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On shrinkage estimators for Pareto II parameters for complete data

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Abstract

Many different fields have found extensive use for the Pareto distributionType II. The Maximum Likelihood Method (ML), and Bayeswill all be used in this study to estimate the parameters of the Pareto distributionType II. Next, we will attempt to determine the parameter's first shrinkage estimator for the estimators of the techniques we are investigating. This study's primary goal is to provide two initial shrinkage estimator comparisons between two estimators: First, there is a shrinkage estimator between the maximum likelihood and the Bayes estimators; second, there is a shrinkage estimator between the first shrinkage estimator and the Bayes estimator. The performance of these estimators was examined using Monte Carlo simulation in order to determine which, as measured by the MSE criterion, one is the best.

Keywords and Phrases: Bayes estimator, Lomax Distribution, Pareto DistributionType II, Maximum Likelihood Method, Monte Carlo simulation, Shrinkage Estimation, Mean Square Error.

Mathematics Subject Classification: 65C05, 62J07

1. Introduction

The 19th century saw the development of the Pareto distributionType II (also called Lomax Distribution that introduced by Lomax (1945) [2]), named for the Italian economist Vilfredo Pareto, which is a heavy-tailed and it is intended to simulate how wealth was distributed across the people [15]. Years after its introduction, significant modifications and alterations were made to it, resulting in several

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variations referred to as Type I, II, III, and IV Pareto distributions, along with the generalized Pareto distribution (GPD).

The focus of this study is on the Pareto Type II distribution, also known as the Lomax distribution. Originally applied to simulate business failure data [18], this extensively utilized distribution has served as a model in various contexts ever since.

This distribution has proven to be valuable in modeling survival times in the presence of censoring. For instance, the exponential Lomax and Weibull Lomax distributions have demonstrated good suitability for right-censored lung cancer survival data [3] and the lifetime of electrical transducers [12]. With times being left-truncated and right-censored, this distribution was found to offer the most suitable fit among various distributions considered. Additionally, this distribution finds applications in modeling wealth distribution [10], queuing service times [7], life testing [8], and the sizes of files on a computer server [9]. To avoid confusion, we will use the term Pareto II instead of Pareto distribution Type II.

Many studies have addressed the estimation of the parameters of the Pareto II distribution using several methods, including the Maximum Likelihood and Bayes methods for complete data. Other studies have also attempted to improve these estimators, including the Shrinkage estimator. Some studies have indicated that the shrinkage estimator can reduce the variance in the estimates resulting from the Pareto II distribution, which improves the accuracy of predictions and makes the models more stable. Some of these studies will be reviewed:

Alkutubi and Ibrahim (2009) [4] present three shrinkage estimators for exponential distribution with maximum likelihood estimator, Bayes estimator, and modify Bayes estimator, where the Bayesian method and the extended Bayesian method performed better in estimation using shrinkage

Prakash (2010) [17] studied the properties of the shrinkage test–estimators of the parameter were studied for an inverse Rayleigh model under the asymmetric loss function. Both the single and double–stage shrinkage test–estimators are considered. He concluded that the performance of both shrinkage test estimators is consistently good with respect to the improved $\hat{\theta}_c$ estimator for the considered set of parameter values based on the presented data. In terms of increasing efficiency, $\hat{\theta}_{SH3}$ is preferred over $\hat{\theta}_{SH2}$ in the region $0.60 \le \delta \le 1.40$. And the two-stage shrinkage test estimator $\hat{\theta}_{DSH}$ is good with respect to the improved ensemble estimator $\hat{\theta}_{PC}$ for the entire considered set of parameter values.

Mohammed et al. (2012) [13] proposed the two-stage shrinkage estimator for the initial test (PTDSSE) to estimate the shape parameter of the Pareto II distribution when the scale parameter is equal to the smallest loss in the area (R) around the available prior knowledge (alpha) around the actual value (alpha) as a primary estimator as well as to reduce the mean square error and the cost of experiments, as it can be used in cases where the time taken for the experiment and the cost of the sample are very expensive and costly, as the use of the two-stage leads to reducing the expected sample size required to obtain the estimator that reduces these costs.

Hamad (2013) [5] considered the two-stage shrinkage estimator to minimize the mean square error of the classical estimator (MLE) of the shape parameter (α) of the generalized exponential distribution in a region (R) around the available prior knowledge (α_0) about the actual value (α) as an initial estimate in case the scale parameter (λ) is known and also to reduce the cost of experiments. This estimator was shown to have a smaller mean square error for a given choice of shrinkage weight factor $\psi(\cdot)$ and for the mentioned region of acceptance R.

Salman and Hussein (2016) [20] used an employee single-stage shrinkage estimator to estimate the Pareto distribution's shape parameter when the scale parameter is known. The proposed estimator is shown to have a smaller mean squared error in a region around theta0 compared to the usual and existing estimators.

