Results in Nonlinear Analysis 8 (2025) No. 1, 124–130
https://doi.org/10.31838/rna/2025.08.01.011
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

Fixed point theorems of multivalued mappings of Integral type contraction in cone metric space and its applications

K. Dinesh[1]*, Ehssan Abdalrhim[2], M. S. Jazmati[3], Mona A. Mohamed[4], Doaa Rizk[5]

[1]Department of Mathematics, K. Ramakrishnan College of Engineering, Tiruchirappalli, Tamilnadu, India; [2,3,4,5]Department of Mathematics, College of Science, Qassim University, Buraydah, 51452, Saudi Arabia.

Abstract

In this research, we use integral type contraction requirements to learning the existence of fixed points for multivalued mappings in the context of cone metric spaces. Cone metric spaces are a generalization of conventional metric spaces that provide a more comprehensive framework for solving challenging issues in fixed point theory by exchanging an ordered Banach space aimed at the real number set. Multivalued mappings pose special difficulties in locating fixed points since they provide each input several outputs. In order to tackle this, we present integral type contraction conditions, which offer a broadened contractive structure that encompasses a variety of non-linear behaviors. This study's main contribution is the construction of novel fixed point theorems over multivalued mappings given these integral type conditions of use, which broadens the application of previously published fixed point results.

Keywords: Cone metric space, Multivalued mappings, Fixed point, Integral type contraction.

Mathematics subject Classification: Primary 47H10, secondary 54H25

1. Introduction and Preliminaries

Many writers have investigated the strong convergence toward a fixed point (FP) of contractive constant over cone metric spaces (CMS) in the last few years. Multivalued mapping has been used by

Email addresses: dinesh.skksv93@gmail.com (K. Dinesh)*, E.suleman@qu.edu.sa (Ehssan Abdalrhim), Jzmaty@qu.edu.sa (M. S. Jazmati), M.HUSSEIN@qu.edu.sa (Mona A. Mohamed), d.hussien@qu.edu.sa (Doaa Rizk)

Received December 20, 2024; Accepted January 29, 2025; Online February 19, 2025

Seong Hoon Cho with Mi sun Kim [2] to demonstrate a number of fixed point (FP) theorems in the context of contractive constants in metric spaces. First, we review the terms and established findings that are required for the continuation. Zhang Xian and Huang Gaung introduced the cone metric space. Given a Banach space and a subset P, P is considered to belong to a cone if and only if it meets the subsequent criteria.

(i) $P \neq \emptyset$ & P is closed;
(ii) $\theta_1\vartheta + \theta_2\omega \in P \ \forall\ \vartheta, \omega \in P$ where the real numbers θ_1, θ_2 are non-ve;
(iii) $P \cap (-P) = \emptyset$;

Regarding the cone, the partial ordering $\leq$ by $\vartheta \leq \omega \Leftrightarrow \omega - \vartheta \in P$

When $\omega - \vartheta \in$ interior of P, it is represented by $\vartheta \ll \omega$.

If a value $\kappa > 0$ such that for any $\vartheta, \omega \in E$, $0 \leq \vartheta \leq \omega \Rightarrow \|\vartheta\| \leq \kappa\ \|\omega\|$, then it is though that the cone P is normal.

If each decreasing sequence that is bounded below and every ascending sequence that is bounded above converges, cone P is therefore considered to be regular.

Definition 1.1:[4] Let X be a non-empty set, and let us assume that the mapping $d : X \times X \to E$ is a cone metric space if and only if it fulfills

(i) $0 \leq d(\vartheta, \omega)\ \forall\ \vartheta, \omega \in X$ and $d(\vartheta, \omega) = 0 \Leftrightarrow$ if $\vartheta = \omega$
(ii) $d(\vartheta, \omega) = d(\omega, \vartheta)$ for all $\vartheta, \omega \in X$
(iii) $d(\vartheta, \omega) = d(\vartheta, z) + d(z, \omega)$ for all $\vartheta, \omega, z \in X$

Example 1.2: Let $E = R^2, P = \{(\vartheta, \omega) \in E; \vartheta, \omega \geq 0\}, X = R$ and $d : X \times X \to E$ as specified by

$$d(\vartheta, \omega) = (|\vartheta - \omega|, \alpha|\vartheta - \omega|)$$

When a constant $\alpha \geq 0$ is present. Then the metric space (X, d) is a cone.

