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Recent generalized exponential distribution using proposed formal with application

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Abstract

It is essential to develop generalizations of current statistical distributions that should be able to adapt and change when simulating actual data. A brand-new continuous distribution family known as the Recent Generalized Exponential distributions using the proposed formal formula RGED is introduced in statistical modeling. Some properties of the statistical features, like the probability density function for PDF, Moments, and the cumulative distribution function (CDF), are obtained. To demonstrate The adaptability of the new generalized family, maximum probability approximation estimates are applied to actual data regarding the model's parameters, specifically patients with head and neck cancer. The new generalized exponential family distribution outperformed other known distribution models identified with the same generalized base distribution, proving its high level and adaptability for analyzing various data.

Keywords: Generalized distribution; Moment; Quantile function; Hazard function; Mode; Maximum likelihood estimation.

Subject Classification: 46N10, 47N10, 90C30

1. Introduction

The statistical distribution used in most situations is the exponential distribution; many fields because it has been developed and modified to generate a new, improved distribution, which is the generalized exponential distribution for variability and its flexibility in representing data compared to traditional distributions, Gupta and Kundu, 1999 [1]. Scientists have proposed more flexible distributions that

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can represent data with any degree of complexity through generalized distributions due to the simplicity of the exponential distribution generalized for the time intervals of events. Generalized distributions have been developed with their applications to describe various phenomena. Several researchers have modified the exponential distribution, such as Gupta and Kundu, 2001 [2], who introduced The distribution function that is either the exponential-exponential or the generalized exponential CDF as

$$\mathrm{F}(x) = (1 - \exp(-\lambda x))^{\alpha},\ x,\lambda,\alpha > 0 \tag{1}$$

Barreto and Cribari, 2009 [3] offered a distribution function generalization of the Exponential-Poisson CDF is

$$F(x)=\frac{\left(1-\exp\left(-\lambda+\lambda\ \exp\left(-\beta x\right)\right)\right)}{1-\exp\left(-\lambda\right)}, \quad x,\lambda,\beta > 0 \tag{2}$$

El-Bassiouny, 2015 [4] proposed the Lomax exponential distribution CDF is

$$F(x) = \int_{0}^{\frac{1}{1-G(x)}} \lambda e^{-\lambda x} dx,\ \lambda > 0 \tag{3}$$

Barreto et al., 2010 [5] introduced the generalized Beta-Exponential family. Nassar and Issa, 2003 [6] have provided generalizations for Gamma, Weibull, Exponential-Gamma, Weibull, Gumbel-Exponential, and Exponential-Frechet distributions. Generalizations can also be obtained from the Beta generalized distributions class by Eugene et al., 2002 [7] and Jones. Sarhan and Apaloo, 2013 [8] proposed the modified exponential distribution.

Zubair et al., 2018 [9] established novel classes for the exponential distribution that are generalized "on generalized classes of exponential distribution using T_X Family framework" CDF of distribution is

$$F_X(x) = \int_{a}^{Q_Y(F_R(x))} f(x)dt \tag{4}$$

Nasir et al., 2017 [10] proposed a study of the generalized exponential Dagum distribution CDF is

$$G_{(x)} = \int_{a}^{-\log\left[1-\left(1-\overline{F}^{d}(x)\right)^{c}\right]} f(t)dt = 1-\left\{1-\left[1-F(x))^{d}\right]^{c}\right\}^{\lambda} \tag{5}$$

Where $\overline{F}(x) = 1 - F(x)$

By [11] studied how to find the family of distributions called the "Exponential-Generalizing Uniform distribution Using the Quantile function E-GUQD". CDF is

$$W(x) = \int_{a}^{Q(F(x))} f(x)dx \tag{6}$$

Muhammad and Al-Kadim [12] conducted a study on "A Transmuted Survival Model" characterized by adaptability, using the transformed exponential survival distribution and CDF of distribution given by.

$$F_{TSE} = 1 - S_{TE}(t) \tag{7}$$

This paper presents a new framework applied to existing distributions to derive a new and useful family of distributionscontributing to survival modeling.

This will be a new family of the recent generalized exponential distribution RGED.

To generate an improved distribution and compare it to the traditional distribution.