Labban (2019) [11] proposed new method (T.O.M) to estimate distribution parameters for complete data. It gave good results compared to the other studied methods based on the MSE criterion.

Salman and Hamad (2019) [19] used different shrinkage estimation methods to study the problem of estimating reliability system in stress-strength model under mismatch and independent of stress and strength and following the Lomax distribution. They adopted maximum likelihood, moment method and shrinkage weight factors based on Monte Carlo simulation. These methods were based on combining the maximum likelihood estimator and moment method as prior information using different shrinkage weight factor, and it was found that the shrinkage estimator using fixed shrinkage weight factor (Sh1) has the lowest statistical index (MAPE) in most cases.

Abd Ali, Jiheel and Al-Hemyar (2022) [1] proposed two-stage Bayesian shrinkage estimator for the shape parameter of the Pareto distribution. It was assumed that the prior knowledge of θ can take the form of an initial estimate θ0 of θ, where the region R was divided into two regions R_1 and R_2, and the bias, bias ratio, mean square error, expected sample size, and relative efficiency were derived. The results indicated that the region R_2 is better than the region R_1, and the shrinkage estimators are preferred due to their higher relative efficiency compared to the classical Bayesian estimator.

Pels, et al. (2023) [16] proposed two new methods for estimating the shape parameter of the generalized Pareto distribution (GPD), using the shrinkage principle to adapt the existing empirical Bayesian to the data-driven prior and the probability moment method to obtain estimators. The results showed that the proposed estimators perform better for a small to moderate number of overfittings in estimating the shape parameter for light-tailed distributions and are competitive when estimating heavy-tailed distributions.

The Pareto II distributionis a continuous probability distribution used in probability theory and statistics. Outside of the field of economics, this scheme is known as the Bradford distribution, and it describes the probability density function (p.d.f.) of a continuous random variable *X* of the Pareto distribution [14]:

$$f(x,\alpha,\beta) = \frac{\alpha\beta}{(1+\beta x)^{\alpha+1}} \qquad \alpha,\beta > 0, \ \ x > 0 \tag{1}$$

With cumulative distribution defined by:

$$F(x,\alpha,\beta) = 1-(1+\beta x)^{-\alpha} \tag{2}$$

One of the requirements for the research is to use the following equation to create sample data from the Pareto distribution:

$$x = \beta^{-1}\left[(1-y)^{-\frac{1}{\alpha}} - 1\right] \tag{3}$$

where y has a uniform distribution with interval (0,1), and arbitrary parameter values α, β.

In this paper we will estimate Lomax distribution parameters using maximum likelihood and Bayes method and then find the first shrinkage and second shrinkage.

2. Estimation Methods

In this paper, to find estimator parameters for Pareto distribution Type II, we will use different sample size (N = 50, 150, 250), and three models of parameter values and shown in Table (1):

Table 1: Models of different values of parameters.

Parameter	Model 1	Model 2	Model 3
Α	1	2	5
β	2	4	3

2.1. Maximum Likelihood Estimatorfor Complete Data

One of the widely adopted methods for estimating unknown parameters within probability distributions is maximum likelihood estimation (MLE). Due to its advantageous asymptotic properties like consistency and unbiasedness, MLE has gained significant popularity. However, these attributes may not hold true for small sample sizes, potentially leading to biased MLEs.

For n independent observations, the likelihood function for Pareto II distribution is:

$$L(\alpha,\beta;\underline{x})=\prod_{i=1}^{n}\frac{\alpha\beta}{(1+\beta x_i)^{\alpha+1}} \tag{4}$$

And log-likelihood become

$$\ln L(\alpha,\beta;\underline{x})=n\ln\alpha+n\ln\beta-(\alpha+1)\sum_{i=1}^{n}\ln(1+\beta x_i) \tag{5}$$

And by maximize equation (5) by taking first partial derivative over α we have

$$\frac{\partial \ln L}{\partial\alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}\ln(1+\beta x_i) \tag{6}$$

Then the maximum likelihood estimation can be obtained from equation (6) as

$$\frac{n}{\alpha}-\sum_{i=1}^{n}\ln(1+\beta x_i)=0$$

And

$$\hat{\alpha}_{ML}=\frac{n}{\sum_{i=1}^{n}\ln(1+\hat{\beta}_{ML}x_i)} \tag{7}$$

And again by maximize equation (5) by taking first partial derivative over β we have:

$$\frac{n}{\beta}-(\alpha+1)\sum_{i=1}^{n}\frac{x_i}{(1+\beta x_i)}=0 \tag{8}$$

As we can see in (8), we cannot get a close form solution for the estimator of β, so numerical procedures are required, which is Newton Raphson to get the maximum likelihood estimator of β which is $\hat{\beta}_{ML}$.

