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Common Fixed Point Theorems in $\mathfrak{M}$-Fuzzy Cone Metric Spaces

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Abstract

This work aims to generalize the Banach contraction theorem to $\mathfrak{M}$-fuzzy cone metric spaces. We construct generalized $\mathfrak{M}$-fuzzy cone contractive conditions for three self mappings with which they have a unique common fixed point.

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1. Introduction

Fuzzy sets that handle uncertainties well was introduced by Zadeh [10]. Huang and Zhang [4] introduced cone and defined cone metric spaces as a generalization of metric spaces [1]. Tarkan Oner et al. [9] introduced fuzzy cone metric spaces that generalized fuzzy metric spaces [2]. These ideas motivated the researchers to come up with several new ideas as they act as a base for introducing new concepts and proving many more new results. The aim here is to construct and prove $\mathfrak{M}$-Fuzzy Cone Banach Contraction Theorem and some common fixed point theorems for three self mappings which satisfy generalized contractive conditions in $\mathfrak{M}$-Fuzzy Cone Metric Spaces and to provide an example to exhibit the same.

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2. Preliminaries

Definition 1. [4] Let $\mathfrak{B}$ be a real Banach space and $\mathcal{C}$ be a subset of $\mathfrak{B}$. $\mathcal{C}$ is called a cone if and only if:

[C1] $\mathcal{C}$ is nonempty, closed and $\mathcal{C} \neq \{0\}$,

[C2] $\rho, \sigma \in \mathbb{R}, \rho, \sigma \geq 0, c_1, c_2 \in \mathcal{C}$ imply $\rho c_1 + \sigma c_2 \in \mathcal{C}$,

[C3] $c \in \mathcal{C}$ and $-c \in \mathcal{C}$ imply $c = 0$.

The cones considered here are subsets of a real Banach space and are with nonempty interiors.

Definition 2. An $\mathfrak{M}$-Fuzzy Cone Metric Space (briefly, $\mathfrak{M}$-FCM Space) is a 3-tuple $(\mathcal{Z}, \mathfrak{M}, *)$ where $\mathcal{Z}$ is an arbitrary set, $*$ is a continuous t-norm, $\mathcal{C}$ is a cone and $\mathfrak{M}$ a fuzzy set in $\mathcal{Z}^3 \times int(\mathcal{C})$ satisfying the following conditions: For all $\zeta, \eta, \omega, \mathsf{u} \in \mathcal{Z}$ and $c, c' \in int(\mathcal{C})$,

[MFC1] $\mathfrak{M}(\zeta, \eta, \omega, c) > 0$,

[MFC2] $\mathfrak{M}(\zeta, \eta, \omega, c) = 1$ if and only if $\zeta = \eta = \omega$,

[MFC3] $\mathfrak{M}(\zeta, \eta, \omega, c) = \mathfrak{M}(p\{\zeta, \eta, \omega\}, c)$, where p is a permutation,

[MFC4] $\mathfrak{M}(\zeta, \eta, \omega, c + c') \geq \mathfrak{M}(\zeta, \eta, \mathsf{u}, c) * \mathfrak{M}(\mathsf{u}, \omega, \omega, c')$,

[MFC5] $\mathfrak{M}(\zeta, \eta, \omega, \cdot) : int(\mathcal{C}) \to [0, 1]$ is continuous.

Then $\mathfrak{M}$ is called an $\mathfrak{M}$-Fuzzy Cone Metric on $\mathcal{Z}$. The function $\mathfrak{M}(\zeta, \eta, \omega, c)$ denotes the degree of nearness between ζ, η and ω with respect to c.

Example 3. *Let $\mathfrak{B} = \mathbb{R}$ and consider the cone $\mathcal{C} = [0, +\infty]$ in $\mathfrak{B}$. Consider an increasing continuous function $g : \mathcal{C} \to \mathcal{C}$ and $a, b > 0$. Let the t-norm $*$ be defined by $\rho * \sigma = \rho\sigma$. Define $\mathfrak{M} : \mathbb{R}^3 \times int(\mathcal{C}) \to [0, 1]$ by*

$$\mathfrak{M}(\zeta, \eta, \omega, c) = \Big(\frac{(min\{f(x), f(y), f(z))^a + \|g(c)\|}{(max\{f(x), f(y), f(z))^a + \|g(c)\|}\Big)^b$$

for all $\zeta, \eta, \omega \in \mathbb{R}$ and $c \in int(\mathcal{C})$. Then $(\mathbb{R}, \mathfrak{M}, *)$ is an $\mathfrak{M}$-FCM Space.

Definition 4. A symmetric $\mathfrak{M}$-FCM Space is an $\mathfrak{M}$-FCM Space $(\mathcal{Z}, \mathfrak{M}, *)$ satisfying

$$\mathfrak{M}(\eta, \omega, \omega, c) = \mathfrak{M}(\omega, \eta, \eta, c), \text{ for all } \eta, \omega \in \mathcal{Z} \text{ and } c \in int(\mathcal{C}).$$

Remark 5. *An $\mathfrak{M}$-FCM Space is symmetric.*

Definition 6. Let $(\mathcal{Z}, \mathfrak{M}, *)$ be an $\mathfrak{M}$-FCM Space. A self mapping $\mathcal{P} : \mathcal{Z} \to \mathcal{Z}$ is said to be $\mathfrak{M}$-Fuzzy Cone Contractive (briefly, $\mathfrak{M}$-FCC) if there exists $k \in (0, 1)$ such that

$$\left(\frac{1}{\mathfrak{M}(\mathcal{P}(\zeta), \mathcal{P}(\eta), \mathcal{P}(\omega), c)} - 1\right) \leq k\left(\frac{1}{\mathfrak{M}(\zeta, \eta, \omega, c)} - 1\right),$$

for all $\zeta, \eta, \omega \in \mathcal{Z}$ and $c \in int(\mathcal{C})$.

Definition 7. In an $\mathfrak{M}$-FCM Space $(\mathcal{Z}, \mathfrak{M}, *)$, $\mathfrak{M}$ is said to be triangular if, for all $\zeta, \eta, \omega, \mathsf{u} \in \mathcal{Z}$ and $c \in$ int$(\mathcal{C})$,

$$\left(\frac{1}{\mathfrak{M}(\zeta, \eta, \omega, c)} - 1\right) \leq \left(\frac{1}{\mathfrak{M}(\zeta, \eta, \mathsf{u}, c)} - 1\right) + \left(\frac{1}{\mathfrak{M}(\mathsf{u}, \omega, \omega, c)} - 1\right).$$

Definition 8. Let $(\mathcal{Z}, \mathfrak{M}, *)$ be an $\mathfrak{M}$-FCM Space, $\zeta' \in \mathcal{Z}$ and $\{\zeta_n\}$ be a sequence in $\mathcal{Z}$.

(i) $\{\zeta_n\}$ is said to converge to ζ' if for all $c \in int(\mathfrak{C})$, $\lim_{n\to+\infty}\left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta',\zeta',c)}-1\right)=0$. It is denoted by $\lim_{n\to+\infty}\zeta_n=\zeta'$ or by $\zeta_n \to \zeta'$ as $n \to +\infty$.

(ii) $\{\zeta_n\}$ is said to be a Cauchy sequence if $\lim_{n\to+\infty}\left(\frac{1}{\mathfrak{M}(\zeta_{n+m},\zeta_n,\zeta_n,c)}-1\right)=0$, for all $c \in int(\mathfrak{C})$ and $m \in \mathbb{N}$.

(iii) $(\mathcal{Z},\mathfrak{M},*)$ is called a complete $\mathfrak{M}$-FCM space if every Cauchy sequence in $\mathcal{Z}$ converges.

Definition 9. Let $(\mathcal{Z},\mathfrak{M},*)$ be an $\mathfrak{M}$-FCM Space. A sequence $\{\zeta_n\}$ in $\mathcal{Z}$ is $\mathfrak{M}$-Fuzzy Cone Contractive if there exists $k \in (0,1)$ such that

$$\left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta_{n+1},\zeta_{n+1},c)}-1\right) \le k\left(\frac{1}{\mathfrak{M}(\zeta_{n-1},\zeta_n,\zeta_n,c)}-1\right), \text{ for all } c \in int(\mathfrak{C}).$$

3. Main Results

Let us first state and prove the $\mathfrak{M}$-fuzzy cone Banach contraction theorem in a complete $\mathfrak{M}$-FCM Space.

