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A note on maximal regularity in relation with measure theory

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Abstract

We present a synthetic method based on double approximation (multilevel approach) to state the maximal regularity of non-autonomous evolution equations, of non-autonomous evolution problems, mainly those driven by closed operators arising from sesquilinear forms, which enjoy some analytic properties. The infinite product of semigroups and elementary results, mainly some classical remarkable sets, in measure theory will play a central role in technical calculus.

Key words and phrases. Approximation techniques; Non-autonomous equations; maximal regularity; semigroups.

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1. Introduction

Maximal regularity continues to grow in importance, brilliance and supremacy in the treatment of non-autonomous evolutionary equations. Many fields are concerned by its robust results: control theory such as [15] and more recently [5] or [9] (mainly the important resut obtained Theorem 3.9), general operator theory such as [24], [2] or [16]. Among other multiple papers interested recently by maximal regularity, we cite [15] where some deterministic results on controllability were developed in infinite dimension spaces and, in a stochastic framework, we refer to [27] and [6].

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From the definition of *maximal regularity* for a given operator A that governs the evolution

$$\begin{cases} \dot{u} + Au = f \quad \text{a.e. on} \quad (0,\tau), \\ u(0) = 0. \end{cases} \tag{1}$$

it should be understood that the derivative $\dot{u}$ and the forcing f have the same regularity as the result of operation action Au. In more comprehensive words, the three mappings $t \mapsto \dot{u}(t)$, $t \mapsto Au(t)$ and $t \mapsto f(t)$ belong to the same functional space.

One might find it advantageous to focus on maximal regularity in the natural spaces of continuous and/or continuously differentiable functions $\mathcal{C}([0, \tau], X) \cap \mathcal{C}^1([0, \tau], D)$. Unfortunately, Baillon proved that this direction is nonsensical, since the class of operators concerned by this strong property is restricted to those which are bounded. We refer to [10] where the authors clarify and give a restitution of this important result.

To dodge this unpredictable situation, studies and research were focused on the L^p-maximal regularity which gives a more suitable frame to deal with natural function spaces, mainly Hölder ones $Lip_\alpha[0, \tau]$; $0 < \alpha < 1$ (see comments in [12]), or Sobolev spaces $W^{p,q}[0, \tau]$. We will give details after the following definition, which makes this notion more precise.

Let τ a positive real integer and consider D a Banach space that is continuously and densely embedded into another Banach space X. We denote this situation by $D \hookrightarrow X$.

Definition 1.1. *Consider a real $p > 1$ and a closed operator $A \in \mathcal{L}(D, X)$. The operator A is said maximal regular or simply A possesses the L^p-maximal regularity property if, for every $f \in L^p(0, \tau; X)$ there exists a unique $u \in W^{1,p}(0, \tau; X) \cap L^p(0, \tau; D)$ verifying (1).*

In this case, we say that A is of category $\mathcal{MR}_p$ $(p \in (1, \infty))$

More space investigations were recently conducted to explore new regularity possibilities in time as developed, for instance in [2] and [3] for a deep treatment in the cases of fractional Lebesgue and Besov spaces. In this optic, Kalton and Portal in [19] treated L^p-maximal regularity of power-bounded operators and related the discrete to the continuous time problem for analytic semigroups. They gave an exhaustive characterization of operators with L^1 and $L^\infty-$maximal regularity. For power-bounded operators, they introduce an unconditional form of Ritt's condition. They describe completely this condition in the case of Banach spaces which are L^1-spaces, $C(K)-$spaces and/or Hilbert spaces. Thereafter, they related this important condition with the existence of an $H^\infty-$calculus.

Maximal regularity has numerous applications, particularly in partial differential equations (PDEs) and nonlinear evolution equations, where the existence, uniqueness, and regularity of solutions naturally follow from this property. Before developing this aspect, we make precise the different function spaces appearing in Definition (1.1).

The set $\mathcal{MR}_p = W^{1,p}(0, \tau; X) \cap L^p(0, \tau; D)$ is defined as a maximal regularity space. The traces of its elements, i.e. $Tr = \{u(0); u \in \mathcal{MR}_p\}$ or indifferently $Tr = \{u(t_0), u \in \mathcal{MR}_p\}$, is the well-known *trace* space.