2.2. Bayes Estimator for Complete Data

The Bayes viewpoint has drawn a lot of attention for statistical inference in recent decades as a strong and legitimate substitute for conventional statistical viewpoints. This section addresses the use of a squared error loss function in Bayesian estimation of the unknown parameters of Pareto Distribution.

According to Jeffry's method, the non-informational initial probability function for each of the two random parameters α, β can be assumed according to the following two formulas:

The prior distribution for α, β. [6]

$$J_1(\alpha)\propto\frac{1}{\alpha}$$

$$J_2(\beta)\propto\frac{1}{\beta}$$

Where α, β > 0, then

$$J(\alpha,\beta)\propto\frac{1}{\alpha\beta} \tag{9}$$

The likelihood function for the Pareto Distribution in (4) can be rewritten as:

$$L(\alpha,\beta;\underline{x})=\alpha^n\beta^n\prod_{i=1}^{n}(1+\beta x_i)^{-(\alpha+1)}$$

Or

$$L(\alpha,\beta;\underline{x})=\alpha^n\beta^n\exp\left\{-(\alpha+1)\sum_{i=1}^{n}\ln(1+\beta x_i)\right\} \tag{10}$$

The posterior

$$h(\alpha,\beta \mid x_1,x_2,\ldots,x_n) \propto \frac{L(\alpha,\beta;\underline{x})J(\alpha,\beta)}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} L(\alpha,\beta;\underline{x})J(\alpha,\beta)d\alpha d\beta}$$

$$h(\alpha,\beta \mid x_1,x_2,\ldots,x_n) \propto \frac{\alpha^n\beta^n \exp\left\{-(\alpha+1)\sum_{i=1}^{n}\ln(1+\beta x_i)\right\}\alpha^{-1}\beta^{-1}}{\int_0^{\infty}\int_0^{\infty}\alpha^n\beta^n \exp\left\{-(\alpha+1)\sum_{i=1}^{n}\ln(1+\beta x_i)\right\}\alpha^{-1}\beta^{-1}d\alpha d\beta} \tag{11}$$

To estimate Pareto distribution parameters we using the square error loss function:

$$L(\underline{\hat{\theta}},\underline{\theta}) = (\underline{\hat{\theta}}-\underline{\theta})^2 \tag{12}$$

Where $\underline{\theta}$ and $\underline{\hat{\theta}}$ represent the vector of parameters and vector of the parameters estimator respectively.

To estimate the unknown parameters α, β we use the average risk (MSE for square error loss) which define by the following:

$$R(\underline{\hat{\theta}},\underline{\theta}) = E[L(\underline{\hat{\theta}},\underline{\theta}) \mid \underline{x}] = E[(\underline{\hat{\theta}}-\underline{\theta})^2 \mid \underline{x}] \tag{13}$$

And then the Bayesian estimators to the Pareto parameters under square error loss function are:

$$\hat{\alpha} = E(\alpha \mid \underline{x}) = \frac{\int_0^{\infty}\int_0^{\infty}\alpha\alpha^{n-1}\beta^{n-1}\exp\left\{-(\alpha+1)\sum_{i=1}^{n}\ln(1+\beta x_i)\right\}d\alpha d\beta}{\int_0^{\infty}\int_0^{\infty}\alpha^{n-1}\beta^{n-1}\exp\left\{-(\alpha+1)\sum_{i=1}^{n}\ln(1+\beta x_i)\right\}d\alpha d\beta}$$

$$\hat{\beta} = E(\beta \mid \underline{x}) = \frac{\int_0^{\infty}\int_0^{\infty}\beta\alpha^{n-1}\beta^{n-1}\exp\left\{-(\alpha+1)\sum_{i=1}^{n}\ln(1+\beta x_i)\right\}d\alpha d\beta}{\int_0^{\infty}\int_0^{\infty}\alpha^{n-1}\beta^{n-1}\exp\left\{-(\alpha+1)\sum_{i=1}^{n}\ln(1+\beta x_i)\right\}d\alpha d\beta}$$

Using Lindley approximation [21] to estimate the Pareto parameters as follows:

$$I(x) = u\left(\hat{\alpha},\hat{\beta}\right) + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}(u_{ij}+2u_ip_j)\sigma_{ij} + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m}L_{ijk}u_l\sigma_{ij}\sigma_{kl} \tag{14}$$