Theorem 1. *Let $(\mathcal{Z},\mathfrak{M},*)$ be a complete $\mathfrak{M}$-FCM Space in which $\mathfrak{M}$-FCC sequences are Cauchy. Let $\mathcal{P}:\mathcal{Z}\to\mathcal{Z}$ be an $\mathfrak{M}$-FCC mapping. Then $\mathcal{P}$ has a unique fixed point.*

Proof. Let $\zeta_0 \in \mathcal{Z}$ and $c \in int(\mathfrak{C})$. Define a sequence $\{\zeta_n\}$ by

$$\zeta_n = \mathcal{P}^n\zeta_0,\ n \in \mathbb{N}.$$

Since $\mathcal{P}$ is $\mathfrak{M}$-FCC, we have

$$\left(\frac{1}{\mathfrak{M}(\mathcal{P}\zeta,\mathcal{P}^2\zeta,\mathcal{P}^2\zeta,c)}-1\right) \le k\left(\frac{1}{\mathfrak{M}(\zeta,\mathcal{P}\zeta,\mathcal{P}\zeta,c)}-1\right),$$

for all $\zeta \in \mathcal{Z}$ and for some $k \in (0,1)$. This gives

$$\left(\frac{1}{\mathfrak{M}(\zeta_{n+1},\zeta_{n+2},\zeta_{n+2},c)}-1\right) \le k\left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta_{n+1},\zeta_{n+1},c)}-1\right).$$

This makes $\{\zeta_n\}$ an $\mathfrak{M}$-FCC sequence and by assumption $\zeta_n \to \zeta$ for some $\zeta \in \mathcal{Z}$.
Now,

$$\left(\frac{1}{\mathfrak{M}(\mathcal{P}\zeta_n,\mathcal{P}\zeta,\mathcal{P}\zeta,c)}-1\right) \le k\left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta,\zeta,c)}-1\right).$$

As $k < 1$,

$$\lim_{n\to+\infty}\left(\frac{1}{\mathfrak{M}(\mathcal{P}\zeta_n,\mathcal{P}\zeta,\mathcal{P}\zeta,c)}-1\right)=0.$$

That is,

$$\left(\frac{1}{\mathfrak{M}(\zeta,\mathcal{P}\zeta,\mathcal{P}\zeta,c)}-1\right)=0, \text{ and which gives}$$
$$\mathcal{P}\zeta=\zeta.$$

Suppose $\mathcal{P}\eta = \eta$, for some $\eta \in \mathcal{Z}$. Then

$$\begin{aligned}\left(\frac{1}{\mathfrak{M}(\zeta,\zeta,\eta,c)}-1\right) &= \left(\frac{1}{\mathfrak{M}(\mathcal{P}\zeta,\mathcal{P}\zeta,\mathcal{P}\eta,c)}-1\right)\\ &\le k\left(\frac{1}{\mathfrak{M}(\zeta,\zeta,\eta,c)}-1\right)\\ &= \left(\frac{1}{\mathfrak{M}(\mathcal{P}\zeta,\mathcal{P}\zeta,\mathcal{P}\eta,c)}-1\right)\\ &\le k^2\left(\frac{1}{\mathfrak{M}(\zeta,\zeta,\eta,c)}-1\right)\\ &\cdots\cdots\cdots\cdots\cdots\cdots\\ &\le k^n\left(\frac{1}{\mathfrak{M}(\zeta,\zeta,\eta,c)}-1\right)\\ &\to 0 \quad \text{as} \quad n\to+\infty.\end{aligned}$$

Therefore $\zeta = \eta$. ■

The following theorem considers three self mappings and proves the existence of their unique fixed point under a generalized contractive condition in a complete $\mathfrak{M}$-FCM Space.

Theorem 2. *Let $(\mathcal{Z},\mathfrak{M},*)$ be a complete $\mathfrak{M}$-FCM Space where $\mathfrak{M}$ is triangular. If $\mathcal{P},\mathcal{Q},\mathcal{R}:\mathcal{Z}\to\mathcal{Z}$ is such that for all $\zeta,\eta,\omega\in\mathcal{Z}$ and $c\in int(\mathcal{C})$,*

$$\left(\frac{1}{\mathfrak{M}(\mathcal{P}\zeta,\mathcal{Q}\eta,\mathcal{R}\omega,c)}-1\right) \le \left\{\begin{array}{c} k_1\left(\frac{1}{\mathfrak{M}(\zeta,\eta,\omega,c)}-1\right)+k_2\left(\frac{1}{\mathfrak{M}(\zeta,\eta,\mathcal{R}\omega,c)}-1\right)\\ +k_3\left(\frac{1}{\mathfrak{M}(\zeta,\mathcal{Q}\eta,\omega,c)}-1\right)+k_4\left(\frac{1}{\mathfrak{M}(\mathcal{P}\zeta,\eta,\omega,c)}-1\right)\end{array}\right\} \tag{2.1}$$

where $k_i\in[0,+\infty]$, $i=1,\ldots,4$ and $k_1+2(k_2+k_3)+k_4<1$. Then $\mathcal{P},\mathcal{Q}$ and $\mathcal{R}$ have a unique common fixed point.

Proof. Let $\zeta_0\in\mathcal{Z}$ be arbitrary. Let the sequence $\{\zeta_n\}$ be defined by

$$\begin{aligned}\zeta_{3n+1} &= \mathcal{P}\zeta_{3n},\\ \zeta_{3n+2} &= \mathcal{Q}\zeta_{3n+1}, \text{and},\\ \zeta_{3n+3} &= \mathcal{R}\zeta_{3n+2} \quad \text{for } n\ge 0.\end{aligned}$$

From (2.1),

$$\begin{aligned}&\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+2},\zeta_{3n+2},c)}-1\right) \le \left(\frac{1}{\mathfrak{M}(\mathcal{P}\zeta_{3n},\mathcal{Q}\zeta_{3n+1},\mathcal{Q}\zeta_{3n+1},c)}-1\right)\\ &\le \left\{\begin{array}{c} k_1\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\zeta_{3n+1},\zeta_{3n+1},c)}-1\right)+k_2\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\zeta_{3n+1},\mathcal{Q}\zeta_{3n+1},c)}-1\right)\\ +k_3\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\mathcal{Q}\zeta_{3n+1},\zeta_{3n+1},c)}-1\right)+k_4\left(\frac{1}{\mathfrak{M}(\mathcal{P}\zeta_{3n},\zeta_{3n+1},\zeta_{3n+1},c)}-1\right)\end{array}\right\}\\ &= \left\{\begin{array}{c} k_1\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\zeta_{3n+1},\zeta_{3n+1},c)}-1\right)+k_2\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\zeta_{3n+1},\zeta_{3n+2},c)}-1\right)\\ +k_3\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\zeta_{3n+2},\zeta_{3n+1},c)}-1\right)+k_4\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+1},\zeta_{3n+1},c)}-1\right)\end{array}\right\}\\ &= \left\{\begin{array}{c} k_1\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\zeta_{3n+1},\zeta_{3n+1},c)}-1\right)\\ +k_2\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\zeta_{3n+1},\zeta_{3n+2},c)}-1\right)+k_3\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\zeta_{3n+2},\zeta_{3n+1},c)}-1\right)\end{array}\right\}\end{aligned}$$

$$\leq \left\{ \begin{array}{c} k_1\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\zeta_{3n+1},\zeta_{3n+1},c)}-1\right) \\ +k_2\left[\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\zeta_{3n+1},\zeta_{3n+1},c)}-1\right)+\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+2},\zeta_{3n+2},c)}-1\right)\right] \\ +k_3\left[\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\zeta_{3n+1},\zeta_{3n+1},c)}-1\right)+\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+2},\zeta_{3n+2},c)}-1\right)\right] \end{array} \right\}$$

$$= \left\{ \begin{array}{c} (k_1+k_2+k_3)\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\zeta_{3n+1},\zeta_{3n+1},c)}-1\right) \\ +(k_2+k_3)\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+2},\zeta_{3n+2},c)}-1\right) \end{array} \right\}.$$

