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Modeling life time data with new alpha power pareto transformation distribution

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Abstract

The need arose to find composite, expanded, or transformed distributions that would be more flexible in representing data using different statistical methods. This research includes a study of the proposed distribution resulting from alpha energy conversion. Using the survival function of the Pareto distribution to obtain the new distribution is the alpha power Pareto transformed (APTP) distribution. The new distribution provides more outstanding suitability than the Pareto and transformed Pareto distributions. Some statistics properties of the proposed distribution, such as quantile function, moments, and survival reliability risk and demand statistics are discussed. The maximum likelihood was used to estimate the parameter, and theactual data sets were used to see how the new distribution performs better in real life.

Key words and phrases: Pareto distribution, survival function, alpha power transformation.

Mathematics Subject Classification: 68Q01, 68T01

1. Introduction

The main purpose of this study is to propose a new probability distribution using the alpha power transformation technique to transform the Pareto probability distribution into a new distribution to improve the statistical properties of the data.

For this purpose, there are many studies based on this technique that are:

The researcher used a method called transformation to develop a Pareto distribution with the Kumaraswamy distribution, where a distribution called the Kumaraswamy Pareto distribution

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emerged [1]. Using the quadratic transformation, the researcher generalized the Pareto distribution and provided a comprehensive description of the properties of this distribution [2].

The CDF and PDF of the Transmuted Pareto distribution is.

$$F(x,a,x_\circ,\lambda)=\left[1-\left(\frac{x_\circ}{x}\right)^a\right]\left[1+\lambda\left(\frac{x_\circ}{x}\right)^a\right] \tag{1}$$

$$f(x,a,x_\circ,\lambda)=\frac{ax_\circ^a}{x^{a+1}}\left[1-\lambda+2\lambda\left(\frac{x_\circ}{x}\right)^a\right] \tag{2}$$

Tahir et al. [3] proposed alpha power Pareto exponential distribution for age data by studying the statistical characteristics. Tahir and Akhter [4] where the researcher used the quadratic equation (QRTM rank transformation map) to generalize the Weibull-Pareto distribution.

The CDF and PDF are as follows.

$$F(x)=\left[1-e^{-\delta\left(\frac{x}{\theta}\right)^\beta}\right]\left[1+\lambda e^{-\delta\left(\frac{x}{\theta}\right)^\beta}\right] \tag{3}$$

$$f(x)=\frac{\beta\delta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1}e^{-\delta\left(\frac{x}{\theta}\right)^\beta}\left[1-\lambda+2\lambda e^{-\delta\left(\frac{x}{\theta}\right)^\beta}\right] \tag{4}$$

to distribution was developed by adding a new parameter to the original distribution using the Alpha-Power method while finding many characteristics of the proposed distribution [5]. The researcher proposed the Exponentiated Generalized Pareto Distribution (EXGPD), which was found by transforming the variable of the generalized Pareto distribution [6].

2. New Alpha Power Transformation Method

Let $S(u)$ be the survival function of a continuous random variable U, then the new Alpha power transformation of $S(u)$ for $u \in R$ is defined as follows:

$$S_A(u)=\begin{cases}\dfrac{1-\alpha^{S(u)}}{1-\alpha} & \text{if } \alpha\neq 1\\ S(u) & \text{if } \alpha=1\end{cases} \tag{5}$$

Since $\dfrac{dS_A(u)}{du}$ = - $f(u)$, *then* $\dfrac{dS_A(u)}{du}$

$$f_A(u;x_\circ,\alpha,\theta)=\begin{cases}\dfrac{-ln(\alpha)\alpha^{S(u)}}{1-\alpha}f(u) & \text{if } \alpha\neq 1\\ f(u) & \text{if } \alpha=1\end{cases} \tag{6}$$

3. New Alpha Power Transformation with Parto Distribution:

Let $U\sim P(\theta)$ Pareto random variable, then the APTP distribution if the survival function of the Pareto distribution.

