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Blow-up dynamics of solutions to a nonlinear wave equation with positive initial energy

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Abstract

This paper investigates the dynamics of a quasi-linear partial differential equation of fourth order characterized by bi-hyperbolic properties, incorporating dynamic boundary conditions. The study focuses on the interplay between the equation's nonlinear source term, the boundary effects, and the initial energy. By applying the concavity method, we derive conditions that lead to the finite-time blow-up phenomenon in solutions with non-negative initial energy. These findings highlight the impact of dynamic boundary conditions on the development of finite-time singularities in higher-order hyperbolic equations.

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1. Introduction

The study of partial differential equations (PDEs) involving higher-order derivatives is central to understanding complex physical and engineering phenomena such as elastic deformations, wave propagation, and plate dynamics. Among these, fourth-order PDEs are particularly significant because of their mathematical complexity and their ability to model intricate systems. This paper examines a quasi-linear PDE of fourth-order, coupled with dynamic boundary conditions that introduce temporal interactions at the domain boundary.

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The equation under consideration is:

$$\begin{cases} v_{tt} + \Delta^2 v - \Delta v = g(-\Delta v), & \text{in} \quad \Omega \times (0,T) \\ \dfrac{\partial v}{\partial \eta} = -a v_t, \quad \dfrac{\partial \Delta v}{\partial \eta} = 0, & \text{on} \quad \Gamma \times (0,T) \\ v_0 = v(x,0), \quad v_1 = v_t(x,0) & \end{cases} \tag{1}$$

where a and t are non-negative, x lies within the domain Ω, and Ω refers to a bounded region with a smooth boundary, represented by $\Gamma = \partial\Omega$.

The mixed hyperbolic-parabolic nature of the studied equation results from the interplay between second and fourth-order derivatives, which gives rise to unique dynamical properties. While the blow-up dynamics of solutions to second-order hyperbolic equations have been extensively explored in previous research (e.g., [1, 2]), the influence of higher-order derivatives and dynamic boundary conditions remains less explored. This gap motivates the present study.

Vasconcellos et al. [3] established the existence and uniqueness of global solutions for the following problem when $n \le 3$:

$$\begin{cases} w_{tt} + \Delta^2 w - \psi(\int_\Omega |\nabla w|^2 \, dx)\Delta w + f(w_t) = 0, & \text{on} \quad \Omega \times (0,T), \\ w = \dfrac{\partial w}{\partial \nu} = 0, & \text{on} \quad \Gamma \times (0,T), \end{cases}$$

Here, ψ is a nonnegative function that is continuously differentiable, and f is a continuous, non-decreasing function that takes real values. A related study by Wu and Tsai [4] analyzed the blow-up characteristics and global existence of a second-order PDE with a dynamic boundary term, given by:

$$w_{tt} - H(s)\Delta w + g(w_t) = h(w), \tag{2}$$

where $H(s)$ is a positive function that is locally Lipschitz continuous. Although their work focused on second-order systems, the inclusion of a fourth-order term $\Delta^2 u$ in (1) introduces additional mathematical challenges and complexity. The existence and finite-time blow-up of solutions to equation (2) were studied in [5-8]. Piskin et al. [9] investigated the following equation for an extensible beam:

$$w_{tt} + \Delta^2 w - H(\|\nabla w\|)\Delta w + |w_t|^{p-1} w_t = |w|^{q-1} w$$

with initial and boundary conditions. They demonstrated that the solution undergoes blow-up in finite time when the initial energy is positive. To analyze the finite-time blow-up in this context, we employ the concavity method, initially introduced by Levine [10]. This method uses the second-order derivative of an auxiliary functional to establish blow-up criteria. Building on Levine's foundational work, Korpusov [11] extended the concavity method to accommodate equations with nonlinear source terms and dynamic boundary effects. Several studies have used this method, as mentioned in [12-16].

In this paper, we apply the Korpusov concavity method to (1), deriving the conditions that are sufficient for the finite-time blow-up of solutions with non-negative initial energy. Our findings contribute to the growing body of work on the impact of boundary dynamics on solution behaviors and extend the understanding of blow-up phenomena in fourth-order hyperbolic equations.

