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Transformation solution for Korteweg-de Vries equation with small delay

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Abstract

In this paper we develop a new approach to get the transformation solution for the mathematical model of waves on shallow fluid; Korteweg-de Vries with a small delay without change the space variables. This method can be base to solve most of nonlinear higher order partial differential equation with time delay.

Key words and phrases: Solitary waves, KdV equation, Small delay, Lie group, Transformation solution.

Mathematics Subject Classification (2010): 34G20, 57S20, 74J35, 74J15

1. Introduction

Currently, the prediction and comprehensive knowledge of the development of mathematical modelling phenomena in life sciences and physiology have significantly intensified [1]. The Korteweg-de Vries (KdV) equation is a mathematical model which describes the velocity of the particles of shallow fluid from the surface to the bottom of the fluid layer. Russell J.S. described the Korteweg-de Vries equation (KdV) starting from fluid dynamics [2]. Next, the KdV is solved, by the same approach of Korteweg but de Vries solved it by inverse scattering transform which is leading to a larger family of solutions [2]. Many researchers solved KdV equation by several ways see [3,4,5,6]. Zhao and Xu [7] entered time delay on the Solitary waves for KdV, so it is became takes the form

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$$u_t(t,x)+u(t-\tau,x)u_x(t,x)+ru_{xx}(t-\tau,x)-u_{xxx}(t,x)=0 \tag{1}$$

where τ is time delay, $\tau>0$. $u(t,x)$ is the waves amplitude when x is positive in time $t, u(t-\tau,x)$ is the waves amplitude of the wave when x is positive in time $t-\tau$. r is small enough.

They proved that time delay make this equation more explained, but in the same hand more complicated. Some researchers studied KdV equation with delay by the stability of it see [8,9,10,11], and the solution of Equation (1) has not been discussed yet, so this paper interdicted a new method to solve KdV with delay. The key idea is to findthe invariance of the KdV equation under transformation of dependent and independent variables. Next modified Olver's method to get the transformation solution which corresponds to the KdV equation.

2. Admitted Lie group

First, one must prove the general infinitesimal generator for KdV equation with small delay.

From Equation (1), define a one-parameter group G on t,x,u as follows:

$$\begin{aligned}\bar{t}&=\varphi^t(t,x,u;\varepsilon)\\ \bar{x}&=\varphi^x(t,x,u;\varepsilon)\\ \bar{u}&=\varphi^u(t,x,u;\varepsilon)\end{aligned} \tag{2}$$

Since $u^\tau=u(t-\tau,x)$, then $\overline{u^\tau}=\varphi^u(t-\tau,x,u_\tau;\varepsilon)$, according to Olver [12] these define a symmetry group. Expanding Equation(2) into Taylor series about ε near 0, this is given the infinitesimal transformation

$$\begin{aligned}\bar{t}&=t+\varepsilon\xi(t,x,u)\\ \bar{x}&=x+\varepsilon\eta(t,x,u)\\ \bar{u}&=u+\varepsilon\zeta(t,x,u)\end{aligned}$$

with delay term $\overline{u^\tau}=u^\tau+\varepsilon\zeta(t,x,u)$

where

$$\xi(t,x,u)=\frac{\partial\varphi^t(t,x,u;\varepsilon)}{\partial\varepsilon}\Big|_{\varepsilon=0}$$

$$\eta(t,x,u)=\frac{\partial\varphi^x(t,x,u;\varepsilon)}{\partial\varepsilon}\Big|_{\varepsilon=0}$$

$$\zeta(t,x,u)=\frac{\partial\varphi^u(t,x,u;\varepsilon)}{\partial\varepsilon}\Big|_{\varepsilon=0}$$

$$\text{and } \zeta(t-\tau,x,u^\tau)=\frac{\partial\varphi^u(t-\tau,x,u^\tau;\varepsilon)}{\partial\varepsilon}\Big|_{\varepsilon=0}.$$

According to Lie [13] the vector $(\xi,\eta,\zeta,\zeta^\tau)$ where $\zeta^\tau=\zeta(t-\tau,x,u^\tau)$ are tangent vector field on G, so can be written them in term of the first order differential operator [14]

$$X=\xi\frac{\partial}{\partial t}+\eta\frac{\partial}{\partial x}+\zeta\frac{\partial}{\partial u}+\zeta^\tau\frac{\partial}{\partial u^\tau} \tag{3}$$

Lie's theorem [14] shows the general is one to one corresponding to the symmetry.