$$S(u)=\left[\frac{x_\circ}{u}\right]^\theta$$

Where $x_\circ$ is positive and $u > x_\circ$, θ is a positive parameterthat can be defined as follows.
The survival function of the distribution APTP is as follows:

$$S_{AP}(u;x_\circ,\alpha,\theta)=\begin{cases}\dfrac{1-\alpha^{\left[\frac{x_\circ}{u}\right]^{\theta}}}{1-\alpha} & \text{if} \quad \alpha\neq 1\\ \left[\dfrac{x_\circ}{u}\right]^{\theta} & \text{if} \quad \alpha=1\end{cases} \tag{7}$$

$$x > x_\circ \text{ and } \theta > 0$$

From Figure 1, it is noted that the survival function exponentially decreases as long as the value of u increases.

3.1. The CDF and PDF of the APTP Distribution

The PDF of APTPD shown in eq (8)

$$g_{AP}(u;x_\circ,\alpha,\theta)=\begin{cases}\dfrac{-\theta x_\circ^{\theta}(\ln\alpha)\alpha^{\left[\frac{x_\circ}{u}\right]^{\theta}}}{(1-\alpha)u^{\theta+1}} & \text{if} \quad \alpha\neq 1\\ \dfrac{-\theta x_\circ^{\theta}}{u^{\theta+1}} & \text{if} \quad \alpha=1\end{cases} \tag{8}$$

$$0<\alpha\left\langle 1,\theta\right\rangle 0$$

Note: $g_{AP}\left(u\right)$ is a PDF since*that*

1. $g_{AP}(u)\geq 0\forall\, u\in R$ for $0<\alpha\left\langle 1. g_{AP}(u)\right\rangle 0,\ \forall u\in(x_\circ,\infty)$
2. $\int_{x_\circ}^{\infty} g_{AP}\, du=\int_{x_\circ}^{\infty}\dfrac{-\theta x_\circ^{\theta}(\ln\alpha)\alpha^{\left[\frac{x_\circ}{u}\right]^{\theta}}}{(1-\alpha)u^{\theta+1}}du=\dfrac{1}{1-\alpha}\left|\alpha^{\left(\frac{x_\circ}{u}\right)^{\theta}}\right|_{x_\circ}^{\infty}=1$

Figure 2 shows that the PDF increases up to the peak, decreasing as the u value rises. It is right skewed.

The CDF of APTP distribution shown in eq (9)

$$G_{AP}\left(u\right)=1-S_{AP}\left(u;x_\circ,\alpha,\theta\right)$$

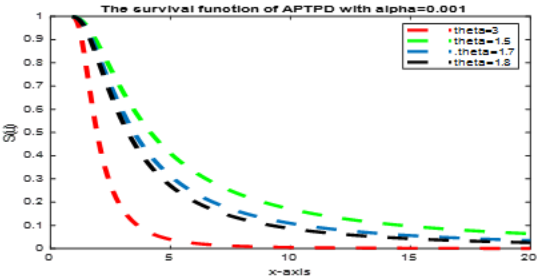

Figure 1: The survival function of APTPD at various values of $\alpha,\theta,x_\circ=0.9$.

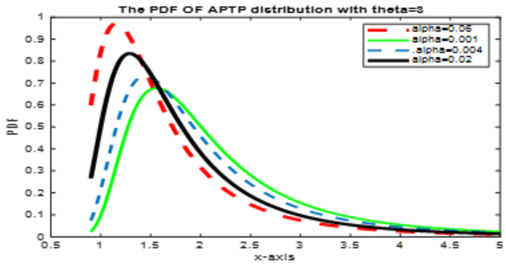

Figure 2: The PDF of APTPD at various values of ±,¸ , $x_{\circ} = 0.9$.