Where m represent the number of parameters, here $m = 2$, and $u(\alpha, \beta)$ is a function of parameters α, β, such that $u(\alpha, \beta) = \alpha$ for α parameter, and $u(\alpha, \beta) = \beta$ for β parameter, and

$$P_i = \frac{\partial \ln J}{\partial \hat{\theta}^{(i)}} \quad , \; \hat{\theta}^{(1)} = \hat{\alpha}, \; \hat{\theta}^{(2)} = \hat{\beta}$$

$$L_{ij} = \frac{\partial^{i+j}\ln L}{\partial\hat{\alpha}^i\partial\hat{\beta}^j}$$

$$\sigma_{ij} = -\frac{1}{L_{ij}} \quad , \; u_{ij} = \frac{\partial^{i+j}u(\hat{\alpha},\hat{\beta})}{\partial\hat{\alpha}^i\partial\hat{\beta}^j} \text{ index } i \text{ for } \hat{\alpha}\text{, and index } j \text{ for } \hat{\beta}$$

All estimators of parameters $\hat{\alpha}$ and $\hat{\beta}$ are represented to the maximum likelihood estimator.

So, for our distribution, we have:

$$P_{\hat{\alpha}} = \frac{\partial \ln J}{\partial \hat{\alpha}} = \frac{-1}{\hat{\alpha}} \quad , \; P_{\beta} = \frac{\partial \ln J}{\partial \hat{\beta}} = \frac{-1}{\hat{\beta}}$$

$$L_{02} = \frac{\partial^2 \ln L}{\partial\hat{\beta}^2} = -\frac{n}{\hat{\beta}^2} + (\hat{\alpha}+1)\sum_{i=1}^{n}\frac{x_i^2}{(1+\hat{\beta}x_i)^2} \quad , \; L_{20} = -\frac{n}{\hat{\alpha}^2}$$

$$L_{03}=\frac{\partial^3 \ln L}{\partial \hat{\beta}^3}=2\frac{n}{\hat{\beta}^3}-2(\hat{\alpha}+1)\sum_{i=1}^{n}\frac{x_i^3}{(1+\hat{\beta}x_i)^3} \quad , L_{30}=\frac{2n}{\hat{\alpha}^3}$$

$$L_{12}=\frac{\partial^3 \ln L}{\partial \hat{\alpha}^1 \partial \hat{\beta}^2}=\sum_{i=1}^{n}\frac{x_i^2}{(1+\hat{\beta}x_i)^2} \quad , L_{21}=0$$

$$\sigma_{\hat{\alpha}\hat{\alpha}}=-\frac{1}{L_{\alpha\alpha}}=\frac{\hat{\alpha}^2}{n} \quad ,\sigma_{\hat{\alpha}\hat{\beta}}=\sigma_{\hat{\beta}\hat{\alpha}}=0 \text{ (because independancy)}$$

$$\sigma_{\hat{\beta}\hat{\beta}}=-\frac{1}{L_{02}}=-\frac{1}{-\frac{n}{\hat{\beta}^2}+(\hat{\alpha}+1)\sum_{i=1}^{n}\frac{x_i^2}{(1+\hat{\beta}x_i)^2}}$$

when $u(\hat{\alpha},\hat{\beta})=\alpha$, then

$u_{\hat{\alpha}}=1 \quad , u_{\hat{\beta}}=0$

$u_{\hat{\alpha}\hat{\alpha}}=0 \quad , u_{\hat{\beta}\hat{\beta}}=0$

So, with substitute all derivatives with $u(\alpha, \beta)= \alpha$, we have $\hat{\alpha}=I(x)$.

In the same way, when $u(\alpha, \beta) = \beta$, then

$$u_{\alpha}=0 \quad , \quad u_{\beta}=1$$

$$u_{\alpha\alpha}=0 \quad , \quad u_{\beta\beta}=0$$

So, with substitute all derivatives with $u(\alpha, \beta) = \beta$, we have $\hat{\beta}=I(x)$.

2.3. First Shrinkage Estimator between Maximum Likelihood Estimator and Bayesian Estimator

The shrinkage estimator can be written by linear combination of MLE estimator and Bayesian estimator:

$$\tilde{\alpha}_1=\omega_1\hat{\alpha}_{ML}+(1-\omega_1)\hat{\alpha}_B \tag{15}$$

So we try to find the value of p to minimize the $MSE(\tilde{\alpha}_1)$.