When it is endowed with the obvious norm$\|.\|_{\mathcal{MR}_p}$ defined by: $\|u\|_{\mathcal{MR}_p} = \|u\|_{W^{1,p}(0,\tau;X)} + \|u\|_{L^p(0,\tau,D)}$, the space $\mathcal{MR}_p$ is a Banach one. We will, in our future works, give more detail on the trace space Tr. For now, let's just say that Tr is complete for the norm$\|x\| = \inf\{\|u\|_{\mathcal{MR}_p}, u \in \mathcal{MR}_p \text{ and } u(0) = x\}$. We do not mention here the possibility of identifying the *trace* space with a Banach interpolation space between D and X.

The maximal regularity, as defined above, gives an efficient tool to establish the well-posedness of

$$\begin{cases} \dot{u} + Au = 0 \quad \text{a.e. on} \quad (0,\tau), \\ u(0) = u_0, \quad u_0 \in Tr. \end{cases}$$

which is a homogeneous Cauchy problem.

Fore more details, we refer to first part of [7] where a nice bridge between homogeneous and inhomogeneous Cauchy problems was built for a single operator (i.e., A is unaffected by time).

A real challenge was how to generalize such results to the non-autonomous case. This means how to obtain similar positive results for the problem

$$(NCP)\begin{cases}\dot{u}+A(t)u=f & \text{a.e. on} \quad (0,\tau),\\ u(0)=u_0\end{cases} \tag{2}$$

When $A(t) = A$ is autonomous, the problems (1) and (2) are obviously equivalent. In other cases, some regularity properties of mapping $t \mapsto A(t)$ are required to establish well-posedness of (2). If $t \mapsto A(t)$ varies slightly, say piecewise constant, a classical splicing technique ensures the well-posedness of (2) under the simple hypothesis of punctual maximal regularity, see for instance [28].

In the general case where $t \mapsto A(t)$ is a continuous function for closed operators, Acquistapace and Terreni [1] pioneered a cohesive method for addressing abstract linear non-autonomous parabolic equations. Their main tool was the bounded Yosida approximation technique of closed operators. In the same direction, Prüss and Schnaubelt [25] integrated the problem when $A(.)$ is continuous, Arendt [7] considered the case when it is relatively continuous, El-Mennaoui and Keyantuo [11] initiated the π−integration theory for a new approximation technique. The definition of π−integration was recently revisited and massively commented in [17]. Sani, see [28], utilized the frozen coefficients method, as initiated in [1], to develop the integral product (or π−integration) to prove otherwise the historical Lions theorem when the ambient space is a Hilbert one.

In fact, it is established that when $X = H$ is a Hilbert space, the generation of a holomorphic semigroup is sufficient to guarantee the maximal regularity of a given operator A. The equivalence between the two properties (generation of analytic semigroups and maximal regularity) was discussed for a long time. According to [4], de Simon was historically the first to demonstrate that maximal regularity implies the generation of a holomorphic semigroup. In a general Banach space, this equivalence does not hold. Fackler [13] provides a counterexample that illustrates the findings of Kalton and Lancien [20] regarding the failure of maximal regularity in various classes of Banach spaces, including L^p-spaces with $1 < p \neq 2 < +\infty$, even for generators of holomorphic semigroups. It is worth mentioning that the maximal regularity does not depend on the parameter p. This means that if A possesses $\mathcal{MR}_p$ for some $p > 1$, then it will be the case for all $p > 1$. So we will denote indifferently $\mathcal{MR}_p$ or just $\mathcal{MR}$.

2. Level 1: Classical Integral Product Approximation

The integral product, as summarized in the work of A. Slavick [26] and further elaborated by Laasri [21] and Sani [28], has proven to be an effective analytical tool for approximating non-autonomous evolution equations. The most significant outcomes are achieved when the operators $A(t)$ that govern the evolution are derived from sesquilinear forms [8]. Hence, we focus on this general scenario and consider only the Hilbertian framework (V, H).