Proposed Formula to Generalized Distribution

Let a r. v. X has are $F(x)$, and $f(x)$, Corresponding, and Y is another r. v. which has a CDF defined as follows:

$$G_Y(x) = F(x)(Two\ F(x) - 1)$$

$$G_Y(x) = 2F^2(x) - F(x)$$

Now derivative function

$$g_Y(x) = 4F(x)\, f(x) - f(x)$$

Which is a PDF since it satisfies the following conditions:

1. $g_Y \geq 0$, $\forall\, x \in (-\infty, \infty)$, since, $0 < F < 1$, f is a PDF then $4Ff > f$
2. $\int_{-\infty}^{\infty} g_Y(x)dx = 1$

Therefore $\int_{-\infty}^{\infty} g_Y(x)dx = 1$

$$\int_{-\infty}^{\infty} g_Y(x)dx = 4\int_{-\infty}^{\infty} F(x)f(x)dx - \int_{-\infty}^{\infty} f(x)dx$$

Let $u = F \Rightarrow du = fdx$
Therefore, we know that.

$$U = F \sim U(0,1),\ 0 < u < 1$$

Then $\int_{-\infty}^{\infty} F(x)f(x)dx = \int_0^1 u\ du$

$$\begin{aligned}\int_{-\infty}^{\infty} g_Y(x)dx &= 4\int_0^1 u\ du - \int_{-\infty}^{\infty} f(x)dx \\ &= 4\left[\frac{u^2}{2}\right]_0^1 - \int_{-\infty}^{\infty} f(x)dx \\ &= 4\left[\frac{1}{2} - \frac{0}{2}\right] - 1 \\ &= 2 - 1 = 1\end{aligned}$$

$$R_Y(x) = 1 - G(x)$$

$$R_Y(x) = 1 - 2F^2(x) + F(x) \tag{8}$$

$$h_Y(x) = \frac{g(x)}{R(x)} = \frac{4F(x)f(x) - f(x)}{1 - [F(x)(2F(x) - 1)]} \tag{9}$$

Cumulative hazard function

$$\begin{aligned}H_Y(x) &= -\ln[1 - G(x)] \\ &= -\ln[1 - 2F^2(x) + F(x)]\end{aligned} \tag{10}$$

Reverse Hazard Function

$$\begin{aligned}\varphi_Y(x) &= \frac{g(x)}{G(x)} \\ &= \frac{4F(x)f(X) - f(X)}{2F^2(X) - F(X)}\end{aligned} \tag{11}$$

Recent Generalized Exponential Distribution Using Proposed Formula
Let X be an exponential r. v. with a scale parameter $\lambda > 0$, then a recent generalized exponential distribution's CDF and PDF are, respectively

The Cumulative Function of RGED

$$G_Y(x) = 2\left(1 - e^{-\lambda x}\right)^2 - \left(1 - e^{-\lambda x}\right)$$

$$G_Y(x) = 1 - 3e^{-\lambda x} + 2e^{-2\lambda x} \tag{12}$$

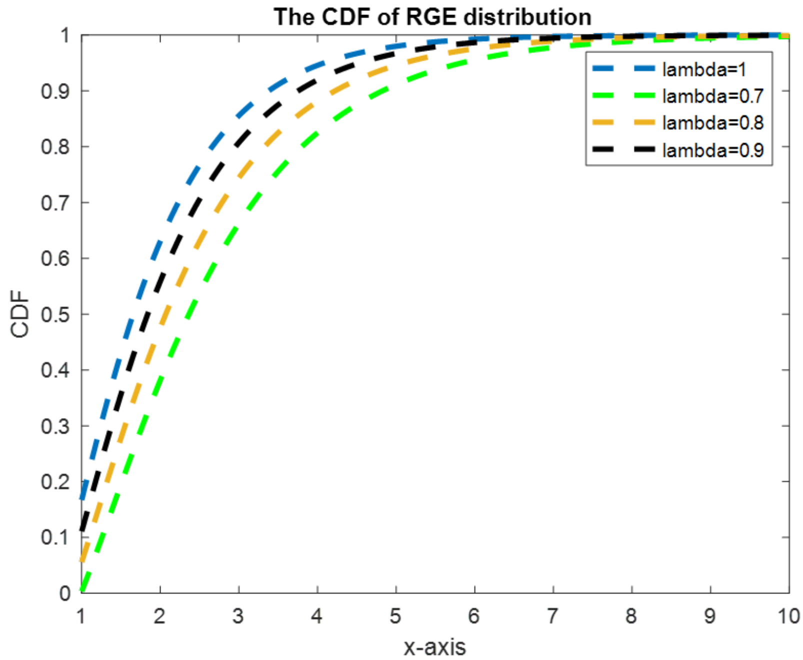

Figure 1: The Cumulative Function of RGED with various values of λ.