Now,

$$\tilde{\alpha}_1-\alpha=\omega_1\hat{\alpha}_{ML}+(1-\omega_1)\hat{\alpha}_B-\alpha \tag{16}$$

and

$$[\tilde{\alpha}_1-\alpha]^2=[\omega_1\hat{\alpha}_{ML}+(1-\omega_1)\hat{\alpha}_B-\alpha]^2 \tag{17}$$

By taking expected to (17) we have:

$$E[\tilde{\alpha}_1-\alpha]^2=E[\omega_1\hat{\alpha}_{ML}+(1-\omega_1)\hat{\alpha}_B-\alpha]^2 \tag{18}$$

Then

$$E[\tilde{\alpha}_1-\alpha]^2=\omega_1^2E[\hat{\alpha}_{ML}-\alpha]^2+(1-\omega_1)^2E[\hat{\alpha}_B-\alpha]^2+\omega_1(1-\omega_1)E[\hat{\alpha}_{ML}-\alpha][\hat{\alpha}_B-\alpha]$$

And

$$MSE(\tilde{\alpha}_1)=\omega_1^2MSE(\hat{\alpha}_{ML})+(1-\omega_1)^2MSE(\hat{\alpha}_B)+2\omega_1(1-\omega_1)\{E(\hat{\alpha}_{ML}\hat{\alpha}_B)-\alpha E(\hat{\alpha}_{ML})-\alpha E(\hat{\alpha}_B)+\alpha^2\}$$

To minimize $MSE(\tilde{\alpha}_1)$ we have to solve $\frac{\partial MSE(\tilde{\alpha}_1)}{\partial \omega_1}=0$, then the value of ω that minimize $MSE(\tilde{\alpha}_1)$ is:

$$\omega_1 = \frac{MSE(\hat{\alpha}_B) - E(\hat{\alpha}_{ML}\hat{\alpha}_B) + \alpha E(\hat{\alpha}_{ML}) + \alpha E(\hat{\alpha}_B) - \alpha^2}{MSE(\hat{\alpha}_{ML}) + MSE(\hat{\alpha}_B) - 2E(\hat{\alpha}_{ML}\hat{\alpha}_B) + 2\alpha E(\hat{\alpha}_{ML}) + 2\alpha E(\hat{\alpha}_B) - 2\alpha^2} \quad (19)$$

Where

$$MSE(\hat{\theta}) = Var(\hat{\theta}) + [E(\hat{\theta}) - \theta]^2$$

Such that

$$E(\hat{\theta}) = \bar{\hat{\theta}} = \frac{1}{r}\sum_{i=1}^{r}\hat{\theta}_i \ , \ Var(\hat{\theta}) = \frac{1}{r}\sum_{i=1}^{r}\left(\hat{\theta}_i - \bar{\hat{\theta}}\right)^2$$

where $\hat{\theta}$ represents a vector of parameters $\hat{\alpha}_{ML}, \hat{\alpha}_B$ and r is the number of simulation, which is 500.

To obtain the value of p that minimized equation (19) we use the Matlab R2021a program.

2.4. Second Shrinkage Estimator between First Shrinkage and Bayesian Estimator

The second shrinkage estimator can be written by linear combination of first shrinkage estimator and Bayesian estimator:

$$\tilde{\alpha}_2 = \omega_2\tilde{\alpha}_1 + (1-\omega_2)\hat{\alpha}_B \quad (20)$$

So we try to find the value of p to minimize the $MSE(\tilde{\alpha}_2)$.

Now,

$$\tilde{\alpha}_2 - \alpha = \omega_2\tilde{\alpha}_1 + (1-\omega_2)\hat{\alpha}_B - \alpha \quad (21)$$

and

$$[\tilde{\alpha}_2 - \alpha]^2 = [\omega_2\tilde{\alpha}_1 + (1-\omega_2)\hat{\alpha}_B - \alpha]^2 \quad (22)$$

By taking expected to (22) we have:

$$E[\tilde{\alpha}_2 - \alpha]^2 = E[\omega_2\tilde{\alpha}_1 + (1-\omega_2)\hat{\alpha}_B - \alpha]^2 \quad (23)$$

Then

$$E[\tilde{\alpha}_2 - \alpha]^2 = \omega_2^2 E[\tilde{\alpha}_1 - \alpha]^2 + (1-\omega_2)^2 E[\hat{\alpha}_B - \alpha]^2 + \omega_2(1-\omega_2)E[\tilde{\alpha}_1 - \alpha][\hat{\alpha}_B - \alpha]$$

And

$$MSE(\tilde{\alpha}_2) = \omega_2^2 MSE(\tilde{\alpha}_1) + (1-\omega_2)^2 MSE(\hat{\alpha}_B) + 2\omega_2(1-\omega_2)\{E(\tilde{\alpha}_1\hat{\alpha}_B) - \alpha E(\tilde{\alpha}_1) - \alpha E(\hat{\alpha}_B) + \alpha^2\}$$

To minimize $MSE(\tilde{\alpha}_2)$ we have to solve $\frac{\partial MSE(\tilde{\alpha}_2)}{\partial \omega_2} = 0$, then the value of ω_2 that minimize $MSE(\tilde{\alpha}_2)$ is:

$$\omega_2 = \frac{MSE(\tilde{\alpha}_1) - E(\tilde{\alpha}_1\hat{\alpha}_B) + \alpha E(\tilde{\alpha}_1) + \alpha E(\hat{\alpha}_B) - \alpha^2}{MSE(\tilde{\alpha}_1) + MSE(\hat{\alpha}_B) - 2E(\tilde{\alpha}_1\hat{\alpha}_B) + 2\alpha E(\tilde{\alpha}_1) + 2\alpha E(\hat{\alpha}_B) - 2\alpha^2} \quad (24)$$