Consider $\tau > 0$ and $\Theta := (0 = \theta_0 < \theta_1 < \ldots < \theta_{n+1} = \tau)$ a subdivision of $[0, \tau]$. We approximate (2) using (3) below, which is obtained by freezing the generators $\mathcal{A}(t)$ over the intervals $[\theta_i, \theta_{i+1}[$for $0 \leq i \leq n$. More specifically, define $\mathcal{A}_\Theta : [0, \tau] \to \mathcal{L}(V, V')$ as follows:

$$\mathcal{A}_\Theta(t) := \begin{cases}\mathcal{A}_i & \text{for } \theta_i \leq t < \theta_{i+1},\\ \mathcal{A}_n & \text{for } t = \tau,\end{cases}$$

where:

$$\mathcal{A}_i u := \frac{1}{\theta_{i+1}-\theta_i}\int_{\theta_i}^{\theta_{i+1}} \mathcal{A}(s)u\,ds \quad (\text{for all } u \in V, \text{and } 0 \leq i \leq n).$$

The legitimacy of the integral on the right-hand side follows from the fact that the mapping $t \longrightarrow \mathcal{A}(t)$ is strongly Bochner-integrable.

It has been shown (see [28]) that for all $u_0 \in H$ and $f \in L^2(0, \tau ; V')$, the non-autonomous problem

$$\begin{cases} \dot{u}_\Theta(t) + \mathcal{A}_\Theta(t)u_\Theta(t) = f(t), \\ u_\Theta(0) = u_0. \end{cases} \tag{3}$$

has a unique solution $u \in \mathcal{MR}(V, V')$, which converges in the same space as $|\Theta|$ tends towards 0, and the limit $u := \lim_{|\Theta| \longrightarrow 0} u_\Theta$ uniquely solves (2) in V'.

According to [28], this result was extended to $\mathcal{MR}(V, H)$. The significance of solutions in $\mathcal{MR}(V, H)$ lies in the necessity of finding realistic solutions in H without resorting to the extrapolated space V'.

An intriguing application of this method is demonstrated in [28], where the invariance of closed convex sets in H by the solution of (2) is proven.

3. Level 2: Mobile-Mean-Integral Approximation

In the previous section, the approximation is, physically speaking, in the steady state since the approximate operator: A_i defined by $\mathcal{A}_i u := \frac{1}{\theta_{i+1} - \theta_i} \int_{\theta_i}^{\theta_{i+1}} \mathcal{A}(s)u ds$ is frozen at the cell $[\theta_i; \theta_{i+1}]$ $\forall i \in \{0, ..., n\}$.

Here, we introduce a more efficient method to approximate the problem (2).

Consider the forms $(a(t, ., .))t\in[0;\tau]$ that fulfills the following assumptions:

(H_1): The forms have an identical domain, i.e., $\forall t \geq 0$ $D(\mathfrak{a}(t; ., .)) = V$.

(H_2): The forms are V −bounded: there exists $M > 0$ such that for all $t \in [0, \tau]$ and $u, v \in V$, we have $|\mathfrak{a}(t; u, v)| \leq M \|u\|_V \|v\|_V$.

(H_3): There are $\alpha > 0$ and $\beta \in \mathbb{R}$ such that for every $t \in [0, \tau]$ and all $u, v \in V$ we have: $\alpha \|u\|_V \leq Re\mathfrak{a}(t; u, u) + \beta \|u\|_H$ which expresses the uniform quasi-coerciveness of $(\mathfrak{a}(t, ., .))_{t\in[0,\tau]}$.

We remind (see[31]) and clarify the sense of the association of $(\mathfrak{a}(t, ., .))_{t\in[0;\tau]}$ with well-known operators on the spaces V and H: when u in V and $t \in [0; \tau]$ are fixed, the mapping $v \mapsto \mathfrak{a}(t, u, v)$ becomes an anti-linear functional on V. Therefore, by classical Riesz theorem, there exists a unique w_u^t in V' such that $\forall y \in V$, we have $\mathfrak{a}(t, u, y) = \langle w_u^t, y\rangle$. Consequently, we define the operator $\mathcal{A}(t) : V \to V'$ by setting $\mathcal{A}(t)(u) = w_u^t$. This yields

$$\forall (u, v) \in V^2 : \mathfrak{a}(t, u, v) = \langle \mathcal{A}(t)u, v\rangle.$$

Indeed, the main focus is on H rather than V', which we only invoke for theoretical considerations, so it is instructive to define the part of $\mathcal{A}(t)$ in the ambient space H. Let's denote this part as $A(t)$, and clarify the practical connection between $\mathcal{A}(t)$ and $A(t)$.