Definition 1.3: Let $\{\vartheta_n\}$ a sequence in $X, \vartheta \in X$ and (X, d) be cone metric space. Then

(i) $\{\vartheta_n\}$ converges to ϑ whenever and for every $\alpha \in E$ with $0 \ll \alpha$, a natural number exists N such that $d(\vartheta_n, \vartheta) \ll \alpha$ for all $n \geq N$
(ii) $\{\vartheta_n\}$ is a Cauchy sequence (CS) whenever and for every $\alpha \in E$ with $0 \ll \alpha$, a natural number exists N such that $d(\vartheta_n, \vartheta_m) \ll \alpha$ for all $n, m \geq N$.

Definition 1.4: Let (X, d) be a metric space. If every CS is convergent in X, then (X, d) be a CCMS. The family of nonempty closed bounded subsets of X is denoted as $CB(X)$. The Hausdorff distance on $CB(X)$ is denoted by $H(.,.)$.

$\Rightarrow A, B \in CB(X)$

$$H(A, B) = \max\left\{\sup_{\theta_1 \in A} d(\theta_1, B),\ \sup_{\theta_2 \in B} d(A, \theta_2)\right\}$$

The distance between the point θ_1 and the subset B is given by $(\theta_1, B) = \inf\{d(\theta_1, \theta_2); \theta_2 \in B\}$. If $\vartheta \in T(X)$, then an element $\vartheta \in X$ is considered a FP in the MV mapping $T : X \to 2^X$.

Definition 1.5: Iff the sets $\{\vartheta_{(t-1)}, \vartheta_t\}_{t=1}^{n}$ are pairwise disjoint and $[\theta_1, \theta_2] = \{\cup_{t=1}^{n} [\vartheta_{t-1}, \vartheta_t) \cup \{\theta_2\}\}$, then the set $\{\theta_1 = \vartheta_0, \vartheta_1, \vartheta_2, \ldots . \vartheta_n = \theta_2\}$ is referred to as a partition for $[\theta_1, \theta_2]$.

Definition 1.6: Cone lower summation and cone upper summation are defined as follows for each partition Q of $[\theta_1, \theta_2]$ and each rising function $\zeta : [\theta_1, \theta_2] \to P$

$$L_n^{con}(\zeta,Q)=\sum_{t=0}^{n-1}\zeta(\vartheta_t)\|\vartheta_t-\vartheta_{t+1}\|$$

$$U_n^{con}(\zeta,Q)=\sum_{t=0}^{n-1}\zeta(\vartheta_{t+1})\|\vartheta_t-\vartheta_{t+1}\|$$

Correspondingly.

Definition 1.7: Assume that P in E is a normal cone (NC). $\zeta:[\theta_1,\theta_2]\to P$ is called with respect to cone P or simplicity on an integrable function on $[\theta_1, \theta_2]$. Cone integrable function iff $\lim_{n\to\infty}L_n^{con}(\zeta,Q)=S^{con}=\lim_{n\to\infty}U_n^{con}(\zeta,Q)$, where S^{con} necessity be unique, for any partition Q of $[\theta_1, \theta_2]$. By using $\int_{\theta_2}^{\theta_1}\zeta(x)d_p(x)$, we demonstrate the common value S^{con}, simplifying it to $\int_{\theta_2}^{\theta_1}\zeta\, d_p$.