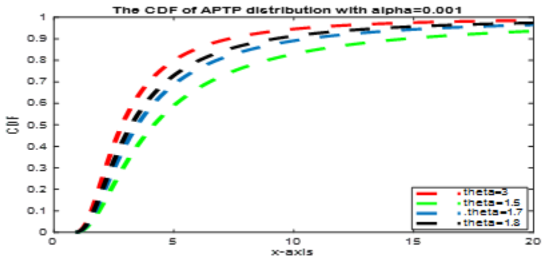

Figure 3: The CDF of APTPD at various values of ±,¸ , $x_{\circ} = 0.9$.

$$G_{AP}(u;x_{\circ},\alpha,\theta)=\begin{cases}\dfrac{\alpha^{\left(\frac{x_{\circ}}{u}\right)^{\theta}}-\alpha}{(1-\alpha)} & \text{if} \quad \alpha\neq 1\\ \alpha^{\left(\frac{x_{\circ}}{u}\right)^{\theta}} & \text{if} \quad \alpha=1\end{cases} \tag{9}$$

$$\forall u\in(x_{\circ},\infty).\ \alpha>0,\theta>0,u>x_{\circ}$$

Figure 3 shows that the CDF monotonically increases as long as the value of x increases.

3.2. The Hazard Function

The hazard function of APTP distribution is shown in eq (10)

$$h_{AP}(u)=\frac{g_{AP}(u.\alpha.\lambda)}{S_{AP}(u)} \tag{10}$$

3.3. The Cumulative Hazard Function

The cumulative hazard function of APTP distribution is shown in eq (11).

$$H_{AP}(u)=-\ln S_{AP}(u)$$

$$H_{AP}(x)=\begin{cases}-\ln\left(\dfrac{1-\alpha^{[\frac{x_{\circ}}{x}]^{\theta}}}{1-\alpha}\right) & \text{if} \quad \alpha\neq 1\\ -ln\left[\dfrac{x_{\circ}}{x}\right]^{\theta} & \text{if} \quad \alpha=1\end{cases} \tag{11}$$

4. Moments

The r[th] moment of APTP distribution about the origin is as follows.

$$E(U^r) = \int_{x_\circ}^{\infty} u^r g_{AP}(u)du = -\int_{x_\circ}^{\infty} u^r \frac{\theta x_\circ^\theta (ln\alpha)\alpha^{\left[\frac{x_\circ}{u}\right]^\theta}}{(1-\alpha)u^{\theta+1}} du$$

1) If $\theta = r$

$$E(U^r) = -\sum_{n=0}^{\infty} \frac{x_\circ^{\theta(1-n)} (\ln\alpha)^{n+1}}{n!(1-\alpha)n} \tag{12}$$

2) If $\theta < r$

$$E(U^r) = \infty$$

3) If $\theta > r$, thus $r-(n+1)\theta =$ negative number

Therefore

$$E(U^r) = \sum_{n=0}^{\infty} \frac{\theta(\ln\alpha)^{n+1} x_\circ{}^{r-n\theta}}{n!(1-\alpha)(r-(n+1)\theta)} \tag{13}$$

The rth moment of NATP distribution about the mean is shown.

$$E(U-\mu)^r = \int_{x_\circ}^{\infty} (u-\mu)^r f_{AP} dx = -\int_{x_\circ}^{\infty} (u-\mu)^r \frac{\theta x_\circ^\theta (ln\alpha)\alpha^{\left[\frac{x_\circ}{u}\right]^\theta}}{(1-\alpha)u^{\theta+1}} du$$

1) If $i = \theta$

$$E(U-\mu)^r = -\sum_{i=0}^{r}\sum_{n=0}^{\infty} \binom{r}{i} (-\mu)^{r-i} \frac{x_\circ{}^{\theta(1-n)} (\ln\alpha)^{n+1}}{n!(1-\alpha)n} \tag{14}$$

2) If $\theta < i$

$$E(U^r) = \infty$$

3) If $\theta > i$, thus $i-(n+1)\theta =$ negative number

$$E(U-\mu)^r = \sum_{i=0}^{r}\sum_{n=0}^{\infty} \binom{r}{i} (-\mu)^{r-i} \frac{\theta x_\circ{}^{i-n\theta} (\ln\alpha)^{n+1}}{n!(1-\alpha)(i-\theta(n+1))} \tag{15}$$