Where

$$MSE(\hat{\theta}) = Var(\hat{\theta}) + [E(\hat{\theta}) - \theta]^2$$

Such that

$$E(\hat{\theta}) = \bar{\hat{\theta}} = \frac{1}{r}\sum_{i=1}^{r}\hat{\theta}_i \ , \ Var(\hat{\theta}) = \frac{1}{r}\sum_{i=1}^{r}(\hat{\theta}_i - \bar{\hat{\theta}})^2$$

where $\hat{\theta}$ represents a vector of parameters $\tilde{\alpha}_1 . \hat{\alpha}_B$ and r is the number of simulation, which is 500.

To obtain the value of ω_2 that minimized equation (25) we use the Matlab R2021a program.

3. Simulation Study

In our simulation study we chosen different sample size (N = 50, 150, 250) with different values of parameters shown in table (1). The simulation program was written using Matlab R2021a program with replication 500. The MSE using to determine the best method of the four estimators (Maximum Likelihood, Bayes, First shrinkage, Second shrinkage) using the following formula:

$$MSE(\hat{\theta}) = \frac{1}{r}\sum_{i=1}^{r}[\widehat{F}(x_i;\hat{\theta}) - F(x_i;\theta)]^2$$

Where $\widehat{F}(x_i;\hat{\theta})$ represent the cumulative function for estimated parameters $\hat{\theta}$, and $F(x_i;\theta)$ represents the true value of cumulative function of true parameters.

The estimators for Maximum likelihood, Bayes, First shrinkage, and Second shrinkage for our models of parameters are shown in Tables (2-4) below:

Table 2: Calculate the estimators of Maximum likelihood, Bayes, First shrinkage, and Second shrinkage for model 1.

	N	$\hat{\theta}_{MLE}$	$\hat{\theta}_B$	ω_1	$\hat{\theta}_1$	ω_2	$\hat{\theta}_2$
$\hat{\alpha}$	50	1.28316	1.29433	0.50451	1.28870	0.4999	1.28524
$\hat{\beta}$		1.38691	1.42233	0.50421	1.40447	0.5000	1.41343
$\hat{\alpha}$	150	1.17510	1.17823	0.50163	1.17803	0.4999	1.17520
$\hat{\beta}$		1.54421	1.55773	0.50422	1.55732	0.5000	1.61343
$\hat{\alpha}$	250	1.13011	1.13212	0.50095	1.13421	0.4999	1.12514
$\hat{\beta}$		1.62712	1.63601	0.50266	1.632230	0.5000	1.66401

Table 3: Calculate the estimators of Maximum likelihood, Bayes, First shrinkage, and Second shrinkage for model 2.

	N	$\hat{\theta}_{MLE}$	$\hat{\theta}_B$	ω_1	$\hat{\theta}_1$	ω_2	$\hat{\theta}_2$
$\hat{\alpha}$	50	1.99820	2.01811	0.50327	2.00521	0.4999	1.98112
$\hat{\beta}$		4.34510	4.47821	0.50304	4.41131	0.4999	4.40822
$\hat{\alpha}$	150	2.02620	2.03065	0.50109	2.02842	0.4999	1.9921
$\hat{\beta}$		4.34520	4.21639	0.50282	4.28131	0.4999	4.24833
$\hat{\alpha}$	250	2.03360	2.03630	0.5006	2.03493	0.4999	2.00751
$\hat{\beta}$		4.34520	4.13071	0.50166	4.23854	0.4999	4.18463

Table 4: Calculate the estimators of Maximum likelihood, Bayes, First shrinkage, and Second shrinkage for model 3.

	N	$\hat{\theta}_{MLE}$	$\hat{\theta}_B$	ω_1	$\hat{\theta}_1$	ω_2	$\hat{\theta}_2$
$\hat{\alpha}$	50	5.28229	5.30133	0.50155	5.29313	0.4999	5.29723
$\hat{\beta}$		3.47066	3.51150	0.50339	3.49629	0.4999	3.47239
$\hat{\alpha}$	150	5.35111	5.35666	0.50052	5.35388	0.4999	5.35527
$\hat{\beta}$		3.22088	3.22945	0.50134	3.22516	0.4999	3.21813
$\hat{\alpha}$	250	5.35342	5.35676	0.500313	5.35508	0.4999	5.35592
$\hat{\beta}$		3.13954	3.14440	0.50077	3.14195	0.4999	3.13793