Therefore,

$$\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+2},\zeta_{3n+2},c)}-1\right) \leq \frac{k_1+k_2+k_3}{1-(k_2+k_3)}\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\zeta_{3n+1},\zeta_{3n+1},c)}-1\right). \tag{2.2}$$

Again, from (2.1),

$$\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\zeta_{3n+3},\zeta_{3n+3},c)}-1\right) \leq \left(\frac{1}{\mathfrak{M}(\mathcal{Q}\zeta_{3n+1},\mathcal{R}\zeta_{3n+2},\mathcal{R}\zeta_{3n+2},c)}-1\right)$$

$$\leq \left\{ \begin{array}{c} k_1\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+2},\zeta_{3n+2},c)}-1\right)+k_2\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+2},\mathcal{R}\zeta_{3n+2},c)}-1\right) \\ +k_3\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\mathcal{R}\zeta_{3n+2},\zeta_{3n+2},c)}-1\right)+k_4\left(\frac{1}{\mathfrak{M}(\mathcal{Q}\zeta_{3n+1},\zeta_{3n+2},\zeta_{3n+2},c)}-1\right) \end{array} \right\}$$

$$= \left\{ \begin{array}{c} k_1\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+2},\zeta_{3n+2},c)}-1\right)+k_2\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+2},\zeta_{3n+3},c)}-1\right) \\ +k_3\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+3},\zeta_{3n+2},c)}-1\right)+k_4\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\zeta_{3n+2},\zeta_{3n+2},c)}-1\right) \end{array} \right\}$$

$$\leq \left\{ \begin{array}{c} k_1\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+2},\zeta_{3n+2},c)}-1\right) \\ +k_2\left[\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+2},\zeta_{3n+2},c)}-1\right)+\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\zeta_{3n+3},\zeta_{3n+3},c)}-1\right)\right] \\ +k_3\left[\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+2},\zeta_{3n+2},c)}-1\right)+\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\zeta_{3n+3},\zeta_{3n+3},c)}-1\right)\right] \end{array} \right\}$$

$$= \left\{ \begin{array}{c} (k_1+k_2+k_3)\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+2},\zeta_{3n+2},c)}-1\right) \\ +(k_2+k_3)\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\zeta_{3n+3},\zeta_{3n+3},c)}-1\right) \end{array} \right\}.$$

This gives,

$$\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\zeta_{3n+3},\zeta_{3n+3},c)}-1\right) \leq \frac{k_1+k_2+k_3}{1-(k_2+k_3)}\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+2},\zeta_{3n+2},c)}-1\right). \tag{2.3}$$

Again, using (2.1),

$$\left(\frac{1}{\mathfrak{M}(\zeta_{3n+3},\zeta_{3n+4},\zeta_{3n+4},c)}-1\right) \leq \left(\frac{1}{\mathfrak{M}(\mathcal{R}\zeta_{3n+2},\mathcal{P}\zeta_{3n+3},\mathcal{P}\zeta_{3n+3},c)}-1\right)$$

$$\leq \left\{ \begin{array}{c} k_1\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\zeta_{3n+3},\zeta_{3n+3},c)}-1\right)+k_2\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\zeta_{3n+3},\mathcal{P}\zeta_{3n+3},c)}-1\right) \\ +k_3\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\mathcal{P}\zeta_{3n+3},\zeta_{3n+3},c)}-1\right)+k_4\left(\frac{1}{\mathfrak{M}(\mathcal{R}\zeta_{3n+2},\zeta_{3n+3},\zeta_{3n+3},c)}-1\right) \end{array} \right\}$$

$$= \left\{ \begin{array}{c} k_1\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\zeta_{3n+3},\zeta_{3n+3},c)}-1\right) \\ +k_2\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\zeta_{3n+3},\zeta_{3n+4},c)}-1\right)+k_3\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\zeta_{3n+4},\zeta_{3n+3},c)}-1\right) \end{array} \right\}$$

$$\leq \left\{ \begin{array}{c} k_1\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\zeta_{3n+3},\zeta_{3n+3},c)}-1\right) \\ +k_2\left[\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\zeta_{3n+3},\zeta_{3n+3},c)}-1\right)+\left(\frac{1}{\mathfrak{M}(\zeta_{3n+3},\zeta_{3n+4},\zeta_{3n+4},c)}-1\right)\right] \\ +k_3\left[\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\zeta_{3n+3},\zeta_{3n+3},c)}-1\right)+\left(\frac{1}{\mathfrak{M}(\zeta_{3n+3},\zeta_{3n+4},\zeta_{3n+4},c)}-1\right)\right] \end{array} \right\}$$

$$= \left\{ \begin{array}{c} (k_1+k_2+k_3)\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\zeta_{3n+3},\zeta_{3n+3},c)}-1\right) \\ +(k_2+k_3)\left(\frac{1}{\mathfrak{M}(\zeta_{3n+3},\zeta_{3n+4},\zeta_{3n+4},c)}-1\right) \end{array} \right\}.$$

This gives,

$$\left(\frac{1}{\mathfrak{M}(\zeta_{3n+3},\zeta_{3n+4},\zeta_{3n+4},c)}-1\right) \leq \frac{k_1+k_2+k_3}{1-(k_2+k_3)}\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\zeta_{3n+3},\zeta_{3n+3},c)}-1\right). \tag{2.4}$$

Put $\mathfrak{M}_n = \left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta_{n+1},\zeta_{n+1},c)}-1\right)$ and $k = \frac{k_1+k_2+k_3}{1-(k_2+k_3)}$. Then from (2.2) to (2.4) we have the following inequalities:
For $n = 0, 1, 2, \ldots,$

$$\begin{aligned} &\mathfrak{M}_{3n+1} \leq k\mathfrak{M}_{3n}, \\ &\mathfrak{M}_{3n+2} \leq k\mathfrak{M}_{3n+1}, \text{ and,} \\ &\mathfrak{M}_{3n+3} \leq k\mathfrak{M}_{3n+2}. \end{aligned}$$

These inequalities together gives that

$$\mathfrak{M}_{n+1} \leq k\mathfrak{M}_n \quad \text{for}\, n = 0, 1, 2, \ldots, \tag{2.5}$$

which makes $\{\zeta_n\}$ an $\mathfrak{M}$-FCC sequence.
Now, $\mathfrak{M}$ is triangular and the space $(\mathcal{Z}, \mathfrak{M}, *)$ is symmetric. Therefore we have,

$$\begin{aligned} \left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta_n,\zeta_m,c)}-1\right) &\leq \left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta_n,\zeta_{n+1},c)}-1\right) + \left(\frac{1}{\mathfrak{M}(\zeta_{n+1},\zeta_m,\zeta_m,c)}-1\right) \\ &= \left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta_{n+1},\zeta_{n+1},c)}-1\right) + \left(\frac{1}{\mathfrak{M}(\zeta_{n+1},\zeta_{n+1},\zeta_m,c)}-1\right) \\ &\leq \left\{ \begin{array}{c} \left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta_{n+1},\zeta_{n+1},c)}-1\right) \\ +\left(\frac{1}{\mathfrak{M}(\zeta_{n+1},\zeta_{n+1},\zeta_{n+2},c)}-1\right)+\left(\frac{1}{\mathfrak{M}(\zeta_{n+2},\zeta_m,\zeta_m,c)}-1\right) \end{array} \right\} \\ &\leq \left\{ \begin{array}{c} \left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta_{n+1},\zeta_{n+1},c)}-1\right) \\ +\left(\frac{1}{\mathfrak{M}(\zeta_{n+1},\zeta_{n+2},\zeta_{n+2},c)}-1\right)+\cdots+\left(\frac{1}{\mathfrak{M}(\zeta_{m-1},\zeta_m,\zeta_m,c)}-1\right) \end{array} \right\} \\ &= \mathfrak{M}_n + \mathfrak{M}_{n+1} + \cdots + \mathfrak{M}_{m-1} \\ &\leq k^n\mathfrak{M}_0 + k^{n+1}\mathfrak{M}_0 + \cdots + k^{m-1}\mathfrak{M}_0 \\ &\leq \frac{k^n}{1-k}\mathfrak{M}_0 \to 0 \,\text{as}\, n \to +\infty. \end{aligned}$$

