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A history of contraction principles

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Abstract

Recently we obtained several extensions of the Banach contraction principle, which appears frequently in many literature. From our 2023 Metatheorem, we deduce Theorem H on the equivalent formulations of completeness of quasi-metric spaces. From Theorem H, we derive the Banach contraction principle, its extended form (Theorem Q), the Rus-Hicks-Rhoades contraction principle (Theorem P), and others. Consequently, our Theorem H contains well-known theorems of Banach, Covitz-Nadler, Oettli-Théra, Rus-Hicks-Rhoades, and some others.

Key words and phrases: fixed point, quasi-metric, Rus-Hicks-Rhoades (RHR) map, T-orbitally complete.

Mathematics Subject Classification: 06A75, 47H10, 54E35, 54H25, 58E30, 65K10

1. Introduction

Among hundreds of extensions of metric spaces, a quasi-metric is the one not necessarily symmetric. In fact, a quasi-metric satisfies all axioms of a metric d except the symmetry $d(x, y) = d(y, x)$ for all x, y in the space. Certain key results in Metric Fixed Point Theory hold for quasi-metric spaces from the beginning; for example, the Banach contraction principle, the Ekeland variational principle, the Caristi fixed point theorem, the Takahashi minimization principle, and their equivalents; see [1–3].

For a quasi-metric space (X, q), a Rus-Hicks-Rhoades (RHR) map $f : X \to X$ is the one satisfying $q(fx, f^2x) \leq \alpha\, q(x, fx)$ for every $x \in X$, where $0 < \alpha < 1$. The fixed point theorems due to Rus [4] in 1973 and Hicks-Rhoades [5] in 1979 are origins of RHR maps. Recently, in [6], we noticed that it has an interesting long history. The RHR maps are closely related to the Banach contraction principle in 1922, but we found that it is more closer to its multi-valued versions due to Nadler [7] in 1969 and

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Covitz-Nadler [8] in 1970. The aim of [9] was to trace such history of the Rus-Hicks-Rhoades theorem, and to show its grown-up versions or equivalents or closely related theorems. Such theorems are too many and could be called its relatives. The metric fixed point theory originates from Banach [10] in 1922 on the study of the Banach contraction $f : X \to X$ on a normed vector space X. Later it was extended to a selfmap f on a complete metric space (X, d) satisfying

$$d(fx, fy) \leq \alpha\, d(x, y) \text{ with } \alpha \in [0, 1)$$

for any $x, y \in X$. Since then there have appeared several hundreds of contractive type conditions and almost one thousand spaces extending or modifying complete metric spaces.

One of such extended contractive type conditions was due to Rus [4] in 1973 and Hicks-Rhoades [5] in 1979 as follows:

$$d(fx, f^2x) \leq \alpha\, d(x, fx) \text{ for every } x \in X,$$

where $\alpha \in [0, 1)$. Such f is called a *weak contraction* or a *Rus-Hicks-Rhoades map* or an *RHR map*, and it has a large number of closely related mapping classes. An RHR map was also called a graphic contraction, iterative contraction, weakly contraction, Banach mapping, … ; see Berinde-Petruşel-Rus [11].

Recently, we noticed in [8],[11]–[14] that the RHR map has an interesting long history. It extends the Banach contraction [10] in 1922, but we found that it is also close to its multi-valued versions due to Nadler [7] in 1969 and Covitz-Nadler [8] in 1970. The aim of our [8] was to trace such history of the Rus-Hicks-Rhoades theorem, and to show its grown-up versions or equivalents or closely related theorems. Such theorems are too many and could be called its relatives.

Our aim in this paper is to collect our extensions of the Banach contraction principle. From our 2023 Metatheorem, we deduce Theorem H on equivalent formulations of the completeness of quasi-metric spaces. From Theorem H, we derive the Banach contraction principle, its extended form (Theorem Q), the Rus-Hicks-Rhoades contraction principle (Theorem P), and others. Consequently, our Theorem H contains well-known theorems of Banach, Covitz-Nadler, Oettli-Théra, Rus-Hicks-Rhoades, and others.