The paper is structured as follows. Section 2 provides the necessary preliminaries, including key definitions and lemmas. Section 3 presents the main results, where the conditions required for the blow-up in finite time are derived and proved. Finally, Section 4 summarizes the findings and explores possible directions for future research.

2. Preliminary Definitions and Lemmas

This section provides some essential results, including lemmas and a theorem on local existence, that will be crucial for analyzing the problem. Throughout the paper $\|.\|$ is used to represent the norms in $L^2(\Omega)$. For further details, see [17,18].

The *Sobolev space* $W^{k,p}(\Omega)$ includes all functions $u \in L^p(\Omega)$ for which the weak derivatives up to order k are also elements of $L^p(\Omega)$, where $1 \le p < \infty$. Additionally, the following function spaces are relevant for the analysis:

$C([0,T);H^k(\Omega))$: the set of continuous functions on $[0,T]$ taking values in the Sobolev space $H^k(\Omega)$.

$C([0,T);L^2(\Omega) \cap L^{p+1}(\Omega \times (0,T))$: the space of functions integrable to the $(p+1)$-th power over $\Omega \times (0,T)$.

In order to establish the blow-up result, we employ the following lemma introduced by M.O. Korpusov, which extends the concavity method of Levine [10] to more complex systems.

Lemma 2.1 *[11] Assume that $\psi(t)$ is a function that is twice continuously differentiable and satisfies*

$$\psi\psi'' - \alpha(\psi')^2 + \gamma\psi'\psi + \beta\psi \ge 0, \quad \alpha > 1, \gamma \ge 0, \beta > 0. \tag{3}$$

Consider the following conditions:

$$(\psi'(0) - \frac{\gamma}{\alpha - 1}\psi(0))^2 > \frac{2\beta}{2\alpha - 1}\psi(0), \quad and \quad \psi'(0) > \frac{\gamma}{\alpha - 1}\psi(0)$$

hold with $\psi(t) \ge 0$, $\psi(0) > 0$, and $\psi'(0) > 0$. Under these conditions, $\psi(t)$ blows up in finite time, satisfying:

$$T \le T_\infty \le \frac{\psi^{1-\alpha}(0)}{A},$$

where

$$A^2 \equiv (\frac{\alpha - 1}{\psi^\alpha(0)})^2[(\psi'(0) - \frac{\gamma}{\alpha - 1}\psi(0))^2 - \frac{2\beta}{2\alpha - 1}\psi(0)],$$

and $\lim_{t \uparrow T_\infty} \psi(t) = +\infty$.

The assumption on the function g is stated as follows:

The function g, along with its antiderivative $G(v) = \int_0^v g(s)ds$, satisfies the following conditions:

$$g(0) = 0, \qquad vg(v) \ge 2(2\delta + 1)G(v), \quad \text{forall} \quad v \in \mathbb{R}, \tag{4}$$

where $\delta > 0$ is a real constant.

Definition 2.2 *Let v be a weak solution of Problem (1.1). The maximal existence time T^* is defined as follows:*

$$T^* = \sup\{T > 0 : v(t) \quad \text{exists on} \quad [0,T]\}.$$

Then, we have the following:

- if $T^* < \infty$,then v undergoes blow-up at a finite time, and T^* represents the blow-up time,
- if $T^* = \infty$, then v a global solution.

Next, we state the local existence theorem of problem (1), whose proof can be found in [15].

Theorem 2.3 *(Local existence theorem) Let $v_0 \in H^2(\Omega)$ and $v_1 \in L^2(\Omega)$. Under these conditions, a unique solution v to (1) exists, satisfying the properties:*

$$v \in C([0,T);H^2(\Omega)), v_t \in C([0,T);L^2(\Omega) \cap L^{p+1}(\Omega \times (0,T)).$$

Furthermore, one of the following conditions is satisfied:

- $T = \infty$,
- $\| \nabla v \| \to \infty, \quad t \to T^-$.