Thus $\{\zeta_n\}$ is Cauchy. As $\mathcal{Z}$ is complete, there exists $\dot{\zeta} \in \mathcal{Z}$ such that

$$\lim_{n\to+\infty}\left(\frac{1}{\mathfrak{M}(\zeta_n,\dot{\zeta},\dot{\zeta},c)}-1\right) = 0. \tag{2.6}$$

Since $\mathfrak{M}$ is triangular,

$$\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\mathcal{P}\dot{\zeta},c)}-1\right) \leq \left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\zeta_{3n+2},c)}-1\right) + \left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\mathcal{P}\dot{\zeta},\mathcal{P}\dot{\zeta},c)}-1\right). \tag{2.7}$$

From (2.1),

$$\begin{aligned}\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\mathcal{P}\dot{\zeta},\mathcal{P}\dot{\zeta},c)}-1\right) &\leq \left(\frac{1}{\mathfrak{M}(\mathcal{Q}\zeta_{3n+1},\mathcal{P}\dot{\zeta},\mathcal{P}\dot{\zeta},c)}-1\right)\\ &\leq \left\{\begin{array}{c} k_1\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\dot{\zeta},\dot{\zeta},c)}-1\right)+k_2\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\dot{\zeta},\mathcal{P}\dot{\zeta},c)}-1\right)\\ +k_3\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\mathcal{P}\dot{\zeta},\dot{\zeta},c)}-1\right)+k_4\left(\frac{1}{\mathfrak{M}(\mathcal{Q}\zeta_{3n+1},\dot{\zeta},\dot{\zeta},c)}-1\right)\end{array}\right\}\\ &= \left\{\begin{array}{c} k_1\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\dot{\zeta},\dot{\zeta},c)}-1\right)+k_2\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\dot{\zeta},\mathcal{P}\dot{\zeta},c)}-1\right)\\ +k_3\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\mathcal{P}\dot{\zeta},\dot{\zeta},c)}-1\right)+k_4\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\dot{\zeta},\dot{\zeta},c)}-1\right)\end{array}\right\}\\ &\to (k_2+k_3)\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\mathcal{P}\dot{\zeta},c)}-1\right)\ \text{as}\, n\to+\infty.\end{aligned}$$

Therefore,

$$\lim_{n\to+\infty}\sup\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\mathcal{P}\dot{\zeta},\mathcal{P}\dot{\zeta},c)}-1\right)\leq (k_2+k_3)\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\mathcal{P}\dot{\zeta},c)}-1\right). \tag{2.8}$$

From (2.7) and (2.8), we have that

$$\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\mathcal{P}\dot{\zeta},c)}-1\right)\leq (k_2+k_3)\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\mathcal{P}\dot{\zeta},c)}-1\right).$$

Since $k_2+k_3<1$, we have

$$\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\mathcal{P}\dot{\zeta},c)}-1\right)=0,\quad \text{and this gives}$$
$$\mathcal{P}\dot{\zeta}=\dot{\zeta}.$$

Since $\mathfrak{M}$ is triangular,

$$\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\mathcal{Q}\dot{\zeta},c)}-1\right)\leq\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\zeta_{3n+3},c)}-1\right)+\left(\frac{1}{\mathfrak{M}(\zeta_{3n+3},\mathcal{Q}\dot{\zeta},\mathcal{Q}\dot{\zeta},c)}-1\right). \tag{2.9}$$

From (2.1),

$$\begin{aligned}\left(\frac{1}{\mathfrak{M}(\zeta_{3n+3},\mathcal{Q}\dot{\zeta},\mathcal{Q}\dot{\zeta},c)}-1\right) &= \left(\frac{1}{\mathfrak{M}(\mathcal{R}\zeta_{3n+2},\mathcal{Q}\dot{\zeta},\mathcal{Q}\dot{\zeta},c)}-1\right)\\ &\leq \left\{\begin{array}{c} k_1\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\dot{\zeta},\dot{\zeta},c)}-1\right)+k_2\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\dot{\zeta},\mathcal{Q}\dot{\zeta},c)}-1\right)\\ +k_3\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\mathcal{Q}\dot{\zeta},\dot{\zeta},c)}-1\right)+k_4\left(\frac{1}{\mathfrak{M}(\mathcal{R}\zeta_{3n+2},\dot{\zeta},\dot{\zeta},c)}-1\right)\end{array}\right\}\\ &= \left\{\begin{array}{c} k_1\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\dot{\zeta},\dot{\zeta},c)}-1\right)+k_2\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\dot{\zeta},\mathcal{Q}\dot{\zeta},c)}-1\right)\\ +k_3\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\mathcal{Q}\dot{\zeta},\dot{\zeta},c)}-1\right)+k_4\left(\frac{1}{\mathfrak{M}(\zeta_{3n+3},\dot{\zeta},\dot{\zeta},c)}-1\right)\end{array}\right\}\\ &\to (k_2+k_3)\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\mathcal{Q}\dot{\zeta},c)}-1\right)\ \text{as}\, n\to+\infty.\end{aligned}$$

Therefore,

$$\lim_{n\to+\infty}\sup\left(\frac{1}{\mathfrak{M}(\zeta_{3n+3},\mathcal{Q}\dot{\zeta},\mathcal{Q}\dot{\zeta},c)}-1\right)\leq (k_2+k_3)\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\mathcal{Q}\dot{\zeta},c)}-1\right). \tag{2.10}$$

From (2.9) and (2.10), we have

$$\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\mathcal{Q}\dot{\zeta},c)}-1\right)\leq(k_2+k_3)\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\mathcal{Q}\dot{\zeta},c)}-1\right).$$

Since $k_2+k_3<1$, we have

$$\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\mathcal{Q}\dot{\zeta},c)}-1\right)=0,\quad \text{and this gives}$$

$$\mathcal{Q}\dot{\zeta}=\dot{\zeta}$$

Since $\mathfrak{M}$ is triangular,

$$\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\mathcal{R}\dot{\zeta},c)}-1\right)\leq\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\zeta_{3n+1},c)}-1\right)+\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\mathcal{R}\dot{\zeta},\mathcal{R}\dot{\zeta},c)}-1\right). \tag{2.11}$$

From (2.1),

$$\begin{aligned}\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\mathcal{R}\dot{\zeta},\mathcal{R}\dot{\zeta},c)}-1\right)&=\left(\frac{1}{\mathfrak{M}(\mathcal{P}\zeta_{3n},\mathcal{R}\dot{\zeta},\mathcal{R}\dot{\zeta},c)}-1\right)\\&\leq\left\{\begin{array}{c}k_1\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\dot{\zeta},\dot{\zeta},c)}-1\right)+k_2\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\dot{\zeta},\mathcal{R}\dot{\zeta},c)}-1\right)\\+k_3\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\mathcal{R}\dot{\zeta},\dot{\zeta},c)}-1\right)+k_4\left(\frac{1}{\mathfrak{M}(\mathcal{P}\zeta_{3n},\dot{\zeta},\dot{\zeta},c)}-1\right)\end{array}\right\}\\&=\left\{\begin{array}{c}k_1\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\dot{\zeta},\dot{\zeta},c)}-1\right)+k_2\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\dot{\zeta},\mathcal{R}\dot{\zeta},c)}-1\right)\\+k_3\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\mathcal{R}\dot{\zeta},\dot{\zeta},c)}-1\right)+k_4\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\dot{\zeta},\dot{\zeta},c)}-1\right)\end{array}\right\}\\&\to(k_2+k_3)\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\mathcal{R}\dot{\zeta},c)}-1\right)\ \text{as}\ n\to+\infty.\end{aligned}$$