As commonly understood, the domain $D(A(t))$ of $A(t)$ is defined by

$$u \in D(A(t)) \iff u \in D(\mathcal{A}(t)) \text{ and } \mathcal{A}(t)u \in H,$$

and for each $u \in D(A(t))$, we have $\mathcal{A}(t)u = A(t)u$.

It is well-known that the operators $-\mathcal{A}(t)$ and $-A(t)$ generate strongly semigroups $(e^{-s\mathcal{A}(t)})_{t\geq 0}$ and $(e^{-sA(t)})_{t\geq 0}$ respectively on V' and H, with the latter being the restriction of the former to H.

Here, we introduce the concept of mobile means to improve the approximation of the problem (1)

Definition 3.1. *Let $\delta > 0$. For every sesquilinear form $\mathfrak{a}(., ., .)$ defined on $[0, \tau] \times V^2 \to \mathbb{C}$, the mobile mean at time $t \in [0, \tau[$ is defined by*

$$\mathfrak{a}_\delta(t,u,v) = \frac{1}{\delta}\int_t^{t+\delta} \mathfrak{a}(s,u,v)ds \quad and \quad \mathfrak{a}(t,u,v) = \mathfrak{a}(\tau,u,v) \; if \; t \geq \tau.$$

By applying the same reasoning as in [28], it is straightforward to prove that for every $t > 0$:

i): The form $\mathfrak{a}_\delta(t, ., .)$ is associated with the operators

$$\mathcal{A}_\delta(t) = \frac{1}{\delta}\int_t^{t+\delta} \mathcal{A}(s,u,v)ds \text{ and } A_\delta(t) = \frac{1}{\delta}\int_t^{t+\delta} A(s,u,v)ds.$$

ii): The form $\mathfrak{a}_\delta(t, ., .)$ is bounded and coercive using the same constants M and α that characterize the forms $\mathfrak{a}(t, ., .)$.

For our purposes, the crucial property of the form $\mathfrak{a}_\delta(t, ., .)$ is

Proposition 3.2. *The operators $\mathcal{A}_\delta(t, ., .)$ associated with $\mathfrak{a}_\delta(t, ., .)$ satisfy*

$$|\mathcal{A}_\delta(t,.,.) - \mathcal{A}_\delta(s,.,.)| \leq \frac{2M}{\delta}|t-s|$$

This simple and surprising result shows that the mapping $t \mapsto \mathcal{A}_\delta(t, ., .)$ is Lipschitz continuous, and its individual operators have the maximal regularity property. On one hand, it permits a simple proof of uniqueness (see comments on the explicit formula (5) bellow) and on the other hand, it warrants the opportunity to apply Prüss-Schnaubelt results [25] on continuous families of maximal regular operators. According to this latter paper, the maximal regularity $(\mathcal{V}, \mathcal{V}')$ is obtained in the following sense:

Theorem 3.3. *Let $\delta \in]0, \delta_0]$; $\epsilon > 0$ and let $A_\delta : [0, \epsilon] \to \mathcal{L}(V, V')$ the function associated with: $a_\delta(t, ., .) : V \times V \to \mathbb{C}$. Then for each $f \in L^p([0, \epsilon], V')$ and $u_0 \in (V',V)_{1-\frac{1}{p},p}$ there is a unique function $u_\delta \in W^{1,p}([0, \epsilon], V') \cap L^p([0, \epsilon], V)$ that uniquely solves the problem:*

$$(P_\delta): \quad \begin{cases} \dot{u}_\delta(t) + A_\delta(t)u_\delta(t) = f(t) \text{ a.e. } t \in [0,\epsilon], \\ u_\delta(0) = u_0. \end{cases}$$

and there is a constant c independent on u_0 and f, depending solely on δ and p, such that

$$\|u_\delta\|_{L^p([0,\tau];\mathcal{V}')} + \|u_\delta\|_{L^p([0,\tau];\mathcal{V})} + \|\dot{u}_\delta\|_{L^p([0,\tau];\mathcal{V}')} \leq c(\|u_0\|_{1-\frac{1}{p},p} + \|f\|_{L^p([0,\tau];\mathcal{V}')})$$