From the definition of infinitesimal generator, Equation(3) is a general infinitesimal generator for Equation (1).

Now, to complete the classification one must prove the determining equation for the KdV equation with delay.

According to group analysis theory, the transformation $\varphi:\Omega\times S\to\Omega$ which transforms a solution of the differential equation to a solution of the same equation is a symmetry of Equation (1),where a set of variables is denoted by Ω and $S\subseteq R$ is a symmetric interval with respect to zero. The transformation φ is consider ε as a parameter which transforms variables t,x,u to new variables $\bar{t},\bar{x},\bar{u}$ of the same space, where

$$\bar{t}=\varphi^t(t,x,u;\varepsilon)=\varphi_\varepsilon^t(t,x,u),\ \bar{x}=\varphi^x(t,x,u;\varepsilon)=\varphi_\varepsilon^x(t,x,u),\ \bar{u}=\varphi^u(t,x,u;\varepsilon)=\varphi_\varepsilon^u(t,x,u).$$

Since $u^\tau=u(t-\tau,x)$, then $\overline{u^\tau}=\varphi^u(t-\tau,x,u^\tau;\varepsilon)=\varphi_\varepsilon^u(t-\tau,x,u^\tau)$.

The set of function φ_ε forms a one-parameters group [14].

Now consider the function

$$u_{xxx}-f(t,x,u,u_t,u^\tau,u_x,u_{xx}^\tau)=F(t,x,u,u_t,u^\tau,u_x,u_{xx}^\tau,u_{xxx})=0 \tag{4}$$

According to Lie [13] the equation with the transformation variables $\bar{t},\bar{x},\bar{u},\overline{u^\tau}$ and its derivatives with respect to ε must vanishes if the transformation is symmetry.

$$\frac{\partial F(t,x,u,u_t,u^\tau,u_x,u_{xx}^\tau,u_{xxx})}{\partial\varepsilon}\Big|_{\varepsilon=0,(4)}=0 \tag{5}$$

Since $\overline{F}$ is a symmetry group (according to Olver [12]), then $\overline{F}=F$ this mean F is invariant. From Theorems in Bluman and Ibragimov [13,14]

$$X^{(3)}F\,|_{(4)}=0 \tag{6}$$

where $X^{(3)}$ is a canonical Lie Backlund operator. Then Equation (6) is called determining equation of Equation(1).

The operator $X^{(3)}=X+\zeta_{(t)}\frac{\partial}{\partial u_t}+\zeta_{(x)}\frac{\partial}{\partial u_x}+\zeta_{(xx)}^\tau\frac{\partial}{\partial u_{xx}^\tau}+\zeta_{(xxx)}\frac{\partial}{\partial u_{xxx}}$ where

$$\zeta_t^{(1)}=\zeta_t+(\zeta_u-\eta_t)u_t-\xi_t u_x-\eta_u u_t^2-\xi_u u_x u_t$$

$$\zeta_x^{(1)}=\zeta_x+(\zeta_u-\xi_x)u_x-\eta_x u_t-\xi_u u_x^2-\eta_u u_x u_t$$

$$\begin{aligned}\zeta_{xx}^{\tau(2)}&=\zeta_{xx}^\tau+(2\zeta_{xu}^\tau-\xi_{xx}^\tau)u_x^\tau-\eta_{xx}^\tau u_t^\tau+(\zeta_{uu}^\tau-2\xi_{xu}^\tau)(u_x^\tau)^2-2\eta_{xu}^\tau u_x^\tau u_t^\tau\\&-\xi_{uu}^\tau(u_x^\tau)^3-\eta_{uu}^\tau(u_x^\tau)^2u_t^\tau+(\zeta_u^\tau-2\xi_x^\tau)u_{xx}^\tau-2\eta_x^\tau u_{xt}^\tau-3\xi_u^\tau u_{xx}^\tau u_x^\tau-\eta_u^\tau u_{xx}^\tau u_t^\tau-2\eta^\tau u_{xt}^\tau u_x^\tau\end{aligned}$$