Therefore,

$$\lim_{n\to+\infty}\sup\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\mathcal{R}\dot{\zeta},\mathcal{R}\dot{\zeta},c)}-1\right)\leq(k_2+k_3)\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\mathcal{R}\dot{\zeta},c)}-1\right). \tag{2.12}$$

From (2.11) and (2.12), we have that

$$\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\mathcal{R}\dot{\zeta},c)}-1\right)\leq(k_2+k_3)\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\mathcal{R}\dot{\zeta},c)}-1\right).$$

Since $k_2+k_3<1$, we have $\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\mathcal{R}\dot{\zeta},c)}-1\right)=0$, and this gives

$$\mathcal{R}\dot{\zeta}=\dot{\zeta}.$$

Thus we have shown that

$$\mathcal{P}\dot{\zeta}=\mathcal{Q}\dot{\zeta}=\mathcal{R}\dot{\zeta}=\dot{\zeta}.$$

Suppose $\mathcal{P}\ddot{\zeta} = \mathcal{Q}\ddot{\zeta} = \mathcal{R}\ddot{\zeta} = \ddot{\zeta}$. Then from (2.1),

$$\begin{aligned}
\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},c)}-1\right) &= \left(\frac{1}{\mathfrak{M}(\mathcal{P}\dot{\zeta},\mathcal{Q}\ddot{\zeta},\mathcal{R}\ddot{\zeta},c)}-1\right)\\
&\leq \left\{\begin{array}{c} k_1\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},c)}-1\right)+k_2\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\ddot{\zeta},\mathcal{R}\ddot{\zeta},c)}-1\right)\\ +k_3\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\mathcal{Q}\ddot{\zeta},\ddot{\zeta},c)}-1\right)+k_4\left(\frac{1}{\mathfrak{M}(\mathcal{P}\dot{\zeta},\ddot{\zeta},\ddot{\zeta},c)}-1\right)\end{array}\right\}\\
&= \left\{\begin{array}{c} k_1\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},c)}-1\right)+k_2\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},c)}-1\right)\\ +k_3\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},c)}-1\right)+k_4\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},c)}-1\right)\end{array}\right\}\\
&= (k_1+k_2+k_3+k_4)\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},c)}-1\right)
\end{aligned}$$

$$\text{That is,}\quad \left(\frac{1}{\mathfrak{M}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},c)}-1\right) \leq (k_1+k_2+k_3+k_4)\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},c)}-1\right).$$

$$\text{Therefore,}\quad \left(\frac{1}{\mathfrak{M}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},c)}-1\right) = 0, \quad \text{since } k_1+k_2+k_3+k_4<1.$$

Hence we can conclude that $\dot{\zeta}$ is the unique common fixed point of $\mathcal{P}, \mathcal{Q}$ and $\mathcal{R}$. ■

Corollary 3. *Let $(\mathcal{Z}, \mathfrak{M}, *)$ be a complete $\mathfrak{M}$-FCM Space where $\mathfrak{M}$ is triangular. If $\mathcal{P} : X \to X$ is such that for all $\zeta, \eta, \omega \in X$ and $c \in int(\mathcal{C})$,*

$$\left(\frac{1}{\mathfrak{M}(\mathcal{P}\zeta,\mathcal{P}\eta,\mathcal{P}\omega,c)}-1\right) \leq \left\{\begin{array}{c} k_1\left(\frac{1}{\mathfrak{M}(\zeta,\eta,\omega,c)}-1\right)+k_2\left(\frac{1}{\mathfrak{M}(\zeta,\eta,\mathcal{P}\omega,c)}-1\right)\\ +k_3\left(\frac{1}{\mathfrak{M}(\zeta,\mathcal{P}\eta,\omega,c)}-1\right)+k_4\left(\frac{1}{\mathfrak{M}(\mathcal{P}\zeta,\eta,\omega,c)}-1\right)\end{array}\right\},$$

where $k_i \in [0, +\infty], i = 1, \ldots, 4$ and $k_1 + 2(k_2 + k_3) + k_4 < 1$. Then $\mathcal{P}$ has unique fixed point.

Corollary 4. *Theorem 2 gives Theorem 1 when $\mathcal{P} = \mathcal{Q} = \mathcal{R}$ and $k_2 = k_3 = k_4 = 0$.*

where , $\mathcal{C} =$ and a continuous t-norm $*$.

Example 5. *Consider$(\mathcal{Z}, \mathfrak{M}, *)$ in which $\mathfrak{M} : \mathcal{Z}^3 \times (0, +\infty) \to [0, 1]$ by*

$$\mathfrak{M}(\zeta,\eta,\omega,c) = \frac{\|c\|}{\|c\| + (|\zeta-\eta| + |\eta-\omega| + |\omega-\zeta|)} \text{ for all } \zeta,\eta,\omega \in \mathcal{Z} \text{ and } c \in int(\mathcal{C})$$

where $\mathcal{Z} = \{1, 2, 3\}$ and $\mathcal{C} = \mathbb{R}^+$. Then it is clear that $(\mathcal{Z}, \mathfrak{M}, *)$ is a complete $\mathfrak{M}$-FCM Space and that $\mathfrak{M}$ is triangular. Consider the self mappings $\mathcal{P}, \mathcal{Q}$ and $\mathcal{R}$ from $\mathcal{Z}$ to $\mathcal{Z}$, given by $P(1) = 1$, $P(2) = 2$, $P(3) = 1$, $Q(1) = 1$, $Q(2) = 2$, $Q(3) = 2$, $R(1) = 3$, $R(2) = 2$ and $R(3) = 2$. Then each one of $\mathcal{P}, \mathcal{Q}$ and $\mathcal{R}$ is not $\mathfrak{M}$-FCC and it is not possible for the $\mathfrak{M}$-fuzzy cone Banach contraction theorem to assure the existence of their respective fixed points. But $\mathcal{P}, \mathcal{Q}$ and $\mathcal{R}$ together satisfies the condition (2.1) with $k_1 = \frac{1}{10}, k_2 = \frac{1}{25}, k_3 = \frac{1}{25}$ and $k_4 = \frac{3}{5}$. Therefore $\mathcal{P}, \mathcal{Q}$ and $\mathcal{R}$ have a unique common fixed point which is 2.

Theorem 6. *Let $(\mathcal{Z}, \mathfrak{M}, *)$ be a complete $\mathfrak{M}$-FCM Space where $\mathfrak{M}$ is triangular. If $\mathcal{P}, \mathcal{Q}, \mathcal{R} : \mathcal{Z} \to \mathcal{Z}$ is such that for all $\zeta, \eta, \omega \in \mathcal{Z}$ and $c \in int(\mathcal{C})$,*

$$\left(\frac{1}{\mathfrak{M}(\mathcal{P}\zeta,\mathcal{Q}\eta,\mathcal{R}\omega,c)}-1\right) \leq k\left(\frac{1}{\Psi(\zeta,\eta,\omega)}-1\right), \tag{6.1}$$

where $\Psi(\zeta, \eta, \omega) = \min\{\mathfrak{M}(\zeta, \mathcal{Q}\eta, \mathcal{R}\omega, c), \mathfrak{M}(\mathcal{P}\zeta, \eta, \mathcal{R}\omega, c), \mathfrak{M}(\mathcal{P}\zeta, \mathcal{Q}\eta, \omega, c)\}$ and $k \in (0, 1)$. Then $\mathcal{P}, \mathcal{Q}$ and $\mathcal{R}$ have unique common fixed point.