In fact, always according to (3.2), and taking into account the important result established in [28] or [30], it is worth mentioning the following (H, V) regularity:

Theorem 3.4. *Consider a symmetric and Lipschitz continuous $\mathfrak{a}$ and consider also:*

$$f \in L^2([0, \tau]; H),\ u_0 \in V.$$

Then, the solution u_δ of (P_δ), converges weakly in $\mathcal{MR}(V, H)$ as $\delta \longrightarrow 0$; moreover, $u := \lim_{\delta \to 0} u_\delta$ uniquely solves (1), and

$$\|u\|_{\mathcal{MR}(V,H)} \leq k\left[\|u_0\|_V + \|f\|_{L^2(0,\tau;H)}\right], \tag{4}$$

in which the constant k depends only on α, c, and M.

We will not give more details on theorems (3.3) and (3.4) since the regularity of the approximate problems (P_λ) and (P_δ) warrant sufficiently of the existence and uniqueness, thanks, among others, to [22] and [28]. It is more fruitful to express the uniqueness, obtained through an explicit solution formula in $\mathcal{MR}(V, V')$ suggested in [23, Chapter 5] as follows:

$$\begin{cases} u_\delta(t) = U^\delta_{u_0}(t) + U^\delta_f(t) + U^\delta u_\delta(t), \\ U^\delta_{u_0}(t) = e^{-tA_\delta(t)}u_0 \\ U^\delta_f(t) = \int_0^t e^{(t-s)A_\delta(t)} f(s)ds \\ U^\delta u_\delta(t) = \int_0^t e^{(t-s)A_\delta(t)}(A_\delta(t) - A_\delta(s))u_\delta(s)ds \end{cases} \tag{5}$$

Using the classical estimate asserting that for every holomorphic semigroup $\mathcal{T}(t)_{t\geq 0}$, one can find a constant $k > 0$ for which

$$\| t\mathcal{T}(t) \| \leq k$$

and thanks to (3.2), one proves easily that the solution of (P_δ) is unique. Let us consider u_δ and $u_{\delta'}$ two eventual solutions of (P_δ) and denote (v_δ) their difference. The two first terms $U^\delta_{u_0}$ and U^δ_f are the same, so one obtains immediately for any ξ large enough that:

$$v_\delta(t) = \int_0^t e^{-(t-s)(A_\delta(t)+\xi I_V)}(A_\delta(t) - A_\delta(s))v_\delta(s)ds.$$

So

$$\|v_\delta\|_{\mathcal{V}} \leq \frac{2Mk}{c}\int_0^t e^{-(t-s)\xi}\|v_\delta(s)\|_{\mathcal{V}}\, ds = \frac{2Mk}{c}(\exp(-(.)\xi) \star \|v_\delta\|(.)\mathbb{1}_{[0,t]}).$$

Gronwall lemma implies that $\lim_{\xi\to\infty} \|v_\delta\|_{\mathcal{V}} = 0$. But this quantity does not depend on ξ, so it is null everywhere, which establishes the desired uniqueness.

In the two following sections, we discuss a new technique to overcome, in a general context, subsets of degeneracy when the mapping $t \mapsto A(t)$ is just assumed to be measurable. We explain how to overcome the overlapping of mutual actions of the operators $A(t)$ at different non-ordered values of time t. We start by recalling, via two classical examples in measure theory, that pathological sets are abundant and may affect the evolution. We explain in the first subsection the opportunity for our investigation on non-measurable sets.