$$\begin{aligned}\zeta_{xxx}^{(3)}&=\zeta_{xxx}+(3\zeta_{xxu}-\xi_{xxx})u_x-\eta_{xxx}u_t+3(\zeta_{xuu}-\xi_{xxu})u_x^2-3\eta_{xxu}u_t u_x\\&+(\zeta_{uuu}-3\xi_{xuu})u_x^3+3(\zeta_{xu}-\xi_{xx})u_{xx}-3\eta_{xx}u_{xt}-3\eta_{xuu}u_x^2u_t+3(\zeta_{uu}-3\xi_{xu}u_x u_{xx})\\&-3\eta_{xu}u_t u_{xx}-6\eta_{xu}u_{xt}u_x-3\eta_x u_{xxt}+(\zeta_u-3\xi_x)u_{xxx}-\xi_{xxx}u_x^4-6\xi_{uu}u_x^2u_{xx}-3\eta_{uu}u_x^2u_{tx}\\&-\eta_{uuu}u_x^3u_t-3\xi_u u_{xx}^2-3\eta_u u_{xxt}u_x-\eta_u u_{xt}u_{xx}-3\eta_{uu}u_{xx}u_x u_t-4\xi_u u_{xxx}u_x-\eta_u u_{xxx}u_t\end{aligned}$$

3. Results and Discussion

The infinitesimal generator $X=\xi(t,x,u)\frac{\partial}{\partial t}+\eta(t,x,u)\frac{\partial}{\partial x}+\zeta(t,x,u)\frac{\partial}{\partial u}+\zeta(t-\tau,x,u)\frac{\partial}{\partial u^\tau}$ generates a one parameter symmetry group if and only if where $F=u_t+u^\tau u_x+ru_{xx}^\tau-u_{xxx}$. i.e.

$$\left(\zeta\frac{\partial}{\partial u}+\zeta^\tau\frac{\partial}{\partial u^\tau}+\zeta_{(t)}\frac{\partial}{\partial u_t}+\zeta_{(x)}\frac{\partial}{\partial u_x}+\zeta_{(xx)}^\tau\frac{\partial}{\partial u_{xx}^\tau}+\zeta_{(xxx)}\frac{\partial}{\partial u_{xxx}}\right)F\,\Big|_{u_{xxx}=u_t+u^\tau u_x+ru_{xx}^\tau}=0$$

where $\zeta^{\tau}=\zeta(t-\tau,x,u)$ and $u^{\tau}=u(t-\tau,x)$. Then

$$\zeta_t^{(1)}+u^{\tau}\zeta_x^{(1)}+u_x\zeta^{\tau}+r\zeta_{xx}^{\tau(2)}-\zeta_{xxx}^{(3)}=0 \tag{7}$$

which leads to:

$$\begin{aligned}
&\zeta_t+(\zeta_u-\eta_t)u_t-\xi_t u_x-\eta_u u_t^2-\xi_u u_x u_t+u^{\tau}[\zeta_x+(\zeta_u-\xi_x)u_x-\eta_x u_t-\xi_u u_x^2-\eta_u u_x u_t]+u_x\zeta^{\tau}\\
&+r[\zeta_{xx}^{\tau}+(2\zeta_{xu}^{\tau}-\xi_{xx}^{\tau})u_x^{\tau}-\eta_{xx}^{\tau}u_t^{\tau}+(\zeta_{uu}^{\tau}-2\xi_{xu}^{\tau})(u_x^{\tau})^2-2\eta_{xu}^{\tau}u_x^{\tau}u_t^{\tau}-\xi_{uu}^{\tau}(u_x^{\tau})^3-\eta_{uu}^{\tau}(u_x^{\tau})^2u_t^{\tau}\\
&+(\zeta_u^{\tau}-2\xi_x^{\tau})u_{xx}^{\tau}-2\eta_x^{\tau}u_{xt}^{\tau}-3\xi_u^{\tau}u_{xx}^{\tau}u_x^{\tau}-\eta_u^{\tau}u_{xx}^{\tau}u_t^{\tau}-2\eta^{\tau}u_{xt}^{\tau}u_x^{\tau}]-[\zeta_{xxx}+(3\zeta_{xxu}-\xi_{xxx})u_x-\eta_{xxx}u_t\\
&+3(\zeta_{xuu}-\xi_{xxu})u_x^2-3\eta_{xxu}u_t u_x+(\zeta_{uuu}-3\xi_{xuu})u_x^3+3(\zeta_{xu}-\xi_{xx})u_{xx}-3\eta_{xx}u_{xt}-3\eta_{xuu}u_x^2u_t\\
&+3(\zeta_{uu}-3\xi_{xu}u_x u_{xx})-3\eta_{xu}u_t u_{xx}-6\eta_{xu}u_{xt}u_x-3\eta_x u_{xxt}+(\zeta_u-3\xi_x)u_{xxx}-\xi_{xxx}u_x^4-6\xi_{uu}u_x^2u_{xx}\\
&-3\eta_{uu}u_x^2u_{tx}-\eta_{uuu}u_x^3u_t-3\xi_u u_{xx}^2-3\eta_u u_{xxt}u_x-\eta_u u_{xt}u_{xx}-3\eta_{uu}u_{xx}u_x u_t-4\xi_u u_{xxx}u_x-\eta_u u_{xxx}u_t]=0
\end{aligned}$$

Equating the coefficient of the various monomial in the first, second and third order partial derivative of u to get the following determining equation for the symmetry group of Equation (1).