5. Moment Generating Function

Proposition 1: The moment-generating function of APTPD is shown.

$$M_u(t) = E\left(e^{tu}\right) = \int_{x_\circ}^{\infty} e^{tu} g_{AP}(u)du = -\int_{x_\circ}^{\infty} e^{tu} \frac{\theta x_\circ^\theta (ln\alpha)\alpha^{[\frac{x_\circ}{u}]^\theta}}{(1-\alpha)u^{\theta+1}} du$$

The MGF is only defined for t ≤ 0

$$M_u\left(0,\theta,x_\circ\right)=-\frac{\theta(ln\alpha)^{n+1}}{\left(1-\alpha\right)n\left(\theta+1\right)} \tag{16}$$

$$M_u\left(t,\theta,x_\circ\right)=-\frac{\theta x_\circ^{\theta(n+1)}(ln\alpha)^{n+1}}{\left(1-\alpha\right)(-t)^{-n(\theta+1)}}\text{“}\left(-n\left(\theta+1\right),-tx_\circ\right) \tag{17}$$

Where $\Gamma\left(a,x\right)=\int_x^\infty t^{a-1}e^{-t}dt$

6. Maximum Likelihood Estimation

It is considered one of the essential estimation methods, and its estimators have excellent properties. Let a random sample. $U_1,U_2,\ldots,U_k$ From the probability distribution defined (6), is taken then the likelihood function will be:

$$L\left(u_1,u_2,\ldots,u_k,\theta,\alpha\right)=\prod_{i=1}^{n}g_{Ap}\left(x_i,\theta,\alpha\right)=\prod_{i=1}^{n}\frac{-\theta x_\circ^\theta\left(ln\alpha\right)\alpha^{[\frac{x_\circ}{u}]^\theta}}{\left(1-\alpha\right)u^{\theta+1}} \tag{18}$$

We take the natural logarithm of both sides of the equation (26)

$$lnL=n\ln\theta+n\theta lnx_\circ+nln\left(-ln\alpha\right)+\sum_{i=1}^{n}(\frac{x_\circ}{u_i})^\theta ln\alpha-nln\left(1-\alpha\right)-\sum_{i=1}^{n}ln(u_i)^{\theta+1}. \tag{19}$$

By differentiating both sides of the equation (19), respect each parameter as

7. Application

The statistical criteria AIC, BIC, HQIC, and CAIC were adopted to determine whether the distribution that was proposed APTPD better represents real data from some selected distributions, such as the Pareto distribution (PD), alpha power Pareto distribution (APPD) and alpha power transformed Pareto distribution (APTD).

Data set: The following data set is the remission times (in months) of a random sample of 128 bladder cancer patients [7].

0.08 , 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34,5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69].

Table 1 shows the parameter values for the proposed distribution and other distributions used and Table 2 shows the quality criteria values for the proposed distribution and other distributions used.

8. The Simulation Concept

Simulation is defined as a representation of real reality, that is, an attempt to find a mirror image of any model or system without taking the system or model itself by using specific models.

Table 1: The MLEs of the parameters (α,θ).

Model	Parameters		
$APTPD(\alpha,\theta)$	$\hat{\alpha}=0.901$	$\hat{\theta}=0.11$	
$PD(\alpha)$	$\hat{\alpha}=0.95$		
$APPD(\alpha,\beta)$	$\hat{\beta}=0.248$	$\hat{\alpha}=1.363$	
$APTD(\alpha,\beta,\theta)$	$\hat{\alpha}=1.017$	$\hat{\beta}=0.969$	$\hat{\theta}=0.0487$

Table 2: Represents the results of statistical tests (BIC, CAIC, AIC) on the data.

