_n - x^*)\| \\ &\quad + 2\beta_n \|T y_n - x^*\| \|j(x_n - x^*) - j(y_n - x^*)\| \\ &\quad + 2\gamma_n \|u_n - x^*\| \|x_{n+1} - x^*\| \\ &\leq (1 - \beta_n)^2 r^2 + 4\beta_n M \varepsilon_0 + 2\gamma_n \cdot M \cdot \frac{3}{2} r \\ &\quad + 2\beta_n \left[ \|y_n - x^*\|^2 - \Phi(\|y_n - x^*\|) \right] , \end{aligned} \tag{0.6}$$

$$\begin{aligned} \|y_n - x^*\|^2 &= \|x_n - x^* - a_n(I - T)x_n - b_n(x_n - v_n)\|^2 \\ &\leq \|x_n - x^*\|^2 - 2a_n \langle (I - T)x_n, j(y_n - x^*) - j(x_n - x^*) \rangle \\ &\quad - 2a_n \langle (I - T)x_n, j(x_n - x^*) \rangle - 2b_n \langle x_n - v_n, j(y_n - x^*) \rangle \\ &\leq \|x_n - x^*\|^2 - 2a_n \Phi(\|x_n - x^*\|) + 2b_n \|x_n - v_n\| \|y_n - x^*\| \\ &\quad + 2a_n \|(I - T)x_n\| \|j(y_n - x^*) - j(x_n - x^*)\| \\ &\leq r^2 + 2M \varepsilon_0 + 2b_n M \cdot \frac{5}{4} r . \end{aligned} \tag{0.7}$$

Substitute (3.7) into (3.6), since  $0 < \beta_n, \gamma_n, a_n, b_n \leq d_0, \gamma_n = o(\beta_n)$  and  $d_0 = \min \left\{ 1, \frac{r}{8M}, \frac{\delta}{2(2M+r)}, \frac{r}{2(2M+r)}, \frac{\delta}{4M}, \frac{\Phi}{r} \right\}$ , we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq (1 - \beta_n)^2 r^2 + 2\beta_n \left[ r^2 + 2M \varepsilon_0 + \frac{5}{2} b_n M r - \Phi(\|y_n - x^*\|) \right] \\ &\quad + 4\beta_n \cdot M \cdot \varepsilon_0 + 3\gamma_n \cdot M \cdot r \\ &\leq r^2 + 2\beta_n \left[ \frac{\beta_n}{2} r^2 + 4M \varepsilon_0 + \frac{5}{2} b_n M r + \frac{3\gamma_n}{2\beta_n} M r \right] - 2\beta_n \Phi \left( \frac{r}{4} \right) \\ &\leq r^2 + 2\beta_n \left[ \frac{\Phi \left( \frac{r}{4} \right)}{2} - \Phi \left( \frac{r}{4} \right) \right] \\ &\leq r^2 \end{aligned}$$

i.e.,  $\|x_{n+1} - x^*\| \leq r$ , a contradiction. Therefore  $x_{n+1} \in G$ . Thus by induction  $\{x_n\}$  is bounded. Then,  $\{y_n\}, \{T y_n\}, \{T x_n\}, \{(I - T)x_n\}$  are also bounded.

**Claim2:**  $x_n \rightarrow x^*$ .

Let  $A_n = \|j(x_{n+1} - x^*) - j(x_n - x^*)\|$ ,  $B_n = \|j(x_n - x^*) - j(y_n - x^*)\|$ , Note that  $x_{n+1} - x_n \rightarrow 0, x_n - y_n \rightarrow 0$  as  $n \rightarrow \infty$  and hence by the uniform continuity of  $j$  on bounded subsets of  $E$  we have that



$$A_n \rightarrow 0, B_n \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Let

$$M_1 = \max \left\{ \begin{array}{l} \sup \|x_n - x^*\|, \sup \|y_n - x^*\|, \sup \|(I - T)x_n\|, \\ \sup \|Ty_n - x^*\|, \sup \|u_n - x^*\|, \sup \|x_n - v_n\| \end{array} \right\},$$

by (3.6), (3.7), we obtain that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq (1 - \beta_n)^2 \|x_n - x^*\|^2 + 2\beta_n \left[ \|y_n - x^*\|^2 - \Phi(\|y_n - x^*\|) \right] \\ &\quad + 2\beta_n \|Ty_n - x^*\| \|j(x_{n+1} - x^*) - j(x_n - x^*)\| \\ &\quad + 2\beta_n \|Ty_n - x^*\| \|j(x_n - x^*) - j(y_n - x^*)\| \\ &\quad + 2\gamma_n \|u_n - x^*\| \|x_{n+1} - x^*\| \\ &\leq (1 - \beta_n)^2 \|x_n - x^*\|^2 + 2\beta_n M_1 A_n + 2\beta_n M_1 B_n + 2\gamma_n M_1^2 \\ &\quad + 2\beta_n \left[ \|y_n - x^*\|^2 - \Phi(\|y_n - x^*\|) \right] \end{aligned} \tag{0.8}$$