This article is organized as follows: Section 2 is for preliminaries on quasi-metric spaces. In Section 3, we introduce our 2023 Metatheorem as the basis of our study on contraction principles. Section 4 devotes to introduce the original Banach contraction theorem and the Cacciopoli theorem. In Section 5 we state the Banach contraction principle due to Bonsal in 1962. We add our own version for quasi-metric spaces. In Section 6, we state a generalized Banach contraction principle (Theorem Q) given in our recent work [14].

Section 7 devotes to introduce the Rus-Hicks-Rhoades (RHR) contraction principle (Theorem P) related to the RHR maps or the weak contractions. In Section 8, as an application of our 2023 Metatheorem, we introduce Theorem H on characterizations of complete quasi-metric spaces. Roughly speaking, Theorem H unifies Theorems P, Q, Theorems due to Covitz-Nadler, Oettli-Théra, Rus-Hicks-Rhoades, and others for quasimetric spaces. Section 9 devotes to introduce some known particular cases of Theorem H.

Finally, Section 10 is for epilogue.

2. Preliminaries

We recall the following:

Definition 2.1. A *quasi-metric* on a nonempty set X is a function $q : X \times X \to \mathbb{R}^+ = [0, \infty)$ satisfying the following conditions for all $x, y, z \in X$:

(a) (self-distance) $q(x, y) = q(y, x) = 0 \iff x = y$;
(b) (triangle inequality) $q(x, z) \leq q(x, y) + q(y, z)$.

A *metric* on a set X is a quasi-metric satisfying

(c) (symmetry) $q(x, y) = q(y, x)$ for all $x, y \in X$.

The convergence and completeness in a quasi-metric space (X, q) are defined as follows:

Definition 2.2. ([15], [16])

(1) A sequence (x_n) in X *converges* to $x \in X$ if

$$\lim_{n\to\infty} q(x_n, x) = \lim_{n\to\infty} q(x, x_n) = 0.$$

(2) A sequence (x_n) is *left-Cauchy* if for every $\varepsilon > 0$, there is a positive integer $N = N(\varepsilon)$ such that $q(x_n, x_m) < \varepsilon$ for all $n > m > N$.

(3) A sequence (x_n) is *right-Cauchy* if for every $\varepsilon > 0$, there is a positive integer $N = N(\varepsilon)$ such that $q(x_n, x_m) < \varepsilon$ for all $m > n > N$.

(4) A sequence (x_n) is *Cauchy* if for every $\varepsilon > 0$ there is positive integer $N = N(\varepsilon)$ such that $q(x_n, x_m) < \varepsilon$ for all $m, n > N$; that is (x_n) is a *Cauchy sequence* if it is left and right Cauchy.

Definition 2.3. ([15], [16])

(1) (X, q) is *left-complete* if every left-Cauchy sequence in X is convergent;
(2) (X, q) is *right-complete* if every right-Cauchy sequence in X is convergent;
(3) (X, q) is *complete* if every Cauchy sequence in X is convergent.

Definition 2.4. Let (X, q) be a quasi-metric space and $T : X \to X$ a selfmap. The *orbit* of T at $x \in X$ is the set

$$O_T(x) = \{x, Tx, \cdots, T^n x, \cdots\}.$$

The space X is said to be *T-orbitally complete* if every right-Cauchy sequence in $O_T(x)$ is convergent in X. A selfmap T of X is said to be *orbitally continuous* at $x_0 \in X$ if

$$\lim_{n\to\infty} T^n x = x_0 \Rightarrow \lim_{n\to\infty} T^{n+1} x = Tx_0$$

for any $x \in X$.

Remark 2.5. Definition 2.2 also works for a topological space X and a function $q: X \times X \to \mathbb{R}^+ = [0, \infty)$ such that $q(x, y) = 0$ implies $x = y$ for $x, y \in X$

Every quasi-metric induces a metric, that is, if (X, q) is a quasi-metric space, then the function $d: X \times X \to [0, \infty)$ defined by

$$d(x, y) = \max\{q(x, y), q(y, x)\}$$

is a metric on X; see Jleli et al. [16].