3. Blow-Up Result

This section focuses on analyzing the existence of blow-up solutions for the initial-boundary value problem described in (1). To achieve this, we define the energy of the solution as:

$$E(t) := \| \nabla v_t \|^2 + \| \Delta v \|^2 + \| \nabla \Delta v \|^2 - 2\langle G(-\Delta v), 1\rangle. \tag{5}$$

The lemma below confirms that the energy functional $E(t)$, as defined in (5), decreases or remains constant over time.

Lemma 3.1 *Assuming condition (4) holds, the energy functional $E(t)$ satisfies $E(t) \le E(0)$ for all $t > 0$.*

Proof. To prove this, we start by multiplying equation (1) by $-2\Delta v_t$ in the $L_2(\Omega)$ space, resulting in the following equation:

$$-2\int_\Omega v_{tt}\Delta v_t dx + 2\int_\Omega \Delta v \Delta v_t dx - 2\int_\Omega \Delta^2 v \Delta v_t dx = -2\int_\Omega g(-\Delta v)\Delta v_t dx. \tag{6}$$

By utilizing Green's formula and incorporating the boundary conditions, we arrive at the following expression for the time derivative of the energy functional:

$$\frac{d}{dt}[\| \nabla v_t \|^2 + \| \Delta v \|^2 + \| \nabla \Delta v \|^2 - 2\langle G(-\Delta v), 1\rangle] = 2\int_\Gamma \frac{\partial v}{\partial \eta} v_{tt} d\tau.$$

Using the boundary condition $\frac{\partial v}{\partial \eta} + a v_t = 0$, which implies

$$\frac{d}{dt}\left(\frac{\partial v}{\partial \eta} + a v_t\right) = 0 \quad \textit{orequivalently}, \quad \frac{\partial v_t}{\partial \eta} = -a v_{tt},$$

we can substitute into the previous expression, obtaining:

$$\frac{d}{dt}E(t) = -2\int_\Gamma a(v_{tt})^2 d\tau. \tag{7}$$

From equation (7), it becomes evident that the energy functional's rate of change is non-positive. As a result, $E(t)$ is a decreasing function of time, ensuring that the inequality $E(t) \le E(0)$ holds for all $t \ge 0$.

Definition 3.2 *A solution v of (1) is called blow-up if there exists a finite time T^* such that*

$$\lim_{t \to T^{*-}} \left[\int_\Omega \nabla v^2 dx + \int_0^t \int_\Gamma a v_t^2 dx d\tau + \int_\Gamma a v_1^2 d\tau\right] = \infty.$$

Let

$$\psi(t) = \| \nabla v \|^2 + \int_0^t \int_\Gamma a v_t^2 ds d\tau + \int_\Gamma a v_1^2 d\tau. \tag{8}$$

In the following theorem, we prove global nonexistence with positive initial energy.

Theorem 3.3 *Let $v_0 \in W_0^{1,m}(\Omega)$, $v_1 \in L^2(\Omega)$ and assume that condition (2.2) holds. If the initial conditions satisfy the following inequality:*

$$E(0) > 0,$$

$$2\int_\Omega \nabla v_0 . \nabla v_1 > 0,$$

and

$$(2\int_\Omega \nabla v_0.\nabla v_1 dx)^2 > \frac{4(2\delta+1)E(0)+2(\delta+1)\int_\Gamma a v_1^2 d\tau}{2\delta+1}\int_\Omega \nabla v_0^2 dx$$

then there exists a finite time T^* such that

$$T^* \leq \frac{1}{A}(\|\nabla v_0\|^2)^{-\delta} = \frac{1}{A}\|\nabla v_0\|^{-2\delta}.$$

Additionally, we have:

$$\limsup_{t\to T^*}\|v(.,t)\| = +\infty.$$

Here, A^2 is given by

$$A^2 = \delta^2\|\nabla v_0\|^{-4\delta-4}(\int_\Omega \nabla v_0.\nabla v_1 dx - \frac{4(2\delta+1)E(0)+2(\delta+1)\int_\Gamma a v_1^2 d\tau}{2\delta+1}\int_\Omega \nabla v_0^2 dx).$$

Proof. We begin by differentiating the function ψ defined in (8):

$$\psi'(t) = 2\langle\nabla v, \nabla v_t\rangle + 2\int_0^t\int_\Gamma a v_t v_{tt} d\tau ds + \int_\Gamma a v_1^2 d\tau. \tag{9}$$