and

$$\begin{aligned} \|y_n - x^*\|^2 &\leq \|x_n - x^*\|^2 + 2a_n \|(I - T)x_n\| \|j(y_n - x^*) - j(x_n - x^*)\| \\ &\quad - 2a_n \Phi(\|x_n - x^*\|) + 2b_n \|x_n - v_n\| \|y_n - x^*\| \\ &\leq \|x_n - x^*\|^2 + 2M_1 B_n + 2b_n M_1^2. \end{aligned} \tag{0.9}$$

Taking (3.9) into (3.8),

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq (1 - \beta_n)^2 \|x_n - x^*\|^2 + 2\beta_n M_1 A_n + 2\beta_n M_1 B_n + 2\gamma_n M_1^2 \\ &\quad + 2\beta_n \left[ \|x_n - x^*\|^2 + 2M_1 B_n + 2b_n M_1^2 - \Phi(\|y_n - x^*\|) \right] \\ &\leq \|x_n - x^*\|^2 + 2\beta_n \left[ \frac{\beta_n}{2} M_1^2 + M_1 A_n + 3M_1 B_n + 2b_n M_1^2 + \frac{\gamma_n}{\beta_n} M_1^2 - \Phi(\|y_n - x^*\|) \right] \\ &\leq \|x_n - x^*\|^2 + 2\beta_n [Z_n - \Phi(\|y_n - x^*\|)], \end{aligned}$$

where  $Z_n = \frac{\beta_n}{2} M_1^2 + M_1 A_n + 3M_1 B_n + 2b_n M_1^2 + \frac{\gamma_n}{\beta_n} M_1^2 \rightarrow 0$  as  $n \rightarrow \infty$ .

Set  $\inf \frac{\Phi(\|y_n - x^*\|)}{\Phi(\|x_n - x^*\|) + 1} = L > 0$ , since  $\Phi$  is a strictly increasing continuous function, then  $L$  exists.

Thus

$$\Phi(\|y_n - x^*\|) \geq L\Phi(\|x_n - x^*\|) + L \geq L\Phi(\|x_n - x^*\|).$$

Then

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \|x_n - x^*\|^2 + 2\beta_n [Z_n - L\Phi(\|x_n - x^*\|)] \\ &\leq \|x_n - x^*\|^2 - 2\beta_n L\Phi(\|x_n - x^*\|) + 2\beta_n Z_n. \end{aligned} \tag{0.10}$$

Let  $\lambda_n := \|x_n - x^*\|$  and  $\rho_n = 2\beta_n Z_n$ , then from inequality (3.10) we obtain that  $\lambda_{n+1} \leq \lambda_n - 2\beta_n L\Phi(\lambda_n) + \rho_n$ , where  $\frac{\rho_n}{\beta_n} \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore, the conclusion of the theorem follows from Proposition 2.7. ■

The following corollary follow trivially, since definition 2.3 and definition 2.4.

**Corollary 11.** *Suppose  $E$  is a real uniformly smooth Banach space. Suppose  $T : E \rightarrow E$  is a bounded generalized  $\Phi$ -accretive map with strictly increasing continuous function  $\Phi : [0, \infty) \rightarrow [0, \infty)$  such that  $\Phi(0) = 0$  and the solution  $x^*$  of the equation  $Tx = 0$  exists. For arbitrary  $x_0 \in E$ , the sequence  $\{x_n\}$  in  $E$  is defined by*

$$\begin{cases} x_{n+1} = (1 - \beta_n - \gamma_n)x_n - \beta_n T y_n + \gamma_n u_n \\ y_n = (1 - a_n - b_n)x_n + a_n T x_n + b_n v_n, \quad n \geq 0, \end{cases}$$

where  $\{\beta_n\}$ ,  $\{\gamma_n\}$ ,  $\{a_n\}$ ,  $\{b_n\} \subseteq [0, 1]$ ,  $\lim_{n \rightarrow \infty} \beta_n = \lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} a_n = 0$ ,  $\gamma_n = o(\beta_n)$  and  $\sum \beta_n = \infty$ . Then, there exists a constant  $d_0 > 0$  such that if  $0 < \beta_n, \gamma_n, a_n, b_n \leq d_0$ ,  $\{x_n\}$  converges strongly to the unique solution of  $Tx = 0$ .

This work is supported by Applied Basic Research Foundation of Sichuan Province of China (Grant No. 2018JY0169).

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