3. The 2023 Metatheorem

Our Metatheorem has a long history. We obtained the following form called the new 2023 Metatheorem in [1,2,17]:

Metatheorem. *Let X be a set, A its nonempty subset, and $G(x, y)$ a sentence formula for $x, y \in X$. Then the following are equivalent:*

(α) *There exists an element $v \in A$ such that the negation of $G(v, w)$ holds for any $w \in X \backslash \{v\}$.*

($\beta 1$) *If $f : A \to X$ is a function such that for any $x \in A$ with $x \neq fx$, there exists a $y \in X \backslash \{x\}$ satisfying $G(x, y)$, then f has a fixed element $v \in A$, that is, $v = fv$.*

(β2) *If* $\mathfrak{F}$ *is a family of functions* $f : A \to X$ *such that for any* $x \in A$ *with* $x \neq fx$, *there exists a* $y \in X \backslash \{x\}$ *satisfying* $G(x, y)$, *then* $\mathfrak{F}$ *has a common fixed element* $v \in A$, *that is,* $v = fv$ *for all* $f \in \mathfrak{F}$.

(γ1) *If* $f : A \to X$ *is a function such that* $G(x, fx)$ *for any* $x \in A$ *with* $x \neq fx$, *then* f *has a fixed element* $v \in A$, *that is,* $v = fv$.

(γ2) *If* $\mathfrak{F}$ *is a family of functions* $f : A \to X$ *satisfying* $G(x, fx)$ *for all* $x \in A$ *with* $x \neq fx$, *then* $\mathfrak{F}$ *has a common fixed element* $v \in A$, *that is,* $v = fv$ *for all* $f \in \mathfrak{F}$.

(δ1) *If* $F : A \multimap X$ *is a multifunction such that, for any* $x \in A \backslash Fx$ *there exists* $y \in X \backslash \{x\}$ *satisfying* $G(x, y)$, *then* F *has a fixed element* $v \in A$, *that is,* $v \in Fv$.

(δ2) *Let* $\mathfrak{F}$ *be a family of multifunctions* $F : A \multimap X$ *such that, for any* $x \in A \backslash Fx$ *there exists* $y \in X \backslash \{x\}$ *satisfying* $G(x, y)$. *Then* $\mathfrak{F}$ *has a common fixed element* $v \in A$, *that is,* $v \in Fv$ *for all* $F \in \mathfrak{F}$.

(ϵ1) *If* $F : A \multimap X$ *is a multifunction satisfying* $G(x, y)$ *for any* $x \in A$ *and any* $y \in Fx \backslash \{x\}$, *then* F *has a stationary element* $v \in A$, *that is,* $\{v\} = Fv$.

(ϵ2) *If* $\mathfrak{F}$ *is a family of multifunctions* $F : A \multimap X$ *such that* $G(x, y)$ *holds for any* $x \in A$ *and any* $y \in Fx \backslash \{x\}$, *then* $\mathfrak{F}$ *has a common stationary element* $v \in A$, *that is,* $\{v\} = Fv$ *for all* $F \in \mathfrak{F}$.

(η) *If* Y *is a subset of* X *such that for each* $x \in A \cap Y$ *there exists a* $z \in X \backslash \{x\}$ *satisfying* $G(x, z)$, *then there exists a* $v \in A \cap Y$.

For the proof, see Park [1],[2],[17]. Each item in Metatheorem has a long history. Especially, (η) is originated from Oettli-Théra [18].

This Metatheorem guarantees the truth of all items when one of them is true. Since 1985, we have shown nearly one hundred examples of $G(x, y)$.

4. Banach [9] in 1922

The work of Cauchy on differential equations has been fundamental to the existence theorems. According to Vasile I. Istrăţescu [19], A. L. Cauchy (1844) was the first mathematician to give a proof for the existence and uniqueness of the solution of a differential equation. In 1877, R. Lipschitz simplified Cauchy's proof using the 'Lipschitz condition.'

Motivated by such circumstances, the origin of the Banach contraction principle was appeared by Banach [9] in 1922:

Theorem 4.1. (Banach) *If* 1^0 $U(X)$ *be a continuous operator in* E, *the counter-domain of* $U(X)$ *is contained in* E_1.

2^0 *There exists a number* $0 < M < 1$ *which implies, for every* X' *and* X'', *the inequality*

$$\|U(X') - U(X'')\| \leq M.\|X' - X''\|.$$

there exists an element X *such that* $X = U(X)$.

Here E and E_1 is a normed space and its complete subset, resp.

