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Generalized cayley graphs and group structure: Insights from the direct products of P_2 and C_3

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Abstract

This work introduces a generalization of Cayley graphs, denoted $Cay_m(\psi, S)$, where ψ is a finite group and S is a non-empty subset of ψ. In this construction, vertices are represented by m-dimensional column vectors with entries in ψ, and adjacency is determined by a matrix-based condition involving the inverse elements and a matrix of elements from S. We focus on elucidating the structure and fundamental properties of $Cay_m(\psi, S)$ when the classical Cayley graph Cay(ψ, S) corresponds to the direct products $P_2 \times P_2$ and $P_2 \times C_2$. Through rigorous analysis, we reveal distinct structural characteristics arising from these specific group structures and their associated generating sets. The generalized Cayley graph, denoted as $Cay_m(\psi, S)$, is a graph where the vertex set consists of all column matrices X_m, with each matrix having elements from the set Ψ. Two vertices X_m and Y_m are adjacent if and only if Y_m^{-1}, the inverse of Y_m, is a column matrix where each entry corresponds to the inverse of the associated element in Ψ. Our findings provide valuable insights into the interplay between algebraic properties of groups and the topological features of their generalized Cayley graph representations. This study contributes to a deeper understanding of generalized Cayley graphs and their potential applications in diverse fields such as network theory, coding theory, and cryptography.

Key words and phrases: Cayley graph, $Cay_m(\Psi, S)$.

Mathematics Subject Classification: 00A71

1. Introduction

Algebraic graph theory has become a prominent area of interest for researchers at the intersection of algebra and graph theory, offering a framework that connects graphs with various algebraic

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structures, such as groups, rings, and modules. A central construct in this field is the *Cayley graph*, introduced by Arthur Cayley in 1878, to elucidate the concept of abstract groups, particularly those generated by a set of elements. In essence, a Cayley graph represents a group's structure through a graphical form, where the vertices and edges encode specific group elements and their relations [1, 2].

Formally, given a group $\Psi$and a subset S that is closed under inverses, with $e\notin S$ (where e is the group's identity), the Cayley graph Cay (Ψ, S) is defined as a simple, undirected graph. Its vertices correspond to the elements of Ψ, and there is an edge between two vertices x and y if and only if $xy^{-1}\in S$. Notably, the Cayley graph is an r-regular graph, where r depends on the size and properties of the subset S. Furthermore, Cay (Ψ, S) is connected if and only if S serves as a generating set for Ψ, highlighting the deep connection between the graph's structure and the algebraic properties of the group [3].

Erfanian et al. recently introduced a new concept in algebraic graph theory known as the *generalized Cayley graph*, denoted $Cay_m(\Psi, S)$ [4]. This novel extension of the classic Cayley graph incorporates m×1 column matrices, providing a broader framework that generalizes the standard Cay(Ψ,S) graph. Formally, $Cay_m(\Psi,S)$ is defined as an undirected, simple graph with a vertex set composed of all m×1 matrices $X= [x_1,x_2,\ldots,x_m]^t$, where each $x_i\in\Psi$ for $1\leq i\leq m$ and for any positive integer $m\geq1$. Two vertices $X= [x_1,x_2,\ldots,x_m]^t$ and $Y=[y_1,y_2,\ldots,y_m]^t$ are adjacent if and only if $X(Y)^t\in M(S)$.

It follows that the traditional Cayley graph Cay (Ψ, S) is recovered when m=1, hence the term *generalized Cayley graph*. In this context, we assume that $S^{-1}\subseteq S$, $e\notin S$ (where e is the identity element), and thatS is a generating set for Ψ. Under these conditions, $Cay_m(\Psi, S)$ is ensured to be a connected graph.

This paper delves into the construction of generalized Cayley graphs through the utilization of Cartesian products. Specifically, we investigate the structural properties of these graphs when the underlying classical Cayley graphs correspond to the direct products $P_2 \times P_2$ and $P_2 \times C_2$.

To establish a foundational framework, we commence by rigorously defining pertinent graph operations, encompassing graph union, Cartesian graph product, Corona product, and generalized Corona product. Subsequently, we introduce a novel generalization of Cayley graphs, denoted $Cay_m(\Psi, S)$, wherein vertices are characterized by m-dimensional column vectors with entries in the finite group Ψ, and adjacency is governed by a matrix-based condition involving inverse elements and a prescribed matrix of elements drawn from a non-empty subset S of Ψ.