1	$\zeta_t+r\zeta_{xx}^{\tau}-\zeta_{xxx}=0$	(8.1)	
u_t	$\zeta_u-\eta_t+\eta_{xxx}=0$	(8.2)	
u_x	$\zeta^{\tau}-\xi_t-3\zeta_{xxu}+\xi_{xxx}=0$	(8.3)	
u_t^2	$-\eta_u=0$	(8.4)	
$u_x u_t$	$3\eta_{xxu}-\xi_u=0$	(8.5)	
u_x^2	$\xi_{xxu}-3\zeta_{xuu}=0$	(8.6)	
u_x^3	$3\xi_{xuu}-\zeta_{uuu}=0$	(8.7)	
u_{xx}	$\xi_{xx}-3\zeta_{xu}=0$	(8.8)	
u_{xt}	$3\eta_{xx}=0$	(8.9)	
$u_x^2u_t$	$3\eta_{xuu}=0$	(8.10)	
$u_x u_{xx}$	$3\xi_{xu}-3\zeta_{uu}=0$	(8.11)	
$u_t u_{xx}$	$3\eta_{xu}=0$	(8.12)	
$u_{xt}u_x$	$6\eta_{xu}=0$	(8.13)	
u_{xxt}	$3\eta_x=0$	(8.14)	
u_{xxx}	$3\xi_x-\zeta_u=0$	(8.15)	(8)
u_x^4	$\xi_{xxx}=0$	(8.16)	
$u_x^2u_{xx}$	$6\xi_{uu}=0$	(8.17)	
$u_x^2u_{tx}$	$3\eta_{uu}=0$	(8.18)	
$u_x^3u_t$	$\eta_{uuu}=0$	(8.19)	
u_{xx}^2	$3\xi_u=0$	(8.20)	
$u_{xxt}u_x$	$3\eta_u=0$	(8.21)	
$u_{xt}u_{xx}$	$\eta_u=0$	(8.22)	
$u_{xx}u_x u_t$	$3\eta_{uu}=0$	(8.23)	
$u_{xxx}u_x$	$4\xi_u=0$	(8.24)	
$u_{xxx}u_t$	$\eta_u=0$	(8.25)	
u^{τ}	$\zeta_x=0$	(8.26)	
$u^{\tau}u_x$	$\zeta_u-\xi_x=0$	(8.27)	
$u^{\tau}u_t$	$-\eta_x=0$	(8.28)	
$u^{\tau}u_x^2$	$-\xi_u=0$	(8.29)	

$u^{\tau} u_x u_t$	$-\eta_u = 0$	(8.30)
u_x^{τ}	$2r\zeta_{xu}^{\tau} - r\xi_{xx}^{\tau} = 0$	(8.31)
u_t^{τ}	$-r\eta_{xx}^{\tau} = 0$	(8.32)
$(u_x^{\tau})^2$	$r\zeta_{uu}^{\tau} - 2r\xi_{xu}^{\tau} = 0$	(8.33)
$u_x^{\tau} u_t^{\tau}$	$-2r\eta_{xu}^{\tau} = 0$	(8.34)
$(u_x^{\tau})^3$	$-r\xi_{uu}^{\tau} = 0$	(8.35)
$(u_x^{\tau})^2 u_t^{\tau}$	$-r\eta_{uu}^{\tau} = 0$	(8.36)
u_{xx}^{τ}	$r\zeta_u^{\tau} - 2r\xi_x^{\tau} = 0$	(8.37)
u_{xt}^{τ}	$-2r\eta_x^{\tau} = 0$	(8.38)
$u_{xx}^{\tau} u_x^{\tau}$	$-3r\xi_u^{\tau} = 0$	(8.39)
$u_{xx}^{\tau} u_t^{\tau}$	$-r\eta_u^{\tau} = 0$	(8.40)
$u_{xt}^{\tau} u_x^{\tau}$	$-2r\eta_u^{\tau} = 0$	(8.41)

First (8.4) and (8.14) require that η be just a function of t. From (8.5) ξ independent on u, and from (8.26) ζ independent on x. (8.11) leads to ζ is linear in u and $\zeta = g(t)u + h(t)$, when g and h are a function of t. From (8.2) and (8.27) $\xi_x = \eta_t = g(t)$.