The curve of the CDF is monotonically increasing with increasing the value of a r .v. X.

The Probability Density Function of RGED

$$g_Y(x,\lambda) = 3\lambda e^{-\lambda x} - 4\lambda e^{-2\lambda x} \tag{13}$$

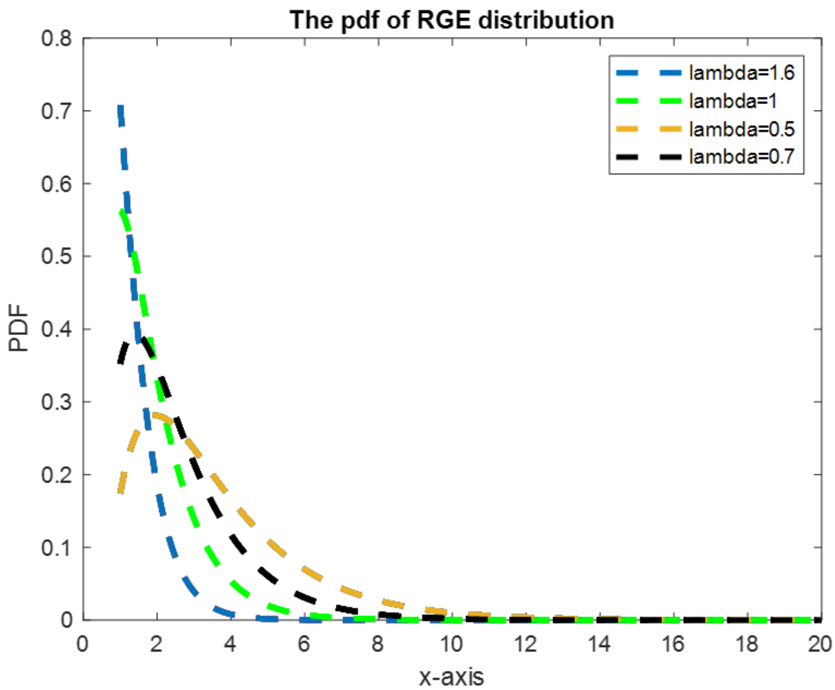

Figure 2: The PDF of RGED with various values of λ.

It is observed that the function continuously decreases until it converges with the x-axis as X approaches infinity.

Survival Rate Function

Role of survival rate for RGED, as

$$S_Y(x) = 1 - [2\left(1-e^{-\lambda x}\right)^2 - \left(1-e^{-\lambda x}\right)$$
$$S_Y(x) = 3e^{-\lambda x} - 2e^{-2\lambda x} \tag{14}$$