Our primary objective lies in elucidating the intricate structure and salient features of $Cay_m(\Psi, S)$ when Cay (Ψ,S) is isomorphic to the direct products. Through meticulous analysis, we unveil distinctive structural characteristics inherent to these specific group structures and their corresponding generating sets. Furthermore, we establish a lemma that provides a crucial link between adjacency in $Cay_m(\Psi,S)$ and adjacency in the classical Cayley Graph $Cay(\Psi, S)$.

This comprehensive investigation yields valuable insights into the interplay between the algebraic properties of groups and the topological attributes of their generalized Cayley graph representations. Moreover, it contributes to a deeper and more nuanced understanding of generalized Cayley graphs, paving the way for their potential exploitation in diverse fields such as network theory, coding theory, and cryptography, where the interplay between algebraic and graphical structures is of paramount importance.

Here are definitions for key graph operations and a lemma relevant to the generalized Cayley graph when analyzing specific Cartesian products.

Description 1: Let Ψ and H represent two graphs. The union of Ψ and H, denoted by $\Psi\cup H$, is a graph whose vertex set is the union of the vertex sets of Ψ and H, and whose edge set is the union of the edge sets of Ψ and H, has a vertex set $V(\Psi\cup H) =V(\Psi)\cup V(H)$ and an edge set $E(\Psi\cup H) =E(\Psi)\cup E(H)$.

Description 2: The Cartesian product of graphs Ψ and H, denoted $\Psi\times H$, is defined as follows:

The vertex set $V(\Psi\times H)$ is the Cartesian product of the vertex sets of Ψ and H:

$$V(\Psi\times H) = V(\Psi) \times V(H)$$

This means each vertex in Ψ×H is an ordered pair (g, h), where g is a vertex from Ψ and h is a vertex from H. The two vertices (g, h) and (g′, h′) in Ψ×H are adjacent if and only if one of the following conditions holds:

1. **g = g′ and hh′ ∈ E(H):** The first vertices in the pairs are the same, and the second vertices are adjacent in H.
2. **h = h′ and gg′ ∈ E(Ψ):** The second vertices in the pairs are the same, and the first vertices are adjacent in Ψ.

Therefore, the edge set E(Ψ×H) consists of all edges of the form (g, h) and (g′, h′) where either condition 1 or 2 is satisfied.The graphs Ψ and H are called the factors of the Cartesian product Ψ×H.

Description 3: The Corona product, denoted by the symbol ∘, is a binary operation defined on the set of all simple graphs. Given two graphs, Ψ and H, their Corona product Ψ∘H is constructed through a specific process of replication and interconnection.

This process begins by taking a single copy of the graph Ψ and |V(Ψ)| distinct copies of the graph H, where |V(Ψ)| represents the cardinality of the vertex set of Ψ. In essence, for each vertex *v* within the graph Ψ, there exists a corresponding copy of the graph H, denoted as the *v*-th copy.

The defining characteristic of the Corona product lies in the establishment of connections between these copies. Specifically, for every vertex *v* in Ψ, edges are introduced to connect *v* to each and every vertex within the corresponding *v*-th copy of H. This construction effectively results in a structure where each vertex of Ψ becomes the center of a "crown" or "halo" formed by a copy of H, hence the nomenclature "Corona".

More formally, the vertex set of the Corona product Ψ∘H is defined as the union of the vertex set of Ψ and the vertex sets of all |V(Ψ)| copies of H. The edge set of Ψ∘H encompasses all the edges originally present in Ψ, all the edges within each of the |V(Ψ)| copies of H, and the newly introduced edges connecting each vertex *v* in Ψ to every vertex in its corresponding *v*-th copy of H.

It is crucial to note that the Corona product is **non-commutative**. This signifies that, in general, the graph of Ψ∘H is not isomorphic to the graph H∘Ψ. The order in which the graphs are combined in the Corona product plays a pivotal role in determining the final structure of the resulting graph due to the inherent asymmetry of the construction.