Next, we differentiate again with respect to t. Using Green's formula, we arrive at:

$$\psi''(t) = 2\|\nabla v_t\|^2 + 2\int_\Omega \nabla v.\nabla v_{tt} dx + 2\int_\Gamma a v_t v_{tt} d\tau.$$

We now rewrite the terms:

$$\psi''(t) = 2\|\nabla v_t\|^2 - 2\int_\Omega v_{tt}\Delta v dx + \underbrace{2\int_\Gamma \frac{\partial v}{\partial\eta} v_{tt} d\tau + 2\int_\Gamma a v_t v_{tt} d\tau}_{A}.$$

Since $A = 2\int_\Gamma v_{tt}(\frac{\partial v}{\partial\eta} + a v_t) d\tau = 0$ (by the boundary condition $\frac{\partial v}{\partial\eta} + a v_t = 0$), we conclude that:

$$\psi''(t) = 2\|\nabla v_t\|^2 - 2\int_\Omega v_{tt}\Delta v dx.$$

Next, we substitute for v_{tt} from the equation (1) to obtain:

$$\psi''(t) = 2\|\nabla v_t\|^2 - 2\int_\Omega (\Delta v - \Delta^2 v + g(-\Delta v))\Delta v dx$$

$$= 2\|\nabla v_t\|^2 - 2\|\Delta v\|^2 + \underbrace{2\langle\Delta^2 v, \Delta v\rangle}_{II} - 2\int_\Omega g(-\Delta v)\Delta v dx.$$

We begin by simplifying the expression for II as follows:

$$II = 2\int_\Omega \Delta^2 v\Delta v dx = -2\int_\Omega \nabla(\Delta v)\nabla(\Delta v) dx + 2\int_\Gamma \frac{\partial(\Delta v)}{\partial\eta}\Delta v d\tau$$

$$= -2\int_\Omega |\nabla\Delta v|^2 dx.$$

Now, substituting this result into the expression for $\psi(t)$, we get:

$$\psi''(t) = 2\|\nabla v_t\|^2 - 2\|\nabla\Delta v\|^2 - 2\|\Delta v\|^2 + 2\langle g(-\Delta v), -\Delta v\rangle.$$

Using the assumption (2.2) and the properties of the function f, we can estimate the term involving $g(-\Delta v)$ as follows:

$$\langle g(-\Delta v), -\Delta v\rangle \geq (2\delta+1)\langle G(-\Delta v), 1\rangle.$$

Thus, we have the inequality:

$$\psi''(t) \geq 2\|\nabla v_t\|^2 - 2\|\Delta v\|^2 - 2\|\nabla\Delta v\|^2 + (4+8\delta)\langle G(-\Delta v), 1\rangle.$$

Simplifying further, we arrive at:

$$\psi''(t) \geq -2(2\delta+1)E(t) + 4(\delta+1)\|\nabla v_t\|^2 + 4\delta\|\Delta v\|^2 + 4\delta\|\nabla\Delta v\|^2 .$$

Finally, from the previous result, it is established that the energy functional follows the evolution equation given by:

$$E(t) = E(0) - 2\int_0^t\int_\Gamma a(v_{tt})^2 d\tau ds.$$

We start with the inequality obtained from the previous result:

$$\psi''(t) \geq -E(0)(4\delta+2) + (8\delta+4)\int_0^t\int_\Gamma a(v_{tt})^2 d\tau ds + 4(\delta+1)\|\nabla v_t\|^2 + 4\delta\|\Delta v\|^2 + 4\delta\|\nabla\Delta v\|^2 .$$