Model	LL	AIC	BIC	HQIC	CAIC
$APTPD(\alpha,\theta)$	-454.4194	912.8387	916.8534	914.3912	913.0695
$PD(\alpha)$	-494.1295	992.2591	994.2664	993.0353	992.3346
$APPD(\alpha,\beta)$	-460.0085	924.0169	928.0316	925.5694	924.2477
$APTD(\alpha,\beta,\theta)$	-491.9073	989.8145	995.8365	992.1433	990.2851

The simulation process starts from building the simulation model and then conducting experiments to study its behavior, based on a set of statistical indicators.

9. Algorithm of Experiments

Step (1): Choose different default values for parameters α,β.

Step (2): Choose small, medium, and large sample sizes: n=25,50,75,100.

Step (3): Generating random data that fits the proposed model APTPD using the inverse transformation method based on the quantile function.

Step (4): Using the Newton-Raphson method to solve nonlinear systems that result from estimation methods to find initial values.

Step (5): We repeat the above steps 500 and 1000 times.

Step (6): --

Step (7): Calculate the estimated mean square error (MSE) values for the survival functions and compare the results.

R represents the times each experiment was repeated and equals R = 500, 1000.

Tables 3 and 4 shows the values of the estimators for survival countries with the mean square error for each function and according to the sample sizes (25,50,75 and 100), where we note the following.

1. The values of SF, HF, and RHF decrease as long as increasing the sample size while the CHF increases.
2. The least mean square error of the survival function in the second model is at a sample size of 25 where $\boldsymbol{MSE} = 9.4946\mathbf{e}-07$
3. The least mean square error of the hazard function in the fourth modelis at a sample size of 100 where $\boldsymbol{MSE} = 0.0064211$

Table 3: The estimated survival functions and the mean square error for each function with $\alpha = 0.5, \beta = 1.3$.

n	Estimating the SF	Estimating the HF	Estimating the CHF	Estimating the RHF
25	0.792303348	10.38609181	0.232810946	39.61997086
	0.253846944	3.329054571	1.371023777	1.132569681
	0.151126671	1.982006132	1.889636916	0.352860642
	0.10758936	1.411033753	2.22943352	0.170114756
	0.083526488	1.095453858	2.482591476	0.099838579
	0.068259811	0.895232616	2.684434108	0.065585245
	0.05771149	0.756891588	2.852298993	0.046356653
	0.049986908	0.655583551	2.995994151	0.034494887
	0.044086061	0.578193692	3.12161162	0.026665876
	0.039431281	0.517145966	3.233195845	0.021228807
	0.035665568	0.467758365	3.333569537	0.017299878
	0.032556409	0.426981483	3.424781039	0.01436878
	0.029945866	0.392743977	3.508363993	0.012124126
	0.027722899	0.363589546	3.585496535	0.010367164
	0.025807159	0.338464426	3.657103335	0.008966197
	0.024139073	0.316587298	3.723923463	0.007831161
	0.022673533	0.297366585	3.786556963	0.006898771
	0.021375761	0.28034615	3.845497672	0.006123507
	0.020218507	0.265168634	3.901156897	0.005471949
	0.019180122	0.251550093	3.953880837	0.004919111
MSE	0.00030275	0.059724	0.41469	0.24566
50	0.792303801	10.38607838	0.232810373	39.62002879
	0.388600724	5.095939378	0.945202878	3.238940267
	0.257416998	3.375894606	1.357057947	1.170256597
	0.192447707	2.523921834	1.647930816	0.601475561
	0.153663867	2.015304355	1.872987744	0.36590599
	0.127890026	1.677293035	2.056584554	0.245965597
	0.109520225	1.436378283	2.211646043	0.176660355
	0.095764757	1.255977365	2.345860545	0.133016677
	0.08507901	1.115834397	2.464174923	0.103762059
	0.076538564	1.003826345	2.56996056	0.083199389
	0.069556312	0.912253658	2.665618614	0.068196497
	0.063741468	0.835991375	2.752919933	0.056915175
	0.058823843	0.771496031	2.83320802	0.048218775
	0.05461065	0.71623931	2.907526358	0.041373741
	0.050960646	0.668368818	2.976701584	0.035889457
	0.047767982	0.626496352	3.041399698	0.031427704
	0.044951769	0.589561074	3.102165168	0.027749188
	0.042449132	0.556738386	3.159448809	0.024680737
	0.040210461	0.527377632	3.2136281	0.022094529
	0.038196084	0.500958537	3.265022281	0.019894548