Definition 1.8: If and only if, for any $\theta_1,\theta_2\in P$, the function $\zeta:P\to E$ is referred to be a sub additive cone integrable function.

$$\int_0^{\theta_1+\theta_2}\zeta\, d_p\le\int_0^{\theta_1}\zeta\, d_p+\int_0^{\theta_2}\zeta\, d_p$$

Example 1.9: Let $E=X=R, d(\vartheta,\omega)=|\vartheta-\omega|, P=(0,\infty)$ and $\zeta(t)=\frac{1}{t+1}$ for all $t>0$. Then for all $\theta_1,\theta_2\in P$

$$\int_0^{\theta_1+\theta_2}\frac{dt}{(t+1)}=In(\theta_1+\theta_2+1),\int_0^{\theta_2}\frac{dt}{(t+1)}=In(a+1),\int_0^{\theta_2}\frac{dt}{(t+1)}=In(\theta_2+1)$$

Since $\theta_1 b\ge 0$, then $\theta_1+\theta_2+1\le\theta_1+\theta_2+1+\theta_1\theta_2=(\theta_1+1)(\theta_2+1)$. Therefore

$$In(\theta_1+\theta_2+1)\le In(\theta_1+1)\le In(\theta_2+1)$$

This demonstrates that ζ is an illustration of a subadditive cone inegrable function.

2. Main Results

Theorem 2.1: Consider a CCMS (complete cone metric space) (X, d) and a MV (multivalued) map $T: X\to CB(X)$ that is satisfied for every $\vartheta, \omega\in X$. For any $\epsilon>0,\int_0^{\epsilon}\zeta(t)dt\gg 0$ is the definition of the function $\zeta:P\cup\{0\}\to P\cup\{0\}$.

$$\int_0^{H(T\vartheta,T\omega)}\zeta(t)dt\le\theta_1\int_0^{[d(\vartheta,T\vartheta)+d(\omega,T\omega)]}\zeta(t)dt+\theta_2\int_0^{([d(\vartheta,T\omega)+d(T\vartheta,\omega)])}\zeta(t)dt$$

for all $\vartheta, \omega\in X$ and $\theta_1+\theta_2<\frac{1}{2}.\theta_1,\theta_2\in[0,\frac{1}{2})$. Afterwards T has a unique FP in X

Proof: for all $\vartheta_0\in X, n\ge 1, \vartheta_1\in T\vartheta_0$ & $\vartheta_{n+1}\in T\vartheta_n$

$$\int_0^{d(\vartheta_{n+1},\vartheta_n)}\zeta(t)dt\le\int_0^{H(T\vartheta_n,T\vartheta_{n-1})}\zeta(t)dt\le\theta_1\int_0^{[d(\vartheta_n,T\vartheta_n)+d(\vartheta_{n-1},T\vartheta_{n-1})]}\zeta(t)dt+\theta_2\int_0^{[d(\vartheta_n,T\vartheta_{n-1})+d(T\vartheta_n,\vartheta_{n-1})]}\zeta(t)dt$$

$$\le\theta_1\int_0^{[d(\vartheta_n,\vartheta_{n+1})+d(\vartheta_{n-1},\vartheta_n)]}\zeta(t)dt+\theta_2\int_0^{[d(\vartheta_n,\vartheta_n)+d(\vartheta_{n+1},\vartheta_{n-1})]}\zeta(t)dt\le\theta_1\int_0^{[d(\vartheta_n,\vartheta_{n+1})+d(\vartheta_{n-1},\vartheta_n)]}\zeta(t)dt$$

$$+\theta_2\int_0^{[d(\vartheta_{n+1},\vartheta_{n-1})]}\zeta(t)dt\le\theta_1\int_0^{[d(\vartheta_n,\vartheta_{n+1})+d(\vartheta_{n-1},\vartheta_n)]}\zeta(t)dt+\theta_2\int_0^{[d(\vartheta_{n+1},\vartheta_n)+d(\vartheta_n,\vartheta_{n-1})]}\zeta(t)dt$$