Next, we use the estimate from earlier to bound the terms:

$$\begin{aligned}\psi''(t) &\geq 4(\delta+1)[\|\nabla v_t\|^2 + \int_0^t\int_\Gamma a(v_{tt})^2 d\tau ds + \frac{1}{2}\int_\Gamma a v_1^2 d\tau] - D\\ &\geq 4(\delta+1)C - D,\end{aligned}$$

where $C = \|\nabla v_t\|^2 + \int_0^t\int_\Gamma a(v_{tt})^2 d\tau ds + \frac{1}{2}\int_\Gamma a v_1^2 d\tau$, and D denotes a constant depending on the initial energy. By multiplying both sides of this inequality by $\psi(t)$, we derive:

$$\psi''(t)\psi(t) - (1+\delta)4C\psi(t) \geq -D\psi(t). \tag{10}$$

From the previous equation, we deduce the following:

$$(1+\delta)[\psi'(t)]^2 = 4(1+\delta)[\langle\nabla v, \nabla v_t\rangle + \int_0^t\int_\Gamma a v_t v_{tt} d\tau ds + \frac{1}{2}\int_\Gamma a v_1^2 d\tau]^2. \tag{11}$$

By applying the Cauchy-Schwarz inequality and simplifying the terms, we get:

$$(1+\delta)[\psi'(t)]^2 \leq 4(1+\delta)[\|\nabla v\|\|\nabla v_t\| + (\int_0^t\int_\Gamma a v_t^2 d\tau ds)^{1/2}(\int_0^t\int_\Gamma a v_{tt}^2 d\tau ds)^{1/2} + \frac{1}{2}\int_\Gamma a v_1^2 d\tau]^2. \tag{12}$$

For convenience, we now define the following notations:

$$A_1 := \|\nabla v\|, \quad A_2 := \{\int_0^t[\int_\Gamma a v_t^2 d\tau]ds\}^{\frac{1}{2}}, \quad B_1 := \|\nabla v_t\|, \quad B_2 := \{\int_0^t[\int_\Gamma a v_{tt}^2 d\tau]ds\}^{\frac{1}{2}},$$

$$Z := \int_\Gamma a v_1^2 d\tau.$$

Using these definitions, and based on the previous inequality, we can conclude the following expression:

$$4(1+\delta)[A_1B_1 + A_2B_2 + \frac{Z}{2}]^2$$

$$= 4(1+\delta)[(A_1^2B_1^2 + A_2^2B_2^2 + \frac{Z^2}{4}) + 2(A_1B_1A_2B_2 + A_1B_1\frac{Z}{2} + A_2B_2\frac{Z}{2})].$$

Next, by applying the Cauchy-Schwarz inequality, we derive the following bounds for the cross terms:

$$A_1B_1Z \le (\frac{A_1^2}{2} + \frac{B_1^2}{2})Z \quad and \quad A_2B_2Z \le (\frac{A_2^2}{2} + \frac{B_2^2}{2})Z.$$

On the other hand, we compute:

$$4(1+\delta)C\psi(t) = 4(1+\delta)[B_1^2 + B_2^2 + \frac{Z}{2}][A_1^2 + A_2^2 + \frac{Z}{2}]$$

$$= 4(1+\delta)[A_1^2B_1^2 + A_2^2B_1^2 + A_1^2B_2^2 + B_1^2Z + A_2^2B_1^2 + B_2^2Z + A_1^2\frac{Z}{2} + A_2^2\frac{Z}{2} + \frac{Z^2}{2}].$$

We also note that:

$$A_1^2B_2^2 + A_2^2B_1^2 = (A_1B_2 - A_2B_1)^2 + 2A_1A_2B_1B_2.$$

Thus, we can rewrite the inequality as:

$$(1+\delta)[\psi'(t)]^2 \le (\delta+1)4C\psi(t). \tag{13}$$

Now, by combining the last two inequalities, we obtain the following:

$$\psi''(t)\psi(t) - (1+\delta)[\psi'(t)]^2 \ge -D\psi(t).$$

This completes the proof of the intended result.

Remark 3.4 *Comparing equation (3) with the earlier inequality, we can immediately identify the following relationships:*

$$\alpha = 1+\delta, \quad \beta = D = (1+4\delta)E(0) + (1+2\delta)\int_\Gamma a v_1^2 d\tau, \quad \gamma = 0.$$

Additionally, we have the following expressions:

$$\frac{2\beta}{2\alpha-1} = \frac{E(0)(8\delta+4) + (4+4\delta)\int_\Gamma a v_1^2 d\tau}{2\delta+1}, \quad \frac{\gamma}{\alpha-1} = 0.$$

By applying Lemma 2.1, it follows that there is a time T^* such that $T^* \le \frac{1}{A}\|\nabla v_0\|^{-2\delta}$.