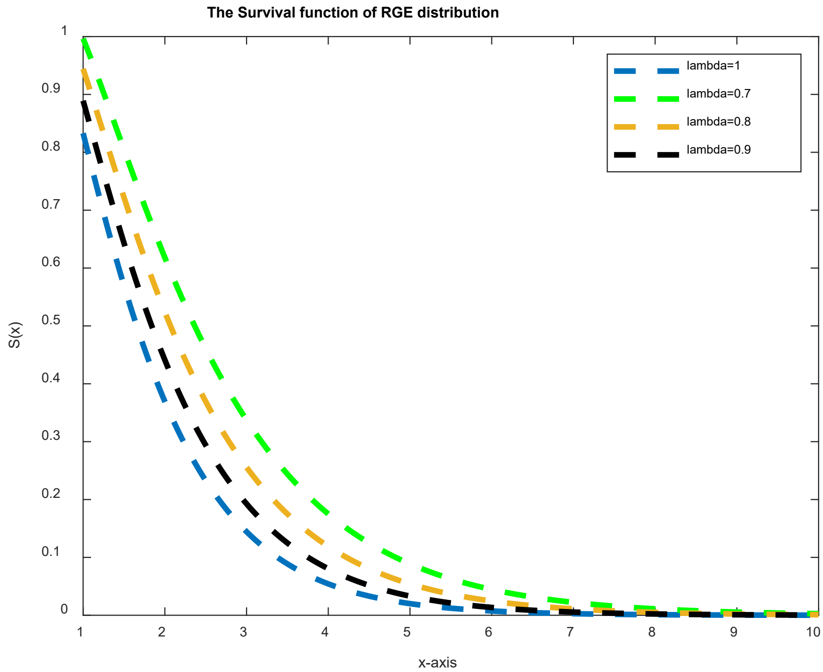

Figure 3: The Survival Function of RGED with modified values of λ.

The Survival Function is skewed to the right and decreases monotonically with increasing time.

Hazard Function

The hazard function of RGED, as

$$h_Y(x) = \frac{g(x)}{R(x)}$$
$$h_Y(x) = \frac{\lambda\left(3-4e^{-\lambda x}\right)}{3-2e^{-\lambda x}} \tag{15}$$

Reverse Hazard Function

The reverse hazard function of RGED, as

$$\varphi_Y(x) = \frac{g(x)}{R(x)}$$
$$\varphi_Y(x) = \frac{3\lambda e^{-\lambda x} - 4\lambda e^{-2\lambda x}}{1-3e^{-\lambda x}+2e^{-2\lambda x}} \tag{16}$$

Cumulative Hazard Function

The role of cumulative hazards of RGED, as

$$H_Y(x) = -\ln\left[3e^{-\lambda x} - 2e^{-2\lambda x}\right] \tag{17}$$

Moment of RGED

In this section, the central moment and the *rth* moment about the origin μ as follows:

Let X~ RGED(λ), the *rth* central moment

$$E(X^r) = \Gamma(r+1)\left[\frac{1}{\lambda^r}\right]$$

$$E(X^r) = \frac{\Gamma(r+1)}{\lambda^r} \tag{18}$$

The *rth* mean moment

$$E(X-\mu)^r = \sum_{(i=0)}^{n} \binom{r}{i} \left(\left(-\left(\frac{\Gamma(r+1)}{\lambda^r} \right) \right)^i \left[\frac{\Gamma(r-i+1)}{\lambda^{r-i}} \right] \right) \tag{19}$$

The Variance of RGED

$$Var(X) = E(X-\mu)^2$$

$$Var(X) = \sum_{(i=0)}^{n} \binom{2}{i} \left(-\left(\frac{\Gamma(2+1)}{\lambda^2} \right)^i \left[\frac{\Gamma(2-i+1)}{\lambda^{2-i}} \right] \right) \tag{20}$$

Moment Generating Function (MGF) of RGED

In this section, the present moment-generating function of RGED is as follows.

The function that generates moments of RGED, as

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \left(\frac{\Gamma(r+1)}{\lambda^r} \right) \tag{21}$$

Quantity Function of RGED

The quantile of RGED is as follows:

$$p = 1 - 3e^{-\lambda x} + 2e^{-2\lambda x} \tag{22}$$

Mode of RGED

Study RGED (3) pdf's and its derivative in relation to x.

$$\frac{\partial g_Y(x)}{\partial x} = 0$$

$$\frac{\partial g_Y(x)}{\partial x} = \frac{\partial\left[3\lambda e^{-\lambda x} - 4\lambda e^{-2\lambda x}\right]}{\partial x}$$

$$= 3\lambda\left(-\lambda e^{-\lambda x}\right) - 4\lambda\left(-2\lambda e^{-2\lambda x}\right)$$

$$0 = -3\lambda^2 e^{-\lambda x} + 8\lambda^2 e^{-2\lambda x}$$

$$[8\lambda^2 e^{-2x} = 3\lambda^2 e^{-\lambda x}] \div 3\lambda^2 e^{-\lambda x}$$

$$\Rightarrow \frac{8e^{-\lambda x}}{3} = 1 \Rightarrow e^{-\lambda x} = \frac{3}{8}$$

$$x = -\frac{\ln \frac{3}{8}}{\lambda} \tag{23}$$

Maximal Likelihood Estimation MLE

Here, we estimamate the RGED's unknown parameters using the MLE method.