$$\le(\theta_1+\theta_2)\int_0^{[d(\vartheta_n,\vartheta_{n+1})+d(\vartheta_{n-1},\vartheta_n)]}\zeta(t)dt\int_0^{d(\vartheta_{n+1},\vartheta_n)}\zeta(t)dt\le\delta\int_0^{d(\vartheta_{n-1},\vartheta_n)}\zeta(t)dt$$

where $\delta=\dfrac{\theta_1+\theta_2}{(1-(\theta_1+\theta_2)}\int_0^{d(\vartheta_{n+1},\vartheta_n)}\zeta(t)dt\le\delta^n\int_0^{d(\vartheta_1,\vartheta_0)}\zeta(t)dt$

For $n > m$ we have

$$\int_0^{d(\vartheta_n,\vartheta_m)}\zeta(t)dt \le \int_0^{d(\vartheta_n,\vartheta_{n-1})+d(\vartheta_{n-1},\vartheta_{n-2})+\cdots+d(\vartheta_{m+1},\vartheta_m)}\zeta(t)dt \le \int_0^{d(\vartheta_n,\vartheta_{n-1})}\zeta(t)dt$$
$$+\int_0^{d(\vartheta_{n-1},\vartheta_{n-2})}\zeta(t)dt+\cdots+\int_0^{d(\vartheta_{m+1},\vartheta_m)}\zeta(t)dt$$
$$\le [\delta^{n-1}+\delta^{n-2}+\cdots+\delta^m]\int_0^{d(\vartheta_1,\vartheta_0)}\zeta(t)dt \le \frac{\delta^m}{(1-\delta)}\int_0^{d(\vartheta_1,\vartheta_0)}\zeta(t)dt$$

Let 0 is an interior of α be given, there exist $N_1 \in \mathbb{N}$ such that $\frac{\delta^m}{(1-\delta)}\int_0^{d(\vartheta_1,\vartheta_0)}\zeta(t)dt \ll \alpha$ for all $m \ge N_1$ this gives $\int_0^{(d(\vartheta_n,\vartheta_m)}\zeta(t)dt \ll \alpha$. For $n \ge m$, $\{\vartheta_n\}$ is a CS in (X, d) is a CCMS, there exists $\Delta_1 \in X$ such that $\vartheta_n \to \Delta_1$.

There exist $N_2 \in \mathbb{N}$ such that $\int_0^{d(\vartheta_n,\Delta_1)}\zeta(t)dt \ll \frac{\alpha(1-\delta)}{3}$, for all $n \ge N_2$. Hence for $n \ge N_2$ we have $\int_0^{d(\vartheta_n,\Delta_1)}\zeta(t)dt \ll \frac{\alpha(1-K)}{3}$ where $k = \theta_1 + \theta_2$