4. Some Pathologic Sets

To make the current paper autonomous and self-contained, we recall in detail some important results from measure theory that are necessary to understand the main result exposed here. In the literature, it is known that a Cantor set is non-countable, but its Lebesgue measure is null. In this section, we give a construction of such measurable sets that share the same property.

4.1. Why Non Measurable Sets

Assume that, according to the next subsection, we have at our disposal such a set (Besicovitch one) denote B. Consider a regular operator A and for each singular time t in B, the operator $A(t) = A$. Now, consider the hole evolution described by $\sum_{t\in B}\mathbb{1}_{\{t\}}(.)A = [\sup_{t\in B}\mathbb{1}_{\{t\}}(.)]A = \mathbb{1}_B(.)A$. So the function indicator $t \mapsto \mathbb{1}_B(t)$ is not measurable, and all results on maximal regularity are ruined.

This construction explains, in elementary words, the opportunity to recall the following results and the expediency of our construction.

4.2. Besicovitch Sets

The main source that we reshuffle to make it easily accessible is the recent paper [18].

Given the importance and originality of this section, we have taken up the work done in the following reference: [14].

Besicovitch discovered these sets, later baptized Besicovitch sets, in 1919 while working on problems of integration on the plan.

Definition 4.1. *A set $B \subset \mathbb{R}^2$ is qualified as a Besicovitch set if:*

- i): *B contains one unit segment in each direction.*
- ii): *$\lambda^2(B) = 0$, where $\lambda^n(\mathcal{B})$ represents the Lebesgue measure, on $\mathbb{R}^n$, of the Borelian $\mathcal{B} \subset \mathbb{R}^n$.*

Remark 4.2. *By the method of duality, we construct a set B containing a straight line in each direction.*

4.2.1. Construction by Duality

Definition 4.3. *The dual of a set $E \subset \mathbb{R}^2$ is the set:*

$$E^* = \{D_{y=ax+b} \mid (a, b) \in E\}$$

$D_{y=ax+b}$ is the line given by its equation $y = ax + b$.

$$\mathcal{D}(E) = \bigcup_{\Delta \in E^*} \Delta$$

Proposition 4.4. • *$\mathcal{D}(E)$ contain a straight line in each direction of the plane $\Longleftrightarrow$ The projection of E onto (Ox) is $\mathbb{R}$.*

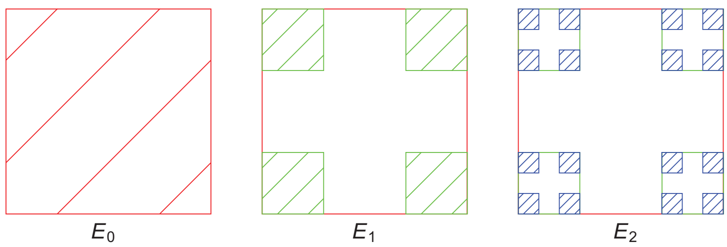
- $\lambda^2(\mathcal{D}(E)) = 0 \Leftrightarrow \lambda^1\left(\{\theta \in]-\frac{\pi}{2},\frac{\pi}{2}[\mid \lambda^1\left(\mathrm{proj}_\theta(E)\right) \neq 0\}\right) = 0.$
- *There must be an angle θ_0 such that $\mathrm{proj}_{\theta_0}(E) = \mathbb{R}$ and the projection of* E *into almost all others directions must have measure zero.*

Where $\mathrm{proj}_\theta(E)$ denotes the orthogonal project of E on the line $D_{y=x}/\tan\theta$.

4.2.2. A Concrete Construction

To realize this construction, we must exhibit a set verifying the previous properties. This set is the "four-corner Cantor set".

Definition 4.5. *We define the 4-corner Cantor set by induction*:



The sets E_0, E_1 and E_2 are hatched.

$$E = \bigcap_{n \in \mathbb{N}} E_n$$

Proposition 4.6. *Let* $\theta_0 = \arctan 1/2$. *For reasons of invariance by rotation, we take* $\theta \in [0, \pi/2]$

$$\mathrm{proj}_{\theta_0}(E) \text{ is a segment and } \forall\theta \neq \theta_0;\ \mathcal{L}^1(\mathrm{proj}_\theta(E)) = 0$$

So we can choose a system of axes (xOy) such that the projection of E on (Ox) is $[-1, 1]$. Thus, $\mathcal{D}(E)$ contains a straight line with a slope of coefficient a for all $a \in [-1, 1]$.