Proof. Let $\zeta_0 \in \mathcal{Z}$ be arbitrary. Define the sequence $\{\zeta_n\}$ as in Theorem (2).
From (6.1),

$$\begin{aligned}
\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+2},\zeta_{3n+2},c)}-1\right) &= \left(\frac{1}{\mathfrak{M}(\mathcal{P}\zeta_{3n},\mathcal{Q}\zeta_{3n+1},\mathcal{Q}\zeta_{3n+1},c)}-1\right)\\
&\leq k\left(\frac{1}{\Psi(\zeta,\eta,\omega)}-1\right),\\
\text{where, } \Psi(\zeta,\eta,\omega) &= \min\left\{\begin{array}{c}\mathfrak{M}(\zeta_{3n},\mathcal{Q}\zeta_{3n+1},\mathcal{Q}\zeta_{3n+1},c),\mathfrak{M}(\mathcal{P}\zeta_{3n},\zeta_{3n+1},\mathcal{Q}\zeta_{3n+1},c),\\ \mathfrak{M}(\mathcal{P}\zeta_{3n},\mathcal{Q}\zeta_{3n+1},\zeta_{3n+1},c)\end{array}\right\}\\
&= \min\left\{\begin{array}{c}\mathfrak{M}(\zeta_{3n},\zeta_{3n+2},\zeta_{3n+2},c),\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+1},\zeta_{3n+2},c),\\ \mathfrak{M}(\zeta_{3n+1},\zeta_{3n+2},\zeta_{3n+1},c)\end{array}\right\}\\
&= \min\left\{\mathfrak{M}(\zeta_{3n},\zeta_{3n+2},\zeta_{3n+2},c),\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+1},\zeta_{3n+2},c)\right\}.
\end{aligned}$$

Case(i) $\Psi(\zeta,\eta,\omega)=\mathfrak{M}(\zeta_{3n},\zeta_{3n+2},\zeta_{3n+2},c)$.

$$\begin{aligned}
\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+2},\zeta_{3n+2},c)}-1\right) &\leq k\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\zeta_{3n+2},\zeta_{3n+2},c)}-1\right)\\
&\leq k\left\{\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\zeta_{3n+2},\zeta_{3n+1},c)}-1\right)+\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\zeta_{3n},\zeta_{3n},c)}-1\right)\right\}.
\end{aligned}$$

Therefore,

$$\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+2},\zeta_{3n+2},c)}-1\right) \leq \frac{k}{1-k}\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\zeta_{3n+1},\zeta_{3n+1},c)}-1\right).$$

Case(ii) $\Psi(\zeta,\eta,\omega)=\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+1},\zeta_{3n+2},c)$.

$$\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+2},\zeta_{3n+2},c)}-1\right) \leq k\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+1},\zeta_{3n+2},c)}-1\right), \text{ and, this gives}$$
$$\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+2},\zeta_{3n+2},c)}-1\right)=0, \text{ which is absurd.}$$

Therefore,

$$\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+2},\zeta_{3n+2},c)}-1\right) \leq \frac{k}{1-k}\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\zeta_{3n+1},\zeta_{3n+1},c)}-1\right). \tag{6.2}$$

From (6.1),

$$\begin{aligned}
\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\zeta_{3n+3},\zeta_{3n+3},c)}-1\right) &= \left(\frac{1}{\mathfrak{M}(\mathcal{Q}\zeta_{3n+1},\mathcal{R}\zeta_{3n+2},\mathcal{R}\zeta_{3n+2},c)}-1\right)\\
&\leq k\left(\frac{1}{\Psi(\zeta,\eta,\omega)}-1\right),\\
\text{where, } \Psi(\zeta,\eta,\omega) &= \min\left\{\begin{array}{c}\mathfrak{M}(\zeta_{3n+1},\mathcal{R}\zeta_{3n+2},\mathcal{R}\zeta_{3n+2},c),\mathfrak{M}(\mathcal{Q}\zeta_{3n+1},\zeta_{3n+2},\mathcal{R}\zeta_{3n+2},c),\\ \mathfrak{M}(\mathcal{Q}\zeta_{3n+1},\mathcal{R}\zeta_{3n+2},\zeta_{3n+2},c)\end{array}\right\}\\
&= \min\left\{\begin{array}{c}\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+3},\zeta_{3n+3},c),\mathfrak{M}(\zeta_{3n+2},\zeta_{3n+2},\zeta_{3n+3},c),\\ \mathfrak{M}(\zeta_{3n+2},\zeta_{3n+3},\zeta_{3n+2},c)\end{array}\right\}\\
&= \min\left\{\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+3},\zeta_{3n+3},c),\mathfrak{M}(\zeta_{3n+2},\zeta_{3n+2},\zeta_{3n+3},c)\right\}.
\end{aligned}$$

Case(i) $\Psi(\zeta,\eta,\omega)=\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+3},\zeta_{3n+3},c)$.

$$\begin{aligned}
\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\zeta_{3n+3},\zeta_{3n+3},c)}-1\right) &\leq k\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+3},\zeta_{3n+3},c)}-1\right)\\
&\leq k\left\{\left(\frac{1}{\mathfrak{M}(\zeta_{3n+3},\zeta_{3n+3},\zeta_{3n+2},c)}-1\right)+\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\zeta_{3n+1},\zeta_{3n+1},c)}-1\right)\right\}.
\end{aligned}$$

Therefore,

$$\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\zeta_{3n+3},\zeta_{3n+3},c)}-1\right)\leq\frac{k}{1-k}\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+2},\zeta_{3n+2},c)}-1\right).$$

Case(ii) $\Psi(\zeta,\eta,\omega)=\mathfrak{M}(\zeta_{3n+2},\zeta_{3n+2},\zeta_{3n+3},c)$.

$$\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\zeta_{3n+3},\zeta_{3n+3},c)}-1\right)\leq k\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\zeta_{3n+2},\zeta_{3n+3},c)}-1\right),\text{ and, this gives}$$

$$\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\zeta_{3n+3},\zeta_{3n+3},c)}-1\right)=0,\text{ which is absurd.}$$

Therefore,

$$\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\zeta_{3n+3},\zeta_{3n+3},c)}-1\right)\leq\frac{k}{1-k}\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+2},\zeta_{3n+2},c)}-1\right). \tag{6.3}$$

Again, from (6.1),

$$\begin{aligned}\left(\frac{1}{\mathfrak{M}(\zeta_{3n+3},\zeta_{3n+4},\zeta_{3n+4},c)}-1\right)&=\left(\frac{1}{\mathfrak{M}(\mathcal{R}\zeta_{3n+2},\mathcal{P}\zeta_{3n+3},\mathcal{P}\zeta_{3n+3},c)}-1\right)\\&\leq k\left(\frac{1}{\Psi(\zeta,\eta,\omega)}-1\right),\\ \text{where, }\Psi(\zeta,\eta,\omega)&=\min\left\{\begin{array}{c}\mathfrak{M}(\zeta_{3n+2},\mathcal{P}\zeta_{3n+3},\mathcal{P}\zeta_{3n+3},c),\mathfrak{M}(\mathcal{R}\zeta_{3n+2},\zeta_{3n+3},\mathcal{P}\zeta_{3n+3},c),\\ \mathfrak{M}(\mathcal{R}\zeta_{3n+2},\mathcal{P}\zeta_{3n+3},\zeta_{3n+3},c)\end{array}\right\}\\&=\min\left\{\begin{array}{c}\mathfrak{M}(\zeta_{3n+2},\zeta_{3n+4},\zeta_{3n+4},c),\mathfrak{M}(\zeta_{3n+3},\zeta_{3n+3},\zeta_{3n+4},c),\\ \mathfrak{M}(\zeta_{3n+3},\zeta_{3n+4},\zeta_{3n+3},c)\end{array}\right\}\\&=\min\left\{\ \mathfrak{M}(\zeta_{3n+2},\zeta_{3n+4},\zeta_{3n+4},c),\mathfrak{M}(\zeta_{3n+3},\zeta_{3n+3},\zeta_{3n+4},c)\ \right\}.\end{aligned}$$

Case(i) $\Psi(\zeta,\eta,\omega)=\mathfrak{M}(\zeta_{3n+2},\zeta_{3n+4},\zeta_{3n+4},c)$.

$$\begin{aligned}\left(\frac{1}{\mathfrak{M}(\zeta_{3n+3},\zeta_{3n+4},\zeta_{3n+4},c)}-1\right)&\leq k\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\zeta_{3n+4},\zeta_{3n+4},c)}-1\right)\\&\leq k\left\{\ \left(\tfrac{1}{\mathfrak{M}(\zeta_{3n+4},\zeta_{3n+4},\zeta_{3n+3},c)}-1\right)+\left(\tfrac{1}{\mathfrak{M}(\zeta_{3n+3},\zeta_{3n+2},\zeta_{3n+2},c)}-1\right)\ \right\}.\end{aligned}$$