Lemma 4: Let $X= [\omega_1,\omega_2,\ldots,\omega_m]^t$and $Y=[y_1,y_2,\ldots,y_m]^t$ be two arbitrary vertices of $Cay_m(\Psi,S)$, where each ω_i and $y_j$belongs to Ψ for *i*, j∈{1,2,...,m}. Then, X and Y are adjacent in $\mathrm{Cay}_m(\Psi,S)$ if and only if ω_i is adjacent to y_j in *Cay* (Ψ,*S*) for all $i,j\in\{1,2,\ldots,m\}$.

This lemma elucidates a fundamental property of adjacency within the Cartesian product of Cayley graphs. It establishes a precise correspondence between adjacency in the higher-dimensional Cayley graph Cay_m(Ψ, S) and adjacency in the underlying Cayley graph *Cay* (Ψ, *S*).

Let us delve into the specifics. Consider two arbitrary vertices, X and Y, within the Cayley graph Cay_m (Ψ, *S*). These vertices can be represented as *m*-dimensional vectors:

$$X= [\omega_1, \omega_2,\ldots,\omega_m]^t \text{ and } Y=[y_1,y_2,\ldots,y_m]^t$$

where each component $\omega_i$and y_j belongs to the group Ψ, and the indices *i* and *j* range from 1 to *m*.

The lemma asserts that these two vertices, X and Y, are adjacent within Cay_m(Ψ, S) if and only if a specific condition holds true. This condition mandates that for every possible pairing of indices *i* and *j*, the corresponding components ω_i and y_j are adjacent within the original Cayley graph *Cay*(Ψ, *S*).

In essence, this lemma establishes a direct link between the adjacency relations in the higher-dimensional Cartesian product of Cayley graphs and the adjacency relations in the foundational Cayley graph. Adjacency in the higher-dimensional structure is contingent upon pairwise adjacency of all corresponding components in the underlying graph. This property provides crucial insights into the structure and connectivity of Cay_m(Ψ, S), enabling a deeper understanding of its properties and characteristics.

Lemma 5. Let $Cay(\Psi,S)=P_2\times P_2$, then $Cay_2(\Psi,S)=K_{4,4}\cup 8P_1$.

Proof: Suppose that $\Psi_1=P_2$ with vertex set $\{x_1,x_2\}$ and $\Psi_2=P_2$ with vertex set $\{x_3,x_4\}$ and $Cay(\Psi,S)=P_2\times P_2$. So, $Cay(\Psi,S)$ is a cycle of length 4 and its vertex set is

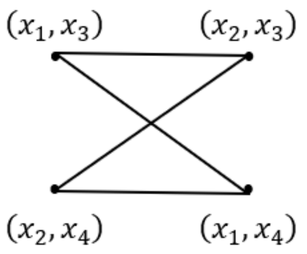
$$V\left(Cay(\Psi,S)\right)=\left\{(x_1,x_3),(x_1,x_4),(x_2,x_3),(x_2,x_4)\right\}$$

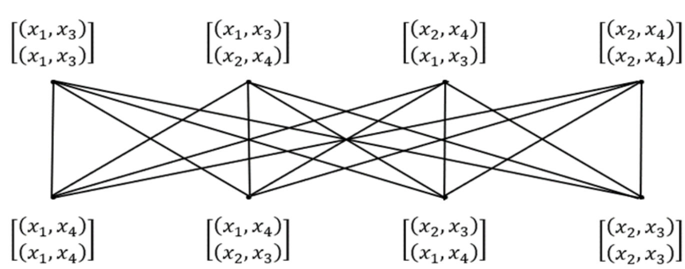
and the set of four edges$\{(x_1,x_3)(x_1,x_4),(x_1,x_3)(x_2,x_3),(x_2,x_3)(x_2,x_4),(x_2,x_4)(x_1,x_4)\}$

Since the Cayley graph is a cycle$(x_1,x_3)-(x_1,x_4)-(x_2,x_4)-(x_2,x_3)-(x_1,x_3)$. Then, we have $4^2=16$ vertices in $Cay_2(\Psi,S)$ and $V\left(Cay_2(\Psi,S)\right)=\left\{\begin{bmatrix}a\\b\end{bmatrix} \mid a,b\in V\left(Cay(\Psi,S)\right)\right\}$