Remark 3.5 *The conditions assumed in Theorem 3.3 are compatible. We begin by selecting an initial function $v_0(x) \in W_0^{1,p}(\Omega)$ for which the following inequality holds:*

$$\int_\Gamma G(\Delta v_0)d\tau + \frac{4a^{1/2}\|\nabla v_0\|^2 \int_\Gamma v_1^2 d\tau + a^2\int_\Gamma v_1^4 d\tau}{8\|\nabla v_0\|^2 + 8a\int_\Gamma v_1^2 d\tau} > \frac{\|\nabla v_0\|^2 \int_\Gamma v_1^2 d\tau}{2(\|\nabla v_0\|^2 + \int_\Gamma a v_1^2 d\tau)}$$

$$+\frac{a(1+\delta)}{2(1+2\delta)}\int_{\Gamma} v_1^2 d\tau+\frac{\|\Delta v_0\|^2}{2}+\frac{\|\nabla\Delta v_0\|^2}{2}. \tag{14}$$

Next, we fix $v_0(x)$ and define $v_1(x)=\lambda v_0(x)$, where $\lambda>0$ is selected large enough to guarantee that the initial energy remains positive:

$$E(0)=\lambda^2\|\nabla v_0\|^2+\|\Delta v_0\|^2+\|\nabla\Delta v_0\|^2-2\int_{\Gamma} G(\Delta v_0)d\tau>0.$$

Additionally, observe that $\psi'(0)=2\lambda\|\nabla v_0\|^2+\int_{\Gamma} a v_1^2>0$. Then we have,

$$4\lambda^2\|\nabla v_0\|^4+4\lambda\|\nabla v_0\|^2\int_{\Gamma} a v_1^2 d\tau+(\int_{\Gamma} a v_1^2 d\tau)^2$$

$$>(4E(0)+4(\frac{1+\delta}{1+2\delta})\int_{\Gamma} a v_1^2 d\tau)).(\|\nabla v_0\|^2+\int_{\Gamma} a v_1^2 d\tau)$$

$$=(4\lambda^2\|\nabla v_0\|^2+4\|\nabla\Delta v_0\|^2+4\|\Delta v_0\|^2-8\int_{\Gamma} G(-\Delta v_0)d\tau+4(\frac{1+\delta}{1+2\delta})\int_{\Gamma} a v_1^2 d\tau)$$

$$.(\|\nabla v_0\|^2+\int_{\Gamma} a v_1^2 d\tau)$$

$$=4\lambda^2\|\nabla v_0\|^4+4\lambda\|\nabla v_0\|^2\int_{\Gamma} a v_1^2 d\tau+(4(\frac{1+\delta}{1+2\delta})\int_{\Gamma} a v_1^2 d\tau+4\|\Delta v_0\|^2$$

$$+4\|\nabla\Delta v_0\|^2-8\int_{\Gamma} G(-\Delta v_0)d\tau).(\|\nabla v_0\|^2+\int_{\Gamma} a v_1^2 d\tau). \tag{15}$$

Now, let $\lambda=1/a^{1/2}$, where $a>0$. A series of transformations in (3.10) is equivalent to (3.11). This demonstrates that the conditions in Theorem 3.3 are consistent for sufficiently small $a>0$.

4. Conclusion

This research explored the blow-up behavior of quasi-linear wave equation characterized by bi-hyperbolic features, subject to dynamic boundary conditions. By employing the concavity method, as extended by Korpusov, we established sufficient criteria for blow-up in finite time under non-negative initial energy. Our results highlight the significant influence of nonlinear source terms and dynamic boundary effects on solution dynamics, offering a deeper understanding of finite-time singularities in higher-order hyperbolic equations.

These findings contribute to the broader study of blow-up phenomena by revealing how dynamic boundary conditions and initial energy interplay in driving singularity formation. Future research could focus on extending the analysis to more general boundary conditions or exploring the long-term behavior of solutions that do not undergo blow-up.

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