Again by Istrăţescu [19], there exists a great number of attempts to weaken the Banach theorem. We mention here the one proposed by R. Cacciopoli who, in 1930, remarked that it is possible to replace the contraction property by the assumption of that of convergence as follows:

Theorem 4.2. (Cacciopoli [20]) *If* (X, d) *be a complete metric space and* $f : X \to X$ *has the property that*

$$d(f(x), f(y)) \leq \|f\| d(x, y) \text{ for } x, y \in X$$

and if $f^n(x) = f(f^{n-1}(x))$, *then*

$$\{f^n(x)\}$$

converges to a fixed point $z_0 = f(z_0)$ *if* $\sum \|f^n\| < \infty$, *and where*

$$d(f^n(x), f^n(y)) \leq \|f^n\| d(x, y).$$

5. The Banach Contraction Principle

A traditional version of the Banach theorem appeared in Bonsall [21] in 1962:

Theorem 5.1. (The contraction mapping theorem) *Let T be a contraction mapping of a complete metric space (E, d) into itself. Then*

(i) *T has a unique fixed point u in E.*
(ii) *If x_0 is an arbitrary point of E, and (x_n) is defined inductively by*

$$x_{n+1} = Tx_n \ (n = 0, 1, 2, \dots),$$

then $\lim_{n\to\infty} x_n = u$ *and*

$$d(x_n, u) \le \frac{k^n}{1-k} d(x_1, x_0)$$

where k is a Lipschitz constant for T.

The following is our version of the Banach contraction principle:

Theorem 5.2. *Let (X, q) be a quasi-metric space and let $T : X \to X$ be a contraction, that is,*

$$q(Tx, Ty) \le \alpha\, q(x, y) \text{ for every } x, y \in X,$$

with $0 < \alpha < 1$. If (X, q) is T-orbitally complete, then T has a unique fixed point $x_0 \in X$. Moreover, for each $x \in X$,

$$\lim_{n\to\infty} T^n(x) = x_0$$

and, in fact, for each $x \in X$,

$$q(T^n x, x_0) \le \frac{\alpha^n}{1-\alpha} q(x, Tx), \ n = 1, 2, \cdots.$$

Almost all text-books or monographs on general topology or fixed point theory do not mention on quasi-metric spaces relative to the Banach principle.

6. The Extended Banach Contraction Principle

The following is the extended Banach contraction principle given in [14]:

Theorem Q. *Let (X, q) be a quasi-metric space and let $T : X \to X$ be a generalized Banach contraction, that is, for each $x \in X$, there exists a $y \in X \backslash \{x\}$ such that*

$$q(Tx, Ty) \le \alpha\, q(x, y) \text{ where } 0 < \alpha < 1. \tag{q}$$

(i) *If X is T-orbitally complete, then, for each $x \in X$, there exists a point $x_0 \in X$ such that*

$$\lim_{n\to\infty} T^n x = x_0$$

and

$$q(T^n x, x_0) \le \alpha \frac{\alpha^n}{1-\alpha} q(x, Tx), \ n = 1, 2, \dots,$$

$$q(T^n x, x_0) \le \frac{\alpha}{1-\alpha} q(T^{n-1} x, T^n x), \ n = 1, 2, \dots$$

(ii) x_0 *is the unique fixed point of* T *(equivalently,* $T : X \to X$ *is orbitally continuous at* $x_0 \in X$*).*

The traditional Banach contraction principle is a particular form of Theorem Q when X is a metric space and (q) holds for all $x, y \in X$. It appears in thousands of publications and should be corrected or replaced by Theorem Q.

7. The Rus-Hicks-Rhoades (RHR) Contraction Principle

The origin of the RHR maps was given as follows by Rus [4] in 1971:

Theorem 7.1. (Rus) *Let* f *be a continuous selfmap of a complete metric space* (X, d) *satisfying*

$$d(fx, f^2x) \le \alpha d(x, fx) \text{ for every } x \in X,$$

where $0 < \alpha < 1$*. Then* f *has a fixed point.*

The following is given by Hicks and Rhoades [5] in 1979:
Let (X, d) be a complete metric space, T a selfmap of X, and $O(x) := \{x, Tx, T^2x, \ldots\}$.