$$\left\{\begin{bmatrix}(x_1,x_3)\\(x_1,x_3)\end{bmatrix},\begin{bmatrix}(x_1,x_3)\\(x_1,x_4)\end{bmatrix},\begin{bmatrix}(x_1,x_3)\\(x_2,x_3)\end{bmatrix},\begin{bmatrix}(x_1,x_3)\\(x_2,x_4)\end{bmatrix},\right.$$
$$\begin{bmatrix}(x_1,x_4)\\(x_1,x_3)\end{bmatrix},\begin{bmatrix}(x_1,x_4)\\(x_1,x_4)\end{bmatrix},\begin{bmatrix}(x_1,x_4)\\(x_2,x_3)\end{bmatrix},\begin{bmatrix}(x_1,x_4)\\(x_2,x_4)\end{bmatrix},$$
$$\begin{bmatrix}(x_2,x_3)\\(x_1,x_3)\end{bmatrix},\begin{bmatrix}(x_2,x_3)\\(x_1,x_4)\end{bmatrix},\begin{bmatrix}(x_2,x_3)\\(x_2,x_3)\end{bmatrix},\begin{bmatrix}(x_2,x_3)\\(x_2,x_4)\end{bmatrix},$$
$$\left.\begin{bmatrix}(x_2,x_4)\\(x_1,x_3)\end{bmatrix},\begin{bmatrix}(x_2,x_4)\\(x_1,x_4)\end{bmatrix},\begin{bmatrix}(x_2,x_4)\\(x_2,x_3)\end{bmatrix},\begin{bmatrix}(x_2,x_4)\\(x_2,x_4)\end{bmatrix}\right\}$$

Consequently. Every vertex in set A is obviously adjacent to every vertex in set B, and vice versa. Thus, the bipartite graph is obtained $K_{4,4}$. We demonstrate that every other vertex is an independent vertex. Assume, without losing generality, that $\begin{bmatrix}(x_1,x_3)\\(x_1,x_4)\end{bmatrix}$ is not isolated. So, there is a vertex $\begin{bmatrix}(a,c)\\(b,d)\end{bmatrix}\in V\left(Cay_2(\Psi,S)\right)$ such that $(x_1,x_3)-(a,c)$, $(x_1,x_3)-(b,d)$, $(x_1,x_4)-(a,b)$ and $(x_1,x_4)-(b,d)$. So, $(a,c)=(x_1,x_4)$ or $(a,c)=(x_2,x_3)$. If $(a,c)=(x_1,x_4)$, then $(a,c)-(x_1,x_4)$ then it implies that $(x_1,x_4)-(x_1,x_4)$ which is a contradiction.Similarly, If $(a,c)=(x_2,x_3)$, then $(a,c)-(x_1,x_4)$which implies that (x_2,x_3)–$(x_1,x_4)$and gain it is a contradiction. Hence, $\begin{bmatrix}(x_1,x_3)\\(x_1,x_4)\end{bmatrix}$ is an isolated vertex. The following procedure may be used to other vertices as well. There are these solitary vertices in an amount of $4^2-8=8$, and hence $Cay_2(\Psi,S)=K_{4,4}\cup 8P_1$. The graph of $Cay_2(\Psi,S)$ in this case, is shown below.





The graph $P_2\times P_2$A component of graph $Cay_2(\Psi,S)$ of $P_2\times P_2$

In the next Theorem, we generalized the Cayley graph for each m=3 when the common Cayley graph is $P_2\times P_2$.

Lemma 6. Let $Cay(\Psi,S)=P_2\times P_2$, then $Cay_3(\Psi,S)=K_{8,8}\cup 48P_1$.

Proof: Suppose that $\Psi_1=P_2$ with vertex set $\{x_1,x_2\}$ and $\Psi_2=P_2$ with vertex set $\{x_3,x_4\}$ and $Cay(\Psi,S)=P_2\times P_2$. So, $Cay(\Psi,S)$ is a cycle of length 4 and its vertex set is

$$V(Cay(\Psi,S))=\{(x_1,x_3),(x_1,x_4),(x_2,x_3),(x_2,x_4)\}$$

and the set of four edges $\{(x_1,x_3)(x_1,x_4),(x_1,x_3)(x_2,x_3),(x_2,x_3)(x_2,x_4),(x_2,x_4)(x_1,x_4)\}$