n	Estimating the SF	Estimating the HF	Estimating the CHF	Estimating the RHF
MSE	0.0003	0.062142	0.41538	0.12514
75	0.792303354	10.38607444	0.232810938	39.619906
	0.469413907	6.155267959	0.756270369	5.445616491
	0.333485805	4.373241388	1.098154978	2.188121327
	0.258599381	3.391308786	1.352475207	1.18288322
	0.211177127	2.769448137	1.555058034	0.741413719
	0.178452028	2.340298548	1.723435464	0.508346483
	0.154508457	2.026300868	1.867506444	0.370294208
	0.136229922	1.786591499	1.993411217	0.281773156
	0.121818608	1.597596111	2.105222161	0.221613594
	0.110164642	1.444760663	2.205779291	0.17886628
	0.100545769	1.318613821	2.297142238	0.147401654
	0.092471728	1.212726356	2.380852325	0.123569596
	0.085598023	1.12258057	2.458093092	0.105085815
	0.079675502	1.044909036	2.529793123	0.090461192
	0.074519504	0.977290077	2.596694393	0.078691201
	0.069990263	0.917890753	2.659399145	0.069078218
	0.065980046	0.865298201	2.718402916	0.061125477
	0.062404472	0.818405817	2.77411834	0.054471445
	0.059196511	0.776334537	2.826892675	0.048847922
MSE	0.00030006	0.061692	0.4157	0.087334
100	0.792304498	10.38605551	0.232809493	39.62010944
	0.523269596	6.861260837	0.647658468	7.531068196
	0.390611298	5.122322211	0.940042337	3.283350872
	0.311607591	4.086497721	1.166010604	1.849793364
	0.259183989	3.399096463	1.350217086	1.189217522
	0.221858569	2.909639911	1.505715175	0.829577404
	0.193930042	2.543392928	1.640257793	0.611907555
	0.172246652	2.259035537	1.758827806	0.470081226
	0.154924365	2.031865673	1.864818249	0.372493877
	0.140767725	1.846208475	1.960644091	0.302463692
	0.128981616	1.691637921	2.048085398	0.2505001
	0.119016628	1.560949575	2.128492066	0.2108768
	0.110480953	1.44900552	2.202912148	0.179970862
	0.103087655	1.352042998	2.272175636	0.155398621
	0.096621784	1.267243196	2.336951052	0.135539353
	0.090919135	1.192452715	2.397784789	0.119259765
	0.085852106	1.125998163	2.455129163	0.105748002
	0.081320037	1.066559474	2.509362839	0.094410087
	0.077242457	1.013081327	2.560806006	0.084803307
	0.073554266	0.964709949	2.609731826	0.076592217
MSE	0.00030028	0.061133	0.41572	0.069697

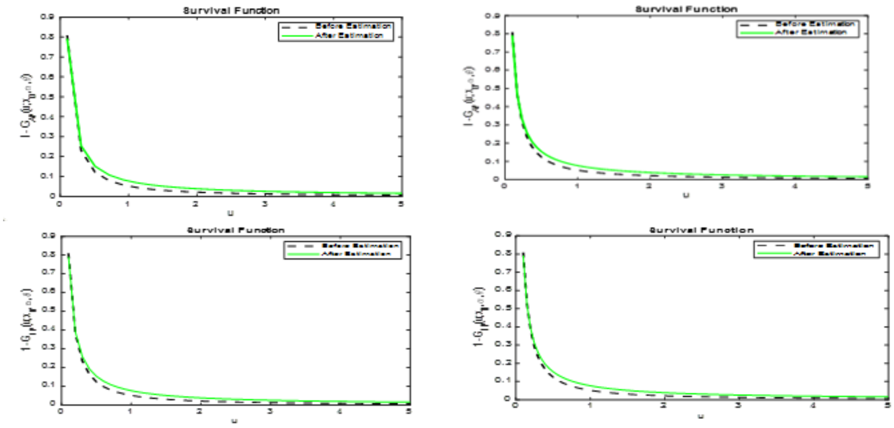

Figure 4: The survival function before and after estimation at a sample size of 25,50,75 and 100.