To obtain the other straight lines, with leading coefficients in the sets $]-\infty, -1[$ and $]1, +\infty[$, it suffices to subject $\mathcal{D}(E)$ to an angle rotation $\pi/2$.

Conclusion: We have a construction of a Besicovitch set.

Now, we come back to the technique that allows us to overcome some pathologies as explained above.

5. Level 3: How to Overcome Pathologic Sets During the Evolution

5.1. A Time Regularized Approximation Problem

In the thesis [21] of Laasri, the walks, as defined above, are taken over intervals that automatically respect the natural evolution of time. In fact, for more general integrable families, such as Lebesgue ones, the simple functions are constant on some finite measurable sets that may naturally overlap. The approximation by Riemann and Bochner integrable functions henceforth loses its meaning, as this may not explain the overlap of time. To overcome this issue, we introduce a method that relies on the concept of time-ordered families, first introduced by G. Schmidt in [29], to suitably approximate the evolution families. To be more precise, we rewrite the following definition, adapted to the context of this paper.

Definition 5.1. *Let* $\tau > 0$ *a strict positive horizon and fix a measurable set E of* $[0, \tau]$. *All the operators* $(A_i)_{i\in E}$ *are assumed to be defined on a dense common subspace* $D \subset X$ *and generate* C_0*−semigroups* $\mathcal{T}_i$ *on* X.

i): *A time-ordered simple function* $A : E \to \mathcal{L}(D, X)$ *is a mapping which can be expressed as:*

$$A(t) = \sum_{i=1}^{n} \mathbb{1}_{Ei}(t) A_i,$$

where $\{A_i\} \subset \mathcal{L}(X)$, E_i *is Borel measurable of* $[0, \tau]$, $E_i \cap E_j = \varnothing$ *for* $i \neq j$, *and* $\sup E_i \leqq \inf E_{i+1}$.

ii): *Let A be a time-ordered simple function, let* λ *be a Borel measure and E a Borel measurable set, so automatically* $\lambda(E) < \infty$. *Then we define the Lebesgue product integral, noted* $\pi^{\mathcal{L}}$, *of A as follows:*

$$\pi^{\mathcal{L}}(A) = \mathcal{T}_n(\lambda(E \cap E_n))\mathcal{T}_{n-1}(\lambda(E \cap E_{n-1}))\ldots\mathcal{T}_1(\lambda(E \cap E_1))\mathcal{T}_0(\lambda(E \cap E_0)).$$

When the product is backward ordered, we obtain an analogous definition which must be the dual of the first. For commutative families, the two definitions are the same and in the general case, we adopt the definition (5.1) above.

In order to extend this definition to an arbitrary strongly integrable function $A(\cdot) : \mathbb{R}^+ \to \mathcal{L}(X)$, it will be instructive to recall these important results proved by Schmidt [29]:

(1) One can find a sequence $\{S_n(\cdot)\}$ of time-ordered simple functions that suitably converges to A (in a given topological sense).
(2) The sequence of walks (product integrals) $(\pi^{\mathcal{L}}(S_n))$ converges to a limit, and that limit is independent of the approximating sequence.

Remark 5.2.

a): *It is easy to see that the definition (5.1) fits with the classical definition of π−integral product in the Riemann-Bochner sense. It is indeed enough to choose $E_i = [\theta_i, \theta_{i+1}]$ for some ordered subdivision $\theta_0 = 0 < \theta_1 < \cdots < \theta_{n+1} = \tau$ and take into account all improvements of the definition 1.1 in* [17] *to retrieve immediately the notion of π−integration.*

b): *The definition (5.1) is more general in the sense that it will allow the overlap of cells E_i with jumps roughly non-measurable or, if they are so, their measures are null. We explain in the next section how to overcome this type of pathologic subsets of Borelian tribe of* $[0, \tau]$.