Since the Cayley graph is a cycle $(x_1,x_3)-(x_1,x_4)-(x_2,x_4)-(x_2,x_3)-(x_1,x_3)$. Then, we have $4^3=64$ vertices in $Cay_3(\Psi,S)$ and $V(Cay_3(\Psi,S))=\left\{\begin{bmatrix}a\\b\\c\end{bmatrix} \mid a,b,c\in V(Cay(\Psi,S))\right\}$. So,

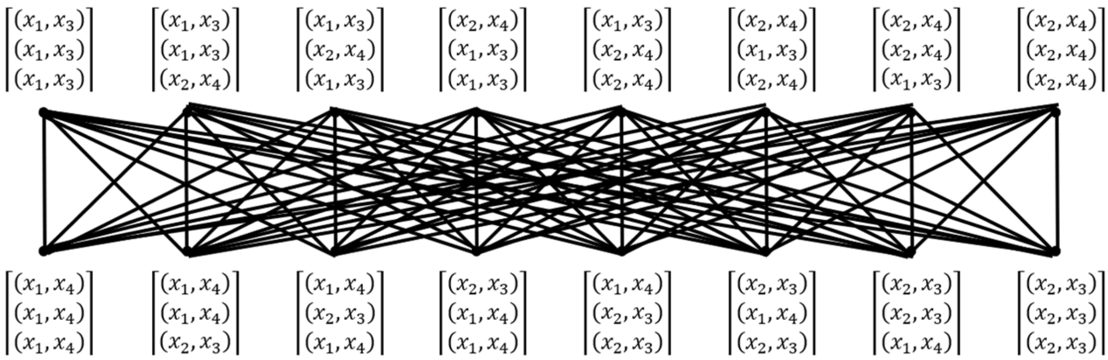
$V(Cay_3(\Psi,S))=\left\{\begin{bmatrix}(x_i,x_j)\\(x_k,x_l)\\(x_r,x_s)\end{bmatrix} \; : \; i,j,k,l,r,s=1,2,3,4\right\}$. Therefore, we have two independent sets

$$A=\left\{\begin{bmatrix}(x_1,x_3)\\(x_1,x_3)\\(x_1,x_3)\end{bmatrix},\begin{bmatrix}(x_1,x_3)\\(x_1,x_3)\\(x_2,x_4)\end{bmatrix},\begin{bmatrix}(x_1,x_3)\\(x_2,x_4)\\(x_1,x_3)\end{bmatrix},\begin{bmatrix}(x_2,x_4)\\(x_1,x_3)\\(x_1,x_3)\end{bmatrix},\begin{bmatrix}(x_1,x_3)\\(x_2,x_4)\\(x_2,x_4)\end{bmatrix},\begin{bmatrix}(x_2,x_4)\\(x_1,x_3)\\(x_2,x_4)\end{bmatrix},\begin{bmatrix}(x_2,x_4)\\(x_2,x_4)\\(x_1,x_3)\end{bmatrix},\begin{bmatrix}(x_2,x_4)\\(x_2,x_4)\\(x_2,x_4)\end{bmatrix}\right\}$$

and

$$B=\left\{\begin{bmatrix}(x_1,x_4)\\(x_1,x_4)\\(x_1,x_4)\end{bmatrix},\begin{bmatrix}(x_1,x_4)\\(x_1,x_4)\\(x_2,x_3)\end{bmatrix},\begin{bmatrix}(x_1,x_4)\\(x_2,x_3)\\(x_1,x_4)\end{bmatrix},\begin{bmatrix}(x_2,x_3)\\(x_1,x_4)\\(x_1,x_4)\end{bmatrix},\begin{bmatrix}(x_1,x_4)\\(x_2,x_3)\\(x_2,x_3)\end{bmatrix},\begin{bmatrix}(x_2,x_3)\\(x_1,x_4)\\(x_2,x_3)\end{bmatrix},\begin{bmatrix}(x_2,x_3)\\(x_2,x_3)\\(x_1,x_4)\end{bmatrix},\begin{bmatrix}(x_2,x_3)\\(x_2,x_3)\\(x_2,x_3)\end{bmatrix}\right\}.$$