The PDF of the likelihood function is:

$$l(x_1, x_2, \ldots, x_n, \lambda) = \prod_{i=1}^{n} g(x_i, \lambda) = \prod_{i=1}^{n} (3\lambda e^{-\lambda x} - 4\lambda e^{-2\lambda x}) \tag{24}$$

The expression for the log-likelihood function for the parameter vector (λ) is as follows:

$$L = lnl(\lambda, x_i) = ln\prod_{i=1}^{n}\left(3\lambda e^{-\lambda x} - 4\lambda e^{-2\lambda x}\right)$$

$$L = lnl(\lambda, x_i) = \prod_{i=1}^{n} ln\left(3\lambda e^{-\lambda x} - 4\lambda e^{-2\lambda x}\right) \tag{25}$$

After that, do the following by calculating the partial derivatives with regard to the unknown parameters (λ):

$$\frac{\partial g}{\partial \lambda} = \sum_{i=1}^{n} \frac{-3\lambda^2 x e^{-\lambda x} + 3e^{-\lambda x} + 8\lambda^2 x e^{-2\lambda x} - 4e^{-2\lambda x}}{3\lambda e^{-\lambda x} - 4\lambda e^{-2\lambda x}}$$

$$\sum_{i=1}^{n} \frac{-3\lambda^2 x e^{-\lambda x} + 3e^{-\lambda x} + 8\lambda^2 x e^{-2\lambda x} - 4e^{-2\lambda x}}{3\lambda e^{-\lambda x} - 4\lambda e^{-2\lambda x}} = 0 \tag{26}$$

The equation (26) above is solved numerically.

2. Application

Using the Bayesian Information Criterion (BIC), Akaike Information Criterion (AIC), Hannan-Quinn Information Criterion (HQIC), and Consistent Akaike Information Criterion (CAIC), the proposed RGED comparison was performed with a range of distributions, including Exponential Weibull Distribution (EWD), Exponential Distribution (ED) and Generalized Exponential Distribution (GED).

Data set: The first dataset,which extracted 55 patients' survival rates from a study on head and neck cancer [13], offered as

6.54, 10.42, 14.48, 16.10, 22.70, 3441.55, 4245.28 49.40 53.62, 63, 64, 83, 84, 91, 108, 112, 129, 133, 133, 139, 140, 140, 146, 149, 154, 157, 160, 160, 165, 146, 149, 154, 157, 160, 160, 165, 173, 176, 218, 225, 241, 248, 273, 277, 297, 405, 417, 420, 440, 523, 583, 594, 1101, 1146, 1417.

Table 1: The MLEs of the parameters (α, λ).

Model	Parameters		
RGED(λ)	$\hat{\lambda}$ = 0.004		
ED(λ)	$\hat{\lambda}$ = 0.003		
GED(α, λ)	$\hat{\alpha}$ = 1.331	$\hat{\lambda}$ = 0.005	
EWD(α, λ, β)	$\hat{\alpha}$ = 2.31	$\hat{\beta}$ = 0.397	$\hat{\lambda}$ = 0.001

Table 2: The statistical test results (BIC, CAIC, AIC) on the data.

Model	LL	AIC	BIC	HQIC	CAIC
RGED(λ)	–263.4436	528.8872	530.8945	529.6634	527.9057
ED(λ)	–381.4081	764.8163	766.8236	765.5925	764.8917
GED(α, λ)	–392.5309	789.0617	793.0764	790.6142	789.2925
EWD(α, λ, β)	–368.5948	743.1896	749.2116	745.5189	

Observing the (2) Table shows that the proposed distribution RGE has an advantage over the rest.

3. Conclusions

The Generalized Exponential Distribution (GED) and its recent extensions offer versatile data modeling and analysis methods across multiple disciplines. Their ability to adapt to different data shapes makes them valuable in engineering, medical research, and environmental science. By leveraging these distributions, practitioners can achieve more accurate predictions and improved decision-making in their respective areas.

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