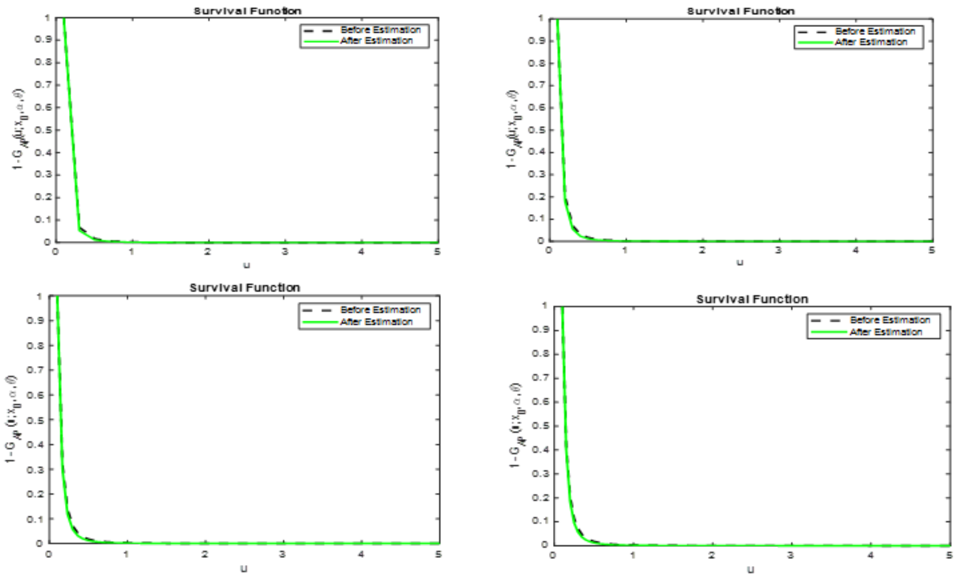

Figure 5: The survival function before and after estimation at a sample size of 25,50,75 and 100.

Table 4: The estimated survival functions and the mean square error for each function.
$\alpha = 0.5, \beta = 2.7$.

n	Estimating the SF	Estimating the HF	Estimating the CHF	Estimating the RHF
25	1	18.3258061	18.3258061	Inf
	0.053395363	9.703242714	9.703242714	0.54733322
	0.011586031	5.881039038	5.881039038	0.068936602
	0.004216895	4.205166805	4.205166805	0.017807839
	0.001981559	3.270758859	3.270758859	0.006494069
	0.001084269	2.675691003	2.675691003	0.002904317
	0.000656402	2.263689717	2.263689717	0.001486866
	0.000427055	1.961588226	1.961588226	0.000838064
	0.000293238	1.730605257	1.730605257	0.000507629
	0.000209968	1.548279086	1.548279086	0.000325158
	0.000155464	1.400703402	1.400703402	0.000217793
	0.000118304	1.278809192	1.278809192	0.000151307
	9.21E-05	1.176430122	1.176430122	0.000108363
	7.31E-05	1.089227442	1.089227442	7.96E-05
	5.90E-05	1.014059701	1.014059701	5.98E-05
	4.83E-05	0.948596442	0.948596442	4.58E-05
	4.00E-05	0.891072382	0.891072382	3.57E-05
	3.35E-05	0.840125886	0.840125886	2.82E-05
	2.84E-05	0.794689842	0.794689842	2.26E-05
	2.42E-05	18.3258061	0.753916119	1.83E-05
MSE	9.4946e-07	0.24382	0.0089586	0.00010194
50	1	18.3258254	0	Inf
	0.180369866	14.15722507	1.712745723	3.115474515
	0.055613606	9.831172382	2.889327391	0.57894412
	0.023692559	7.446361026	3.74259424	0.180704706
	0.012173049	5.977973445	4.408530864	0.073666915
	0.007055504	4.989350329	4.953947216	0.035452518
	0.004446626	4.279947927	5.415609564	0.019116333
	0.002980218	3.746606563	5.815758865	0.01119908
	0.002093667	3.331204375	6.168838019	0.006989067
	0.001526548	2.998594703	6.48474601	0.004584499
	0.001147053	2.72630584	6.77055967	0.003130807
	0.000883595	2.499311356	7.031512026	0.002210331
	0.000695012	2.307187231	7.27158107	0.001604639
	0.00055649	2.142477214	7.493861119	0.001192931
	0.000452462	1.999707827	7.700806378	0.000905202
	0.000372827	1.874770889	7.894396044	0.000699226
	0.000310835	1.764523072	8.076249429	0.000548645
	0.000261858	1.666518503	8.247708202	0.000436505
	0.000222653	1.578825575	8.409896427	0.000351608
	0.000190899	1.499898586	8.563765192	0.000286384