It is clear that every vertex in set A is adjacent to all vertices in set B and vice versa. Thus, we get the bipartite graph$K_{8,8}$. We demonstrate that every other vertex is an independent vertex. Absent loss of generality, suppose that $\begin{bmatrix}(x_i,x_j)\\(x_k,x_l)\\(x_r,x_s)\end{bmatrix}$ is not isolated where $i,k,r=1,2$ and $j,l,s=3,4$. So, there is a vertex $\begin{bmatrix}(a,b)\\(c,d)\\(e,f)\end{bmatrix}\in V(Cay_3(\Psi,S))$ such that$(x_i,x_j)-(a,b)$, $(x_i,x_j)-(c,d)$, $(x_i,x_j)-(e,f)$ and $(x_k,x_l)-(a,b)$, $(x_k,x_l)-(c,d)$, $(x_k,x_l)-(e,f)$ and $(x_r,x_s)-(a,b)$, $(x_r,x_s)-(c,d)$, $(x_r,x_s)-(e,f)$. but (x_i,x_j) is of degree 2. So, $(a,b)=(x_i,x_j)$ or $(a,b)=(x_k,x_l)$ or $(a,b)=(x_r,x_s)$.If $(a,b)=(x_i,x_j)$, then it implies that $(x_i,x_j)-(x_i,x_j)$ which is a contradiction. Likewise, If $(a,b)=(x_k,x_l)$ and $(a,b)=(x_r,x_s)$ we get the acontradiction. Hence, the rest vertices $\begin{bmatrix}(x_i,x_j)\\(x_k,x_l)\\(x_r,x_s)\end{bmatrix}$ are isolated vertices. We can prove by the same method as above for more vertices. There are these solitary vertices in an amount of $|V(Cay_3(\Psi,S)|-(|A|+|B|)=4^3-(8+8)=48$, and hence $Cay_3(\Psi,S)=K_{8,8}\cup 48P_1$. The graph of $Cay_3(\Psi,S)$ is shown below.



A component of graph $Cay_3(\Psi,S)$ of $P_2 \times P_2$

In the following theorem, we extend the concept of the Cayley graph for all $m \geq 2$ when the shared Cayley graph is $P_2 \times P_2$.

Theorem 7. Let $Cay(\Psi,S) = P_2 \times P_2$, then the generalized Cayley graph $Cay_m(\Psi,S)$ is the graph $K_{2^m,2^m} \cup \left(2^{m+1}\left(2^{m-1}-1\right)\right)P_1$ for all $m \geq 2$.

Proof: Suppose that $\Psi_1 = P_2$ with vertex set $\{x_1,x_2\}$ and $\Psi_2 = P_2$ with vertex set $\{x_3,x_4\}$ and $Cay(\Psi,S) = P_2 \times P_2$. So, $Cay(\Psi,S)$ is a cycle of lenɰth 4 and its vertex set is

$$V\left(Cay(\ddot{\ } ,S)\right) = \left\{(x_1,x_3),(x_1,x_4),(x_2,x_3),(x_2,x_4)\right\}$$

and the set of edges is $(x_1,x_3)-(x_1,x_4)-(x_2,x_4)-(x_2,x_3)-(x_1,x_3)$. So, $V = V\left(Cay_m(\Psi,S)\right) = \left\{\left[a_1,a_2,\ldots,a_m\right]^t \mid a_1,a_2,\cdots,a_m \in V\left(Cay(G,S)\right)\right\}$ Therefore, $\left|V\left(Cay_m(\Psi,S)\right)\right| = 4^m$. Consider the subsets A and B of V as follows:

$A = \left\{\left[a_1,a_2,\ldots,a_m\right]^t : a_i \in \{x_1,x_3\}, i = 1,2,\ldots,m\right\}$ a n d $B = \left\{\left[a_1,a_2,\ldots,a_m\right]^t : a_i \in \{x_2,x_4\},\ i = 1,2,\ldots,m\right\}$.

It is evident that A and B constitute independent sets, wherein each vertex within one set is connected to a vertex in the opposite set, employing an analogous technique to that utilized in the proof of the preceding lemma. Consequently, the bipartite structure is wholly generated by the union of the disjoint sets A∪B, with any residual vertices existing solely as isolated entities. Consequently, $Cay_m(\Psi,S) = K_{2^m,2^m} \cup \left(2^{m+1}\left(2^{m-1}-1\right)\right)P_1$ for all $m \geq 2$.

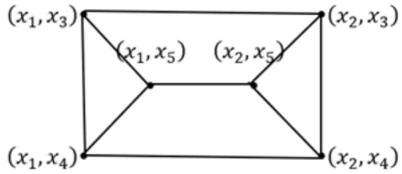
In the subsequent lemma, we derive the generalized Cayley graph for the specific case of n=2 when $Cay(G, S) = P_2 \times C_3$.