Theorem 7.2. (Hicks-Rhoades) *Let* $0 < h < 1$*. Suppose there exists an* x *in* X *such that*

$$d(Ty, T^2y) < h\, d(y, Ty) \text{ for each } y \in O(x).$$

Then

(i) $\lim_n T^n x = z$ *exists*

(ii) $d(T^n x, z) < \frac{h^n}{1-h} d(x, Tx)$*, and*

(iii) z *is a fixed point of* T *if and only if* $G(x) := d(x, Tx)$ *is* T*-orbitally lower semi-continuous at* z*.*

Instead of $y \in O(x)$, we may assume (X, d) is a T-orbitally complete quasi-metric space.
The following in Park [3],[8] is called the weak contraction principle or the Rus-Hicks-Rhoades (RHR) contraction principle:

Theorem P. *Let* (X, q) *be a quasi-metric space and let* $T : X \to X$ *be an RHR map; that is,*

$$q(Tx, T^2x) \le \alpha\, q(x, Tx) \text{ for every } x \in X,$$

where $0 < \alpha < 1$.

(i) *If* X *is* T*-orbitally complete, then, for each* $x \in X$*, there exists a point* $x_0 \in X$ *such that*

$$\lim_{n\to\infty} T^n x = x_0$$

and

$$q(T^n x, x_0) \le \frac{\alpha^n}{1-\alpha} q(x, Tx),\ n = 1, 2, \cdots,$$

$$q(T^n x, x_0) \le \frac{\alpha}{1-\alpha} q(T^{n-1}x, T^n x),\ n = 1, 2, \cdots.$$

(ii) x_0 *is a fixed point of* T*, and, equivalently,*
(iii) $T : X \to X$ *is orbitally continuous at* $x_0 \in X$*.*

This was first proved in [3] by analyzing a typical proof of the Banach contraction principle given by Art Kirk ([22], Theorem 2.2). We reprove this for the completeness:

Proof. Step 1. *For each* $x \in X$, $\{T^n x\}$ *is right Cauchy*:

Adding $q(x, Tx)$ to both sides of the inequality $q(Tx, T^2x) \le \alpha\, q(x, Tx)$ yields

$$q(x, Tx) + q(Tx, T^2x) \le q(x, Tx) + \alpha\, q(x, Tx)$$

which can be rewritten

$$q(x, Tx) - \alpha\, q(x, Tx) \le q(x, Tx) - q(Tx, T^2x).$$

This in turn is equivalent to

$$q(x, Tx) \le (1 - \alpha)^{-1}[q(x, Tx) - q(Tx, T^2x)].$$

Now define the function $\varphi : X \to [0, \infty)$ by setting $\varphi(x) = (1 - \alpha)^{-1} q(x, Tx)$, for $x \in X$.

This gives us the basic inequality

$$q(x, Tx) \le \varphi(x) - \varphi(Tx),\ x \in X.$$

Therefore $\{T^n x\}$ is a right-Cauchy sequence by Lemma 3.1 in Kirk [10].

Step 2. *T-orbital completeness*:

Since X is T-orbitally complete, for any $x \in X$ there exists $x_0 \in X$ such that

$$\lim_{n\to\infty} T^n x = x_0.$$

Step 3. *Orbital continuity at* x_0:

If T is orbitally continuous at x_0, then

$$x_0 = \lim_{n\to\infty} T^n x = \lim_{n\to\infty} T^{n+1} x = Tx_0.$$

Thus x_0 is a fixed point of T. Conversely, if x_0 is fixed, then clearly T is orbitally continuous at x_0.

Step 4. *Convergence for* $\{T^n x\}$:

The last part of Kirk's original proof in [22] is added for completeness. Returning to the inequality

$$q(T^n x, T^{m+1} x) \le \varphi(T^n x) - \varphi(T^{m+1} x),$$

upon letting $m \to \infty$ we see that

$$q(T^n x, x_0) \le \varphi(T^n x) = (1 - \alpha)^{-1} q(T^n x, T^{n+1} x).$$

Since $(1-\alpha)^{-1} q(T^n x, T^{n+1} x) \le \dfrac{\alpha^n}{1-\alpha} q(x, Tx)$, we obtain

$$q(T^n x, x_0) \le \frac{\alpha^n}{1-\alpha} q(x, Tx).$$

This provides an estimate on the rate of convergence for the sequence $\{T^n x\}$ which depends only on $q(x, Tx)$.