5.2. How to Dodge Singular Operators

We are now able to explain how to shirk from singular operators $A(t)$ when t varies in a pathological set.

The following theorem provides the answer to this question, as presented in Proposition 1 of [29].

Theorem 5.3. *Let $A(\cdot) : \mathbb{R}^1 \to \mathcal{B}(X)$ be strongly integrable over any finite interval with respect to a Borel measure λ. Then, a sequence $\{S_n(\cdot)\}$ of time-ordered simple functions can be found such that for any finite interval J*

$$\sup_n \int_J \|S_n(\cdot)\| d\lambda < \infty$$

and for any x in X

$$\int_J \|[S_n(\cdot) - A(.)]x\| d\lambda \longrightarrow 0.$$

Let us give in details the technique of overcoming degeneracy:

Consider a mapping $t \mapsto A(t)$ that describes an evolution equation of tye: (2)

In which $t \mapsto A(t)$ is assumed to be measurable and possesses pointwise maximal regularity.

First, we partition the interval $[0, \tau]$ into three subsets $[0, \tau] = R \cup B \cup F$

where:

i): $t \in R$ means that $t \in [0, \tau]$ and $t \mapsto A(t)$ is regular on some interval $[t - \delta, t + \delta]$ for some $\delta > 0$.

ii): B is the set of pathological operator indices t such that $t \mapsto A(t)$, $t \in B$ satisfies $\lambda(B) = 0$. A prototype to keep in mind is the one-dimensional Besicovitch set as described above.

iii): F is a finite set where eventually $t \mapsto A(t)$ has not maximal regularity property.

Thanks to theorem (5.3), There exists a sequence $\{S_n(\cdot)\}$ of time-ordered simple functions such that for any finite interval J

$$\sup_n \int_J \|S_n(\cdot)\| d\lambda < \infty$$

and for any x in X

$$\int_J \|[S_n(\cdot) - A(.)]x\| d\lambda \longrightarrow 0.$$

here each S_n may be written as follows:

$$S_n(.) = \sum_{p=0}^{m_n} \mathbb{1}_{E_{p,n}}(.) S_{p,n}$$

$\{S_{p,n}\} \subset \mathcal{L}(X)$, $E_{p,n}$ is Borel measurable of $[0, \tau]$ such that:

$$\bigcup_{p=0}^{m_n} E_{p,n} = [0,\tau], \qquad E_{p,n} \cap E_{p',n} = \varnothing \quad \text{for} \quad p \neq p', \quad \text{and} \quad \sup E_{p,n} \leqq \inf E_{p+1,n}.$$

It is enough at this stage to consider, for each integer n, the partition:

$$E_{p,n} = R_{p,n} \cup B_{p,n} \cup F_{p,n}, \quad \text{where} \quad R_{p,n} = R \cap E_{p,n} \quad \text{and} \quad B_{p,n} = B \cap E_{p,n} \quad \text{and} \quad F_{p,n} = F \cap E_{p,n}.$$

To conclude, it suffices to write

$$\int_0^\tau \|[S_n(\cdot) - A(.)]x\| d\lambda = \sum_{p=0}^{m_n} \left[I_{p,n} + J_{p,n} + K_{p,n} \right]$$

where

$$I_{p,n} = \int_{R_{p,n}} \|[S_n(\cdot) - A(.)]x\| d\lambda$$

$$J_{p,n} = \int_{B_{p,n}} \|[S_n(\cdot) - A(.)]x\| d\lambda$$

and

$$K_{p,n} = \int_{F_{p,n}} \|[S_n(\cdot) - A(.)]x\| d\lambda.$$

Since the two last terms are null, it is allowed to apply the results of the last section, mainly the theorem (3.4) to establish the well-posedness of the problem (2) in the ambient space H when this latter one denotes a separable Hilbert space. To this end, one may simply apply the Lusin theorem to approximate $t \mapsto A(t)$ on the regular part R by sequence of continuous functions and by part by affine ones, which are obviously Lipshitz continuous and pursued as in [28] or with mobile means approximation as detailed in the third section mainly theorem (3.4) above.

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