Now, from (8.37) $\zeta_u^{\tau} = 2\xi_x^{\tau}$, then by periodic theorem $\xi = \xi^{\tau}$, and $\eta = \eta^{\tau}$, this mean $\xi_x = \xi_x^{\tau}$. By above $\zeta_u^{\tau} = 2\xi_x = 2g(t)$, thus $\zeta^{\tau} = 2g(t)u + k(t-\tau, x)$ for some k is arbitrary function.

(8.1) gives $g_t u + h_t + r(2g_{xx}u + k_{xx}^{\tau}) = 0$, where $k^{\tau} = k(t-\tau, x)$. Equating the coefficients of various monomials to get

$$g_t + 2rg_{xx} = 0 \tag{9}$$

$$h_t + rk_{xx}^{\tau} = 0$$

Since g is a function of t, then $g_{xx} = 0$, also since $\eta_t = g$ this mean $\eta_{tt} = g_t$.

Equation(9) gives $\eta_{tt} = 0$, by integrating $\eta = c_1 t + c_2$. Since $\eta_t = \xi_x$, then $\xi_x = c_1$, and $\xi = c_1 x + c_3$, for some c_1, c_2, c_3 are arbitrary constants.

From above $g = c_1$, then $\zeta = c_1 u + h$ and $\zeta^{\tau} = 2c_1 u + k^{\tau}$. That is the general infinitesimal generator for Equation (1) is

$$X = (c_1 x + c_3)\frac{\partial}{\partial t} + (c_1 t + c_2)\frac{\partial}{\partial x} + (c_1 u + h)\frac{\partial}{\partial u} + (2c_1 u + k^{\tau})\frac{\partial}{\partial u^{\tau}}$$

Thus the Lie algebra of infinitesimal symmetries of Equation (1) is spanned by the following infinitesimal generators corresponding to each parameterc_i,

$$X_1 = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + 2u\frac{\partial}{\partial u^{\tau}}$$

$$X_2 = \frac{\partial}{\partial x}$$

$$X_3 = \frac{\partial}{\partial t}$$

$$X_4 = h\frac{\partial}{\partial u} + k^{\tau}\frac{\partial}{\partial u^{\tau}}.$$

Not that X_4 is an infinite dimensional Lie subalgebra.

Using Lie equation on this space to get

$$X_1 : (\varepsilon t, \varepsilon t + x, e^{\varepsilon} u, e^{\varepsilon} u + u^{\tau})$$

$$X_2 : (t, x + \varepsilon, u, u^{\tau})$$

$$X_3 : (t + \varepsilon, x, u, u^{\tau})$$

$$X_4 : (t, x, \varepsilon v(t) + u, \varepsilon w(t - \tau, x) + u^{\tau})$$

Following Olver [12], if $u(t,x) = F(t,x)$ is a solution, then $\bar{u} = gF(t,x)$ is also a solution for any g where g is a group element. So whole families of solutions constructed by transformation of a known solution by any g. By above $u(t,x) = F(t,x) = f_1(t,x) + f_2(t,x)$ is a solution of Equation (1), then‘ $\bar{u} = gF(t,x)$ when‘ $u = f_1$ and‘ $u^{\tau} = f_2$ is also a solution for Equation (1).

The family of the transformation solutions of Equation (1) is

$$u_1 = f_1(\frac{t}{\varepsilon}, x - t)(e^{\varepsilon} - 1) + f_2(\frac{t}{\varepsilon}, x - t)$$

$$u_2 = F(t, x - \varepsilon)$$

$$u_3 = F(t - \varepsilon, x)$$

$$u_4 = F(t,x) + \varepsilon(v(t) + w(t - \tau, x))$$

Finally, note that under special conditions the transformation solutions can be used to get the general solution to the corresponding KdV with small delay.

4. Conclusion

This work succeeded to solve KdV equation with small delay by employing an approach based on the prolongation of this equation and periodic property of ξ's theorem which help us to introduced the invariant for KdV with delay to classify these equations as Lie algebra without changing the space variables. The development of Olvers method led us to obtain the transformation solution for KdV equation with time delay. Under special conditions the transformation solutions can be used to solve the corresponding KdV. These findings enhance our understanding of the KdV equation with delay. Furthermore, we can consider this research as a base to study many scientific branches which use delay partial differential equations.

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