Theorem 8. Let $Cay(\Psi,S) = P_2 \times C_3$, then $Cay_2(\Psi,S)$ has $\left((P_2 \times C_3) \circ 2P_1\right) \cup 18P_1$ as a subgraph.

Proof: Assume that $\Psi_1 = P_2$ with apex set $\{x_1,x_2\}$ and $\Psi_2 = C_3$ with vertex set $\{x_3,x_4,x_5\}$ and $Cay(\Psi,S) = P_2 \times C_3$. So, $V\left(Cay(\ddot{\ } ,S)\right) = \left\{(x_1,x_3),(x_1,x_4),(x_1,x_5),(x_2,x_3)(x_2,x_4),(x_2,x_5)\right\}$ and $\left|V(Cay(G,S)\right| = 6$ and $E\left(Cay(\ddot{\ } ,S)\right) = \{(x_1,x_3)(x_1,x_4),(x_1,x_3)(x_2,x_3),(x_2,x_3)(x_2,x_4),$

$$(x_2,x_4)(x_1,x_4),(x_2,x_4)(x_2,x_5),(x_2,x_3)(x_2,x_5),(x_1,x_3)(x_1,x_5),,(x_1,x_4)(x_1,x_5),(x_2,x_5)(x_1,x_5)$$

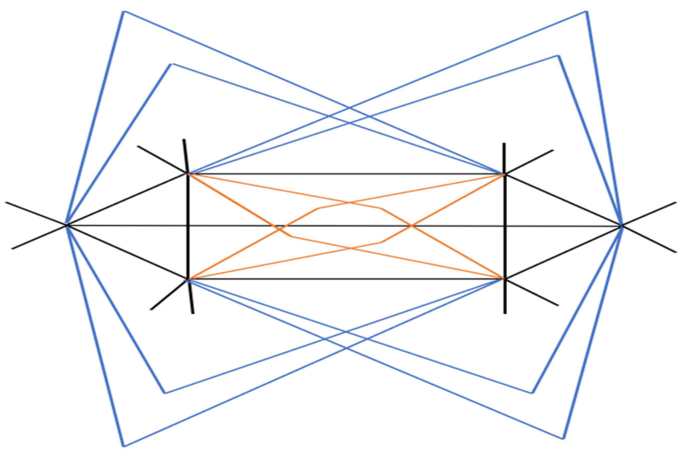
The graph $Cay(\ddot{\ } ,S) = P_2 \times C_3$ shown in below.



Then, we have $6^2 = 36$ vertices in $Cay_2(\Psi, S)$ and $V(Cay_2(\Psi,S)) = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \mid a, b \in V(Cay(\Psi,S)) \right\} =$