From the proof of Theorem P, we have the following in [3]:

Theorem 7.3. *Let* (X, q) *be a quasi-metric space and let* $T : X \to X$ *be a map satisfying*

$$q(x, Tx) \le \varphi(x) - \varphi(Tx),\ x \in X,$$

for a real-valued function $\varphi : X \to [0, \infty)$ *such that*

$$\varphi(x) = (1 - \alpha)^{-1} q(x, Tx) \text{ with } 0 < \alpha < 1.$$

(i) *If X is T-orbitally complete, then, for each $x \in X$, there exists a point $x_0 \in X$ such that*

$$\lim_{n\to\infty} T^n x = x_0$$

and

$$q(T^n x, x_0) \le \frac{\alpha^n}{1-\alpha} q(x, Tx), n = 1, 2, \cdots.$$

(ii) *$T : X \to X$ is orbitally continuous at $x_0 \in X$ in* (i) *if and only if x_0 is a fixed point of T.*

This is a particular form of the Caristi type fixed point theorem.

More early in our [1],[2],[8] we obtained Theorem H which gives equivalent formulations of the completeness of quasi-metric spaces.

8. Completeness of Quasi-metric Spaces

Recently, as a basis of Ordered Fixed Point Theory [1], [2], we obtained the 2023 Metatheorem and Theorem H including Nadler's fixed point theorem [6] in 1969 and its extended version by Covitz-Nadler [7] in 1970.

Let (X, q) be a quasi-metric space and $\mathrm{Cl}(X)$ denote the family of all nonempty closed subsets of X (not necessarily bounded). For $A, B \in \mathrm{Cl}(X)$, set

$$H(A, B) = \max\{\sup\{q(a, B) : a \in A\}, \sup\{q(b, A) : b \in B\}\},$$

where $q(a, B) = \inf\{q(a, b) : b \in B\}$. Then H is called a generalized Hausdorff distance and it may have infinite values.

Recently, as a basis of Ordered Fixed Point Theory [1], we obtained the new 2023 Metatheorem and the following more general equivalent formulations of Nadler's fixed point theorem [6] in 1970 established by Covitz-Nadler [7] in 1970.

The fixed point theory for multivalued operators in metric structures has attracted the attention of several mathematicians. In 1973, Markin [23] initiated the theory of fixed point of multivalued mappings to satisfy contractive and nonexpansive conditions by employing the Hausdorff metric structure.

From Theorem P and our 2023 Metatheorem, we obtained the following new version:

Theorem H. ([2], [8]) *Let (X, q) be a quasi-metric space and $0 \le h < 1$. Then the following statements are equivalent:*

(0) *(X, q) is complete.*

(α) *For a multimap $T : X \to \mathrm{Cl}(X)$, there exists an element $v \in X$ such that $H(Tv, Tw) > h\, q(v, w)$ for any $w \in X\backslash\{v\}$.*

(β) *If $\mathfrak{F}$ is a family of maps $f : X \to X$ such that, for any $x \in X\backslash\{fx\}$, there exists a $y \in X\backslash\{x\}$ satisfying $q(fx, fy) \le h\, q(x, y)$, then $\mathfrak{F}$ has a common fixed element $v \in X$, that is, $v = fv$ for all $f \in \mathfrak{F}$.*

(γ) *If $\mathfrak{F}$ is a family of maps $f : X \to X$ satisfying $q(fx, f^2x) \le h\, q(x, fx)$ for all $x \in X\backslash\{fx\}$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = fv$ for all $f \in \mathfrak{F}$.*

(δ) *Let $\mathfrak{F}$ be a family of multimaps $T : X \to \mathrm{Cl}(X)$ such that, for any $x \in X\backslash Tx$, there exists $y \in X\backslash\{x\}$ satisfying $H(Tx, Ty) \le h\, q(x, y)$. Then $\mathfrak{F}$ has a common stationary element $v \in X$, that is, $\{v\} = Tv$ for all $T \in \mathfrak{F}$.*

(ϵ) *If $\mathfrak{F}$ is a family of multimaps $T : X \to \mathrm{Cl}(X)$ satisfying $H(Tx, Ty) \le h\, q(x, y)$ for all $x \in X$ and any $y \in Tx\{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in X$, that is, $\{v\} = Tv$ for all $T \in \mathfrak{F}$.*

(η) *If Y is a subset of X such that for each $x \in XY$ there exists a $z \in X\backslash\{x\}$ satisfying $H(Tx, Tz) \le h\, q(x, z)$ for some $T : X \to \mathrm{Cl}(X)$, then there exists a $v \in X \cap Y = Y$.*

Remark 8.1. (1) When $\mathfrak{F}$ is a singleton, (β)–(ϵ) are denoted by (β1)–(ϵ1), respectively, They are also logically equivalent to (α)-(η).