n	Estimating the SF	Estimating the HF	Estimating the CHF	Estimating the RHF
MSE	1.6803e-05	0.10762	0.69988	0.00056186
75	1	18.32578522	0	Inf
	0.301479212	16.3078376	1.199054214	7.03840761
	0.117286897	12.44169259	2.143132237	1.653139066
	0.056358717	9.873384265	2.876018357	0.589685169
	0.031146351	8.144848323	3.469058189	0.261837589
	0.018955559	6.919468007	3.965658047	0.133696678
	0.012371752	6.010116071	4.392339471	0.075287099
	0.008512815	5.310080368	4.766182638	0.045591845
	0.006104353	4.755176458	5.098753084	0.029205559
	0.004524631	4.304789419	5.398219362	0.019566111
	0.003445952	3.932067531	5.670555035	0.01359657
	0.002684512	3.618586984	5.920256353	0.009740288
	0.002131796	3.351302537	6.150790395	0.007159557
	0.001720938	3.120725905	6.364885966	0.005379833
	0.001409206	2.919794551	6.564728821	0.004120399
	0.001168427	2.743144991	6.752096527	0.003208915
	0.000979514	2.586631895	6.928453991	0.002536126
	0.000829222	2.447001428	7.095022632	0.002030791
	0.000708166	2.321664165	7.252831456	0.00164529
MSE	1.751e-05	0.11457	0.70072	0.00042567
100	1	18.32583048	0	Inf
	0.399770438	17.4412725	0.916864802	11.61639744
	0.182972419	14.21642718	1.698419856	3.183753061
	0.096623855	11.71615587	2.336929617	1.253143731
	0.056732353	9.894397334	2.869410629	0.595093498
	0.036007043	8.539718297	3.324040722	0.318975363
	0.024235033	7.501984506	3.719956031	0.186326473
	0.017072554	6.684875102	4.070283112	0.116110191
	0.012471702	6.026147599	4.384293034	0.076105482
	0.009384136	5.48444728	4.668734641	0.051954348
	0.007236216	5.031445978	4.928656819	0.036674013
	0.00569636	4.647174341	5.167927922	0.026623635
	0.004563918	4.3171882	5.389573731	0.019793631
	0.003712671	4.030798913	5.596003805	0.015020796
	0.00306053	3.779936097	5.789167223	0.011604122
	0.00255255	3.558396819	5.970662349	0.009106231
	0.002151014	3.361337539	6.141815896	0.00724587
	0.001829448	3.184922517	6.303740995	0.005837329
	0.0015689	3.026075523	6.457380581	0.00475507
	0.00135556	2.882301244	6.603540334	0.003912437
MSE	1.7404e-05	0.11658	0.70113	0.0006644

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