$$\left\{ \begin{bmatrix}(x_1,x_3)\\(x_1,x_3)\end{bmatrix}, \begin{bmatrix}(x_1,x_3)\\(x_1,x_4)\end{bmatrix}, \begin{bmatrix}(x_1,x_3)\\(x_1,x_5)\end{bmatrix}, \begin{bmatrix}(x_1,x_3)\\(x_2,x_3)\end{bmatrix}, \begin{bmatrix}(x_1,x_3)\\(x_2,x_4)\end{bmatrix}, \begin{bmatrix}(x_1,x_3)\\(x_2,x_5)\end{bmatrix}, \right.$$
$$\begin{bmatrix}(x_1,x_4)\\(x_1,x_3)\end{bmatrix}, \begin{bmatrix}(x_1,x_4)\\(x_1,x_4)\end{bmatrix}, \begin{bmatrix}(x_1,x_4)\\(x_1,x_5)\end{bmatrix}, \begin{bmatrix}(x_1,x_4)\\(x_2,x_3)\end{bmatrix}, \begin{bmatrix}(x_1,x_4)\\(x_2,x_4)\end{bmatrix}, \begin{bmatrix}(x_1,x_4)\\(x_2,x_5)\end{bmatrix},$$
$$\begin{bmatrix}(x_1,x_5)\\(x_1,x_3)\end{bmatrix}, \begin{bmatrix}(x_1,x_5)\\(x_1,x_4)\end{bmatrix}, \begin{bmatrix}(x_1,x_5)\\(x_1,x_5)\end{bmatrix}, \begin{bmatrix}(x_1,x_5)\\(x_2,x_3)\end{bmatrix}, \begin{bmatrix}(x_1,x_5)\\(x_2,x_4)\end{bmatrix}, \begin{bmatrix}(x_1,x_5)\\(x_2,x_5)\end{bmatrix},$$
$$\begin{bmatrix}(x_2,x_3)\\(x_1,x_3)\end{bmatrix}, \begin{bmatrix}(x_2,x_3)\\(x_1,x_4)\end{bmatrix}, \begin{bmatrix}(x_2,x_3)\\(x_1,x_5)\end{bmatrix}, \begin{bmatrix}(x_2,x_3)\\(x_2,x_3)\end{bmatrix}, \begin{bmatrix}(x_2,x_3)\\(x_2,x_4)\end{bmatrix}, \begin{bmatrix}(x_2,x_3)\\(x_2,x_5)\end{bmatrix},$$
$$\begin{bmatrix}(x_2,x_4)\\(x_1,x_3)\end{bmatrix}, \begin{bmatrix}(x_2,x_4)\\(x_1,x_4)\end{bmatrix}, \begin{bmatrix}(x_2,x_4)\\(x_1,x_5)\end{bmatrix}, \begin{bmatrix}(x_2,x_4)\\(x_2,x_3)\end{bmatrix}, \begin{bmatrix}(x_2,x_4)\\(x_2,x_4)\end{bmatrix}, \begin{bmatrix}(x_2,x_4)\\(x_2,x_5)\end{bmatrix},$$
$$\left. \begin{bmatrix}(x_2,x_5)\\(x_1,x_3)\end{bmatrix}, \begin{bmatrix}(x_2,x_5)\\(x_1,x_4)\end{bmatrix}, \begin{bmatrix}(x_2,x_5)\\(x_1,x_5)\end{bmatrix}, \begin{bmatrix}(x_2,x_5)\\(x_2,x_3)\end{bmatrix}, \begin{bmatrix}(x_2,x_5)\\(x_2,x_4)\end{bmatrix}, \begin{bmatrix}(x_2,x_5)\\(x_2,x_5)\end{bmatrix} \right\}$$

Therefore, each vertex $\begin{bmatrix}(x_i,x_j)\\(x_i,x_j)\end{bmatrix}$ has deɥree 4 and it is adjacent to the vertices $\begin{bmatrix}(x_i,x_{j+1})\\(x_{i+1},x_j)\end{bmatrix}$ and $\begin{bmatrix}(x_{i+1},x_j)\\(x_i,x_{j+1})\end{bmatrix}$ and $\begin{bmatrix}(x_i,x_{j+1})\\(x_i,x_{j+1})\end{bmatrix}$ and $\begin{bmatrix}(x_i,x_{j+2})\\(x_i,x_{j+2})\end{bmatrix}$. The other vertices are isolated. The graph $Cay_2(\Psi,S)$ is shown in below.



Conclusions

In this paper, we introduced a novel generalization of Cayley graphs, denoted $Cay_m(\Psi, S)$, based on m-dimensional column vectors and a matrix-based adjacency condition. Our primary focus was on understanding the structure of $Cay_m(\Psi, S)$ when the classical Cayley graph $Cay(\Psi, S)$ is isomorphic to the direct products $P_2 \times P_2$ and $P_2 \times C_2$. Through detailed analysis, we revealed distinct structural properties and established connections between the algebraic properties of the groups and the topological features of their generalized Cayley graph representations.

This research serves as a foundation for further exploration of generalized Cayley graphs under diverse group structures and matrix conditions. Future investigations could delve into:

- Analyzing $Cay(\Psi, S)$ for other direct products: Extending the study to different group families and exploring the impact of group properties on graph structure.
- Investigating the properties of $Cay(\Psi, S)$ for specific matrices: Examining how different choices of the matrix in the adjacency condition influence graph properties.
- Exploring applications: Investigating the potential applications of these generalized Cayley graphs in areas such as network design, coding theory, and cryptography, where the interplay between algebraic and graphical structures is crucial.

By addressing these open questions, we can gain a more comprehensive understanding of generalized Cayley graphs and their potential to solve real-world problems.

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