Therefore,

$$\left(\frac{1}{\mathfrak{M}(\zeta_{3n+3},\zeta_{3n+4},\zeta_{3n+4},c)}-1\right)\leq\frac{k}{1-k}\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\zeta_{3n+3},\zeta_{3n+2},c)}-1\right).$$

Case(ii) $\Psi(\zeta,\eta,\omega)=\mathfrak{M}(\zeta_{3n+3},\zeta_{3n+3},\zeta_{3n+4},c)$.

$$\left(\frac{1}{\mathfrak{M}(\zeta_{3n+3},\zeta_{3n+4},\zeta_{3n+4},c)}-1\right)\leq k\left(\frac{1}{\mathfrak{M}(\zeta_{3n+3},\zeta_{3n+3},\zeta_{3n+4},c)}-1\right),\text{ and, this gives}$$

$$\left(\frac{1}{\mathfrak{M}(\zeta_{3n+3},\zeta_{3n+4},\zeta_{3n+4},c)}-1\right)=0,\ \text{which is absurd.}$$

Therefore, $$\left(\frac{1}{\mathfrak{M}(\zeta_{3n+3},\zeta_{3n+4},\zeta_{3n+4},c)}-1\right)\leq\frac{k}{1-k}\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\zeta_{3n+3},\zeta_{3n+2},c)}-1\right). \tag{6.4}$$

From (6.2), (6.3) and (6.4), we obtain

$$\left(\frac{1}{\mathfrak{M}(\zeta_{n+1},\zeta_{n+2},\zeta_{n+2},c)}-1\right)\leq\frac{k}{1-k}\left(\frac{1}{\mathfrak{M}(\zeta_{n},\zeta_{n+1},\zeta_{n+1},c)}-1\right),\text{ and, this gives,}$$

$$\left(\frac{1}{\mathfrak{M}(\zeta_{n+1},\zeta_{n+2},\zeta_{n+2},c)}-1\right)\leq\left(\frac{k}{1-k}\right)^{n}\left(\frac{1}{\mathfrak{M}(\zeta_{0},\zeta_{1},\zeta_{1},c)}-1\right).$$

The above two inequalities imply that $\{\zeta_n\}$ is $\mathfrak{M}$-FCC and Cauchy. Therefore there is an element $\dot{\zeta} \in \mathcal{Z}$ such that

$$\lim_{n\to+\infty} \left(\frac{1}{\mathfrak{M}(\zeta_n, \dot{\zeta}, \dot{\zeta}, t)} - 1 \right) = 0. \tag{6.5}$$

Since $\mathfrak{M}$ is triangular,

$$\left(\frac{1}{\mathfrak{M}(\dot{\zeta}, \dot{\zeta}, \mathcal{P}\dot{\zeta}, t)} - 1 \right) \le \left(\frac{1}{\mathfrak{M}(\dot{\zeta}, \dot{\zeta}, \zeta_{3n+2}, t)} - 1 \right) + \left(\frac{1}{\mathfrak{M}(\zeta_{3n+2}, \mathcal{P}\dot{\zeta}, \mathcal{P}\dot{\zeta}, t)} - 1 \right). \tag{6.6}$$

From (6.1),

$$\begin{aligned}
\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2}, \mathcal{P}\dot{\zeta}, \mathcal{P}\dot{\zeta}, t)} - 1 \right) &= \left(\frac{1}{\mathfrak{M}(\mathcal{Q}\zeta_{3n+1}, \mathcal{P}\dot{\zeta}, \mathcal{P}\dot{\zeta}, t)} - 1 \right) \\
&\le k \left(\frac{1}{\Psi(\zeta, \eta, \omega)} - 1 \right), \\
\text{where, } \Psi(\zeta, \eta, \omega) &= \min\left\{ \ \mathfrak{M}(\zeta_{3n+1}, \mathcal{P}\dot{\zeta}, \mathcal{P}\dot{\zeta}, t), \mathfrak{M}(\mathcal{Q}\zeta_{3n+1}, \dot{\zeta}, \mathcal{P}\dot{\zeta}, t), \mathfrak{M}(\mathcal{Q}\zeta_{3n+1}, \mathcal{P}\dot{\zeta}, \dot{\zeta}, t) \ \right\} \\
&= \min\left\{ \ \mathfrak{M}(\zeta_{3n+1}, \zeta_{3n+1}, \mathcal{P}\dot{\zeta}, t), \mathfrak{M}(\zeta_{3n+2}, \dot{\zeta}, \mathcal{P}\dot{\zeta}, t), \mathfrak{M}(\zeta_{3n+2}, \mathcal{P}\dot{\zeta}, \dot{\zeta}, t) \ \right\} \\
&\to \mathfrak{M}(\dot{\zeta}, \dot{\zeta}, \mathcal{P}\dot{\zeta}, t) \quad \text{as} \quad n \to +\infty.
\end{aligned}$$

Therefore,

$$\lim_{n\to+\infty} \sup \left(\frac{1}{\mathfrak{M}(\zeta_{3n+2}, \mathcal{P}\dot{\zeta}, \mathcal{P}\dot{\zeta}, t)} - 1 \right) \le k \left(\frac{1}{\mathfrak{M}(\dot{\zeta}, \dot{\zeta}, \mathcal{P}\dot{\zeta}, t)} - 1 \right). \tag{6.7}$$

From (6.5), (6.6) and (6.7), we have that

$$\left(\frac{1}{\mathfrak{M}(\dot{\zeta}, \dot{\zeta}, \mathcal{P}\dot{\zeta}, t)} - 1 \right) \le k \left(\frac{1}{\mathfrak{M}(\dot{\zeta}, \dot{\zeta}, \mathcal{P}\dot{\zeta}, t)} - 1 \right). \tag{6.8}$$

Therefore,

$$\left(\frac{1}{\mathfrak{M}(\dot{\zeta}, \dot{\zeta}, \mathcal{P}\dot{\zeta}, t)} - 1 \right) = 0$$

.

This gives $\mathcal{P}\dot{\zeta} = \dot{\zeta}$. Since $\mathfrak{M}$ is triangular,

$$\left(\frac{1}{\mathfrak{M}(\dot{\zeta}, \dot{\zeta}, \mathcal{Q}\dot{\zeta}, t)} - 1 \right) \le \left(\frac{1}{\mathfrak{M}(\dot{\zeta}, \dot{\zeta}, \zeta_{3n+3}, t)} - 1 \right) + \left(\frac{1}{\mathfrak{M}(\zeta_{3n+3}, \mathcal{Q}\dot{\zeta}, \mathcal{Q}\dot{\zeta}, t)} - 1 \right). \tag{6.9}$$

From (6.1),

$$\begin{aligned}
\left(\frac{1}{\mathfrak{M}(\zeta_{3n+3}, \mathcal{Q}\dot{\zeta}, \mathcal{Q}\dot{\zeta}, t)} - 1 \right) &= \left(\frac{1}{\mathfrak{M}(\mathcal{R}\zeta_{3n+2}, \mathcal{Q}\dot{\zeta}, \mathcal{Q}\dot{\zeta}, t)} - 1 \right) \\
&\le k \left(\frac{1}{\Psi(\zeta, \eta, \omega)} - 1 \right), \\
\text{where, } \Psi(\zeta, \eta, \omega) &= \min\left\{ \ \mathfrak{M}(\zeta_{3n+2}, \mathcal{Q}\dot{\zeta}, \mathcal{Q}\dot{\zeta}, t), \mathfrak{M}(\mathcal{R}\zeta_{3n+2}, \dot{\zeta}, \mathcal{Q}\dot{\zeta}, t), \mathfrak{M}(\mathcal{R}\zeta_{3n+2}, \mathcal{Q}\dot{\zeta}, \dot{\zeta}, t) \ \right\} \\
&= \min\left\{ \ \mathfrak{M}(\zeta_{3n+2}, \mathcal{Q}\dot{\zeta}, \mathcal{Q}\dot{\zeta}, t), \mathfrak{M}(\zeta_{3n+3}, \dot{\zeta}, \mathcal{Q}\dot{\zeta}, t), \mathfrak{M}(\zeta_{3n+3}, \mathcal{Q}\dot{\zeta}, \dot{\zeta}, t) \ \right\} \\
&\to \mathfrak{M}(\dot{\zeta}, \dot{\zeta}, \mathcal{Q}\dot{\zeta}, t) \quad \text{as} \quad n \to +\infty.
\end{aligned}$$