(2) Theorem H unifies several theorems appeared in this paper as follows:

(0) $\Longrightarrow$ (β1) implies the generalized Banach contraction principle.

(0) $\Longrightarrow$ (γ1) is the Rus-Hicks-Rhoades contraction principle. In some sense, this shows that the Banach contraction principle does not characterize the metric completeness. But so does the RHR theorem or (0) $\Longleftrightarrow$ (γ1).

(0) $\Longrightarrow$ (δ1) extends the generalized Banach contraction principle and theorems of Nadler [6] and Covitz-Nadler [7].

(0) $\Longrightarrow$ (ϵ1) implies theorems of Nadler [6] and Covitz-Nadler [7].

(0) $\Longrightarrow$ (η) originates from Oettli-Théra [18].

(3) Note that all ten statements in Theorem H are equivalent to the Covitz-Nadler theorem [7] in 1970 and Theorem H gives its elementary proof.

9. Orbital completeness for Theorem H

The completeness in Theorem H can be extended by T-orbital completeness for a self-map $T : X \to X$. For example, from the single-valued version of H(α1), we have the following generalization of (0) $\Longrightarrow$ (α) [23] in Theorem H:

Theorem H(α1)*. *Let (X, q) be a quasi-metric space, $f : X \to X$ a map, and $0 < r < 1$. If X is f-orbitally complete, then there exists an element $v \in X$ such that $q(fv, fw) > r\, q(v, w)$ for any $w \in X\backslash\{v\}$.*

Similarly, we have Theorems H(β1)*–H((ϵ1)*).

The following form of the RHR theorem is a useful consequence of Theorems P and H:

Theorem H(γ1)***.** *Let (X, q) be a quasi-metric space, and $0 < \alpha < 1$. If $f : X \to X$ is a map satisfying*

$$q(f(x), f^2(x)) \le \alpha\, q(x, f(x)) \text{ for all } x \in X\backslash\{f(x)\},$$

and X is f-orbitally complete, then f has a fixed element $v \in X$, that is, $v = f(v)$.

Note that Theorem H(γ1)* $\Longrightarrow$ Theorem 7.1 $\Longrightarrow$ Theorem 7.2. Therefore, the continuity in Theorem 7.1 is redundant.

Moreover, we have the following from Theorem H:

Theorem H(δ1)***.** *Let (X, δ) be a quasi-metric space, and $0 < \alpha < 1$. Let $T : X \to \mathrm{Cl}(X)$ be a multimap such that, for any $x \in X\backslash T(x)$, there exists $y \in X\backslash\{x\}$ satisfying*

$$H(T(x), T(y)) \le \alpha\, \delta(x, y).$$

If X is T-orbitally complete, then T has a fixed element $v \in X$, that is, $v \in T(v)$.

10. Epilogue

In this paper, we recalled the way from the Banach contraction to our Theorem H. There are several new contraction principles between them unknown to many researchers working in artificial metric type spaces.

Actually, the proof of Theorem H covers the corresponding ones of Banach [R], Rus [31], Hicks-Rhoades [7], Nadler [11], Covitz-Nadler [5], Oettli-Théra [12] and others.

Moreover, there are a large number of characterizations of metric completeness. It is well-known that the Banach contraction does not characterize. However, so does its slight generalized form (β1) and the RHR map in (γ1).

Consequently, all ten statements in Theorem H are close relatives of Theorems of Rus [47] and Hicks-Rhoades [13]. In our previous works [3],[6]–[9],[12], we applied Theorems P and H(γ1) to a large number of early extensions or relatives of theorems of Rus in 1973 and Hicks-Rhoades in 1979. This is rather surprising and all of them also extends the Banach contraction principle in 1922.

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