Table 5: MSE values of Maximum likelihood, Bayes, First shrinkage, and Second shrinkage for the Model 1. ($\times 10^{-3}$)

N	$\hat{\theta}_{MLE}$	$\hat{\theta}_B$	$\hat{\theta}_1$	$\hat{\theta}_2$	Ranking the Best Estimator
50	0.1369590	0.1617070	0.1361260	0.1349760	$\hat{\theta}_2$, $\hat{\theta}_1$, $\hat{\theta}_{MLE}$, $\hat{\theta}_B$
150	0.0872673	0.0685117	0.0763354	0.0684692	$\hat{\theta}_2$, $\hat{\theta}_B$, $\hat{\theta}_1$, $\hat{\theta}_{MLE}$
250	0.0677067	0.0532938	0.0599225	0.0392344	$\hat{\theta}_2$, $\hat{\theta}_1$, $\hat{\theta}_{MLE}$, $\hat{\theta}_B$

Table 6: MSE values of Maximum likelihood, Bayes, First shrinkage, and Second shrinkage for the Model 2. ($\times 10^{-3}$)

N	$\hat{\theta}_{MLE}$	$\hat{\theta}_B$	$\hat{\theta}_1$	$\hat{\theta}_2$	Ranking the Best Estimator
50	0.4014640	0.6583860	0.5216890	0.3314930	$\hat{\theta}_2$, $\hat{\theta}_1$, $\hat{\theta}_{MLE}$, $\hat{\theta}_B$
150	0.1629720	0.2087430	0.1850740	0.1236000	$\hat{\theta}_2$, $\hat{\theta}_B$, $\hat{\theta}_1$, $\hat{\theta}_{MLE}$
250	0.1005860	0.1216220	0.1108150	0.1005620	$\hat{\theta}_2$, $\hat{\theta}_1$, $\hat{\theta}_{MLE}$, $\hat{\theta}_B$

Table 7: MSE values of Maximum likelihood, Bayes, First shrinkage, and Second shrinkage for the Model 3. ($\times 10^{-2}$)

N	$\hat{\theta}_{MLE}$	$\hat{\theta}_B$	$\hat{\theta}_1$	$\hat{\theta}_2$	Ranking the Best Estimator
50	0.2558906	0.2859551	0.2706498	0.2558528	$\hat{\theta}_2$, $\hat{\theta}_1$, $\hat{\theta}_{MLE}$, $\hat{\theta}_B$
150	0.1237130	0.1302365	0.1269485	0.1237081	$\hat{\theta}_2$, $\hat{\theta}_B$, $\hat{\theta}_1$, $\hat{\theta}_{MLE}$
250	0.0847752	0.0879434	0.0863428	0.0847732	$\hat{\theta}_2$, $\hat{\theta}_1$, $\hat{\theta}_{MLE}$, $\hat{\theta}_B$

4. Discussion

We note from Table (5) that the second shrinkage estimator $\hat{\theta}_2$ between first shrinkage $\hat{\theta}_1$ (between Maximum likelihood and Bayes estimator) is the best estimator because it has the lowest error than the rest of the estimators, according to MSE, for all sample sizes. In second place is the first shrinkage estimator $\hat{\theta}_1$ between Maximum likelihood and Bayes estimator for most sample sizes. In third place comes the maximum likelihood estimator for most sample sizes. The Bayes estimator comes in last place for most sample sizes, according to values of the parameters in Model 1.

In Table (6) we can see that the second shrinkage estimator $\hat{\theta}_2$ is the best estimator because it has the lowest error than the rest of the estimators, according to MSE, for all sample sizes. In second place is the maximum likelihood estimator for all sample sizes. In third place comes the first shrinkage estimator $\hat{\theta}_1$ for all sample sizes. The Bayes estimator comes in last place for all sample sizes, according to values of the parameters in Model 2.

In Table (7) we can also seethat the second shrinkage estimator $\hat{\theta}_2$ is the best estimator because it has the lowest error than the rest of the estimators, according to MSE, for all sample sizes. In second place is the maximum likelihood estimator for all sample sizes. In third place comes the first shrinkage estimator $\hat{\theta}_1$ for all sample sizes. The Bayes estimator comes in last place for all sample sizes, according to values of the parameters in Model 3.

5. Conclusion

From the previous tables, we note that the second shrinkage was the better estimation method than the other methods under study. It is also clear to us that by increasing the number of shrinkages, we will arrive at better estimates of the distribution parameters.

References

[1] Abd Ali, M. H., Jiheel, A. K., & Al-Hemyar, Z. (2022). Two-stage shrinkage Bayesian estimators for the shape parameter of Pareto distribution dependent on Katti's regions. *Iraqi Journal for Computer Science and Mathematics,* 3(2), 42–54. https://doi.org/10.52866/ijcsm.2022.02.01.005

[2] Abdullah, M., Iqbal, Z., Ali, A., Zakria, M., & Ahmad, M. (2016). Size biased Lomax distribution. 23, 32–49.