Therefore,

$$\lim_{n\to+\infty} \sup \left(\frac{1}{\mathfrak{M}(\zeta_{3n+3}, \mathcal{Q}\dot{\zeta}, \mathcal{Q}\dot{\zeta}, t)} - 1 \right) \le k \left(\frac{1}{\mathfrak{M}(\dot{\zeta}, \dot{\zeta}, \mathcal{Q}\dot{\zeta}, t)} - 1 \right). \tag{6.10}$$

From (6.5), (6.9) and (6.10), we have that

$$\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\mathcal{Q}\dot{\zeta},t)}-1\right)\leq k\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\mathcal{Q}\dot{\zeta},t)}-1\right).$$

$$\text{Therefore, }\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\mathcal{Q}\dot{\zeta},t)}-1\right)=0,\text{ and this gives,}$$

$$\mathcal{Q}\dot{\zeta}=\dot{\zeta}. \tag{6.11}$$

Since $\mathfrak{M}$ is triangular,

$$\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\mathcal{R}\dot{\zeta},t)}-1\right)\leq\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\zeta_{3n+1},t)}-1\right)+\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\mathcal{R}\dot{\zeta},\mathcal{R}\dot{\zeta},t)}-1\right). \tag{6.12}$$

From (6.1),

$$\begin{aligned}\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\mathcal{R}\dot{\zeta},\mathcal{R}\dot{\zeta},t)}-1\right)&=\left(\frac{1}{\mathfrak{M}(\mathcal{P}\zeta_{3n},\mathcal{R}\dot{\zeta},\mathcal{R}\dot{\zeta},t)}-1\right)\\&\leq k\left(\frac{1}{\Psi(\zeta,\eta,\omega)}-1\right),\\\text{where, }\Psi(\zeta,\eta,\omega)&=\min\left\{\ \mathfrak{M}(\zeta_{3n},\mathcal{R}\dot{\zeta},\mathcal{R}\dot{\zeta},t),\mathfrak{M}(\mathcal{P}\zeta_{3n},\dot{\zeta},\mathcal{R}\dot{\zeta},t),\mathfrak{M}(\mathcal{P}\zeta_{3n},\mathcal{R}\dot{\zeta},\dot{\zeta},t)\ \right\}\\&=\min\left\{\ \mathfrak{M}(\zeta_{3n},\mathcal{R}\dot{\zeta},\mathcal{R}\dot{\zeta},t),\mathfrak{M}(\zeta_{3n+1},\dot{\zeta},\mathcal{R}\dot{\zeta},t),\mathfrak{M}(\zeta_{3n+1},\mathcal{R}\dot{\zeta},\dot{\zeta},t)\ \right\}\\&\to\mathfrak{M}(\dot{\zeta},\dot{\zeta},\mathcal{R}\dot{\zeta},t)\quad\text{as}\quad n\to+\infty.\end{aligned}$$

Therefore,

$$\lim_{n\to+\infty}\sup\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\mathcal{R}\dot{\zeta},\mathcal{R}\dot{\zeta},t)}-1\right)\leq k\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\mathcal{R}\dot{\zeta},t)}-1\right). \tag{6.13}$$

From (6.5), (6.12) and (6.13), we have

$$\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\mathcal{R}\dot{\zeta},t)}-1\right)\leq k\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\mathcal{R}\dot{\zeta},t)}-1\right).$$

$$\text{Therefore, }\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\mathcal{R}\dot{\zeta},t)}-1\right)=0,\text{ and this gives}$$

$$\mathcal{R}\dot{\zeta}=\dot{\zeta}. \tag{6.14}$$

From (6.8), (6.11) and (6.14), we get $\mathcal{P}\dot{\zeta}=\mathcal{Q}\dot{\zeta}=\mathcal{R}\dot{\zeta}=\dot{\zeta}$.
Suppose $\mathcal{P}\ddot{\zeta}=\mathcal{Q}\ddot{\zeta}=\mathcal{R}\ddot{\zeta}=\ddot{\zeta}$. Then from (6.1),

$$\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},t)}-1\right)=\left(\frac{1}{\mathfrak{M}(\mathcal{P}\dot{\zeta},\mathcal{Q}\ddot{\zeta},\mathcal{R}\ddot{\zeta},t)}-1\right)\leq k\left(\frac{1}{\Psi(\zeta,\eta,\omega)}-1\right),$$

$$\begin{aligned}\text{where, }\Psi(\zeta,\eta,\omega)&=\min\left\{\ \mathfrak{M}(\dot{\zeta},\mathcal{Q}\ddot{\zeta},\mathcal{R}\ddot{\zeta},t),\mathfrak{M}(\mathcal{P}\dot{\zeta},\ddot{\zeta},\mathcal{R}\ddot{\zeta},t),\mathfrak{M}(\mathcal{P}\dot{\zeta},\mathcal{Q}\ddot{\zeta},\ddot{\zeta},t)\ \right\}\\&=\min\left\{\ \mathfrak{M}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},t),\mathfrak{M}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},t),\mathfrak{M}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},t)\ \right\}\\&=\mathfrak{M}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},t).\end{aligned}$$

Therefore,

$$\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},t)}-1\right)\leq k\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},t)}-1\right).$$

$$\text{Hence, }\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},t)}-1\right)=0,\text{ , and this gives,}$$

$$\dot{\zeta}=\ddot{\zeta}.$$

Hence we can conclude that $\mathcal{P},\mathcal{Q}$ and $\mathcal{R}$ have a unique common fixed point. ■

Example 7. *Consider the $\mathfrak{M}$-FCM Space given in Example (5) with $\mathcal{Z} = [0, +\infty]$ and the self mappings $\mathcal{P}, \mathcal{Q}$ and $\mathcal{R}$ from $\mathcal{Z}$ to $\mathcal{Z}$, given by $\mathcal{P}\zeta = \frac{2}{3}\zeta + 1$, $\mathcal{Q}\eta = \frac{1}{3}\eta + 2$, and $\mathcal{R}\omega = 3$. It is easily seen that condition* (6.1) *holds and therefore $\mathcal{P}, \mathcal{Q}$ and $\mathcal{R}$ have a unique common fixed point and it is* 3.

Corollary 8. *Let $(\mathcal{Z}, \mathfrak{M}, *)$ be a complete $\mathfrak{M}$-FCM Space where $\mathfrak{M}$ is triangular. If $\mathcal{P} : \mathcal{Z} \to \mathcal{Z}$ is such that for all $\zeta, \eta, \omega \in \mathcal{Z}$ and $c \in int(\mathcal{C})$,*

$$\left(\frac{1}{\mathfrak{M}(\mathcal{P}\zeta, \mathcal{P}\eta, \mathcal{P}\omega, c)} - 1\right) \leq k\left(\frac{1}{\Psi(\zeta, \eta, \omega)} - 1\right)$$

where $\Psi(\zeta, \eta, \omega) = \min\{\mathfrak{M}(\zeta, \mathcal{P}\eta, \mathcal{P}\omega, c), \mathfrak{M}(\mathcal{P}\zeta, \eta, \mathcal{P}\omega, c), \mathfrak{M}(\mathcal{P}\zeta, \mathcal{P}\eta, z, c)\}$ and $k \in (0, 1)$. Then $\mathcal{P}$ has a unique fixed point.

Conclusion:

We constructed $\mathfrak{M}$-fuzzy cone Banach contraction theorem and theorems which assure the common fixed points for three self mappings under generalized fuzzy contractive conditions in $\mathfrak{M}$-fuzzy cone metric spaces. This work can be either extended or generalized to various kinds of other spaces.

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