[3] Abujarad, M. H., & Khan, A. A. (2021). Bayesian survival analysis of Lomax family models with Stan (R). *Journal of Modern Applied Statistical Methods,* 19(1), Article 12. https://doi.org/10.22237/jmasm/1608553800

[4] Alkutubi, H., & Ibrahim, N. K. (2009). Comparison of three shrinkage estimators of parameter exponential distribution. *International Journal of Applied Mathematics,* 22(5), 785–792.

[5] Hamad, A. M. (2013). Double stage shrinkage estimator of two parameters generalized exponential distribution. *International Mathematical Forum,* 8(23), 1143–1153. http://dx.doi.org/10.12988/imf.2013.3351

[6] Hadi, S., & Salih, S. (2010). Bayes estimator for the reliability function of the Pareto model of Type I failure. *Journal of Economics and Administrative Sciences,* 16(57), 145. https://doi.org/10.33095/jeas.v16i57.1434

[7] Harris, C. M. (1968). The Pareto distribution as a queue service discipline. *Operations Research,* 16(2), 307–313. https://doi.org/10.1287/opre.16.2.307

[8] Hassan, A., & Al-Ghamdi, A. (2009). Optimum step-stress accelerated life testing for Lomax distribution. *Journal of Applied Sciences Research,* 12, 2153–2164.

[9] Holland, O., Golaup, A., & Aghvami, A. H. (2006). Traffic characteristics of aggregated module downloads for mobile terminal reconfiguration. *IEE Proceedings: Communications,* 153(5), 683–690. https://doi.org/10.1049/ip-com:20045155

[10] Khan, N., Khalil, A., & Rehman, A. (2023). A new extended generalized Pareto distribution: Its properties and applications. *Journal of Xi'an Shiyou University, Natural Science Edition,* 19(6), 171–190.

[11] Labban, J. A. (2019). On 2-parameter estimation of Lomax distribution. *Journal of Physics: Conference Series,* 1294(3), 1–5. https://doi.org/10.1088/1742-6596/1294/3/032018

[12] Mitra, D., Kundu, D., & Balakrishnan, N. (2021). Likelihood analysis and stochastic EM algorithm for left truncated right censored data and associated model selection from the Lehmann family of life distributions. *Japanese Journal of Statistics and Data Science,* 4(2), 1019–1043. https://doi.org/10.1007/s42081-021-00115-1

[13] Rahim, R. (2024). Adaptive Algorithms for Power Management in Battery-Powered Embedded Systems. SCCTS Journal of Embedded Systems Design and Applications, 1(1), 20-24.

[14] Para, B. A., & Jan, T. R. (2018). On three parameter weighted Pareto Type II distribution: Properties and applications in medical sciences. *Applied Mathematics & Information Sciences Letters,* 6(1), 13–26. https://doi.org/10.18576/amisl/060103

[15] Pareto, V. (1897). The new theories of economics. *Journal of Political Economy,* 5(4), 485–502.

[16] Pels, W. A., Adebanji, A. O., Twumasi-Ankrah, S., & Minkah, R. (2023). Shrinkage methods for estimating the shape parameter of the generalized Pareto distribution. *Journal of Applied Mathematics, 2023*. https://doi.org/10.1155/2023/9750638

[17] Prakash, G. (2010). Shrinkage estimation in the inverse Rayleigh distribution. *Journal of Modern Applied Statistical Methods,* 9(1), 209–220. https://doi.org/10.22237/jmasm/1272687540

[18] Muralidharan, J. (2024). Machine Learning Techniques for Anomaly Detection in Smart IoT Sensor Networks. Journal of Wireless Sensor Networks and IoT, 1(1), 10–14.

[19] Salman, A. N., & Hamad, A. M. (2019). On estimation of the stress-strength reliability based on Lomax distribution. *IOP Conference Series: Materials Science and Engineering,* 571(1), 1–7. https://doi.org/10.1088/1757-899X/571/1/012038

[20] Salman, A. N., & Hussein, A. A. (2016). Single stage shrinkage estimator for the shape parameter of the Pareto distribution. *International Journal of Mathematics Trends and Technology,* 35(3), 156–162. https://doi.org/10.14445/22315373/IJMTT-V35P521

[21] Sharma, V. K., Singh, S. K., & Singh, U. (2017). Classical and Bayesian methods of estimation for power Lindley distribution with application to waiting time data. *Communications for Statistical Applications and Methods,* 24(3), 193–209. https://doi.org/10.5351/CSAM.2017.24.3.19