$$\int_0^{d(T\Delta_1,\Delta_1)}\zeta(t)dt \le \int_0^{H(T\vartheta_n,T\Delta_1)}\zeta(t)dt + \int_0^{d(T\vartheta_n,\Delta_1)}\zeta(t)dt \le \theta_1\int_0^{[d(\vartheta_n,T\vartheta_n)+d(\Delta_1,T\Delta_1)]}\zeta(t)dt$$
$$+\theta_2\int_0^{[d(\vartheta_n,T\Delta_1)+d(T\vartheta_n,\Delta_1)]}\zeta(t)dt + \int_0^{d(\vartheta_{n+1},\Delta_1)}\zeta(t)dt \le \theta_1\int_0^{[d(\vartheta_n,\vartheta_{n+1})+d(\Delta_1,T\Delta_1)]}\zeta(t)dt$$
$$+\theta_2\int_0^{[d(\vartheta_n,T\Delta_1)+d(\vartheta_{n+1},\Delta_1)]}\zeta(t)dt + \int_0^{(d(\vartheta_{n+1},\Delta_1)}\zeta(t)dt \le \theta_1\int_0^{[d(\vartheta_n,\vartheta_{n+1})+d(\Delta_1,T\Delta_1)]}\zeta(t)dt$$
$$+\theta_2\int_0^{[d(\vartheta_n,T\Delta_1)+d(\Delta_1,T\Delta_1)+d(\vartheta_{n+1},\Delta_1)]}\zeta(t)dt + \int_0^{d(\vartheta_{n+1},\Delta_1)}\zeta(t)dt \le \theta_1\int_0^{d(\vartheta_n,\vartheta n+1)}\zeta(t)dt$$
$$+\theta_1\int_0^{d(\Delta_1,T\Delta_1)}\zeta(t)dt + \theta_2\int_0^{d(\vartheta_n,T\Delta_1)}\zeta(t)dt + \theta_2\int_0^{d(\Delta_1,T\Delta_1)}\zeta(t)dt + \theta_2\int_0^{d(\vartheta_{n+1},\Delta_1)}\zeta(t)dt$$
$$+\int_0^{d(\vartheta_{n+1},\Delta_1)}\zeta(t)dt(1-k)\int_0^{d(T\Delta_1,\Delta_1)}\zeta(t)dt \le k\int_0^{d(\vartheta_n,T\Delta_1)}\zeta(t)dt + k\int_0^{d(\vartheta_{n+1},\Delta_1)}\zeta(t)dt$$
$$+\int_0^{d(\vartheta_{n+1},\Delta_1)}\zeta(t)dt \le \int_0^{d(\vartheta_n,T\Delta_1)}\zeta(t)dt + \int_0^{d(\vartheta_{n+1},\Delta_1)}\zeta(t)dt + \int_0^{d(\vartheta_{n+1},\Delta_1)}\zeta(t)dt\int_0^{d(T\Delta_1,\Delta_1)}\zeta(t)dt$$
$$\le \frac{\left[\int_0^{d(\vartheta_n,T\Delta_1)}\zeta(t)dt + \int_0^{d(\vartheta_{n+1},\Delta_1)}\zeta(t)dt + \int_0^{d(\vartheta_{n+1},\Delta_1)}\zeta(t)dt\right]}{(1-k)} d(T\Delta_1,\Delta_1) \ll \frac{\alpha}{3}+\frac{\alpha}{3}+\frac{\alpha}{3} d(T\Delta_1,\Delta_1) \ll c$$

For all $n \ge N_2, \int_0^{d(T\Delta_1,\Delta_1)}\zeta(t)dt \ll \frac{\alpha}{\Upsilon}$ for all $\Upsilon \ge 1$, we get $\frac{\alpha}{\Upsilon} - \int_0^{d(T\Delta_1,\Delta_1)}\zeta(t)dt \in P$ and $\Upsilon \to \infty$ we get $\frac{\alpha}{\Upsilon} \to 0$ and P is closed $\int_0^{-d(T\Delta_1,\Delta_1)}\zeta(t)dt \in P$ but $\int_0^{d(T\Delta_1,\Delta_1)}\zeta(t)dt \in P$

$\therefore \int_0^{d(T\Delta_1,\Delta_1)}\zeta(t)dt = 0$ and so $\Delta_1 \in T\Delta_1$.

Corollary 2.2: Let (X, d) be a CCMS and the mapping $T : X \to CB(X)$ be MV map sufficient for each $\vartheta, y \in X$. The function $\zeta : P \cup \{0\} \to P \cup \{0\}$ is defined as for each $\epsilon > 0, \int_0^{\epsilon}\zeta(t)dt \gg 0$ $\int_0^{d(T\vartheta,T\omega)}\zeta(t)dt \le \theta_1\int_0^{[d(\vartheta,T\vartheta)+d(\omega,T\omega)]}\zeta(t)dt$ for all $\vartheta, \omega \in X$ and $\theta_1 \in [0, \frac{1}{2})$. Then T has a unique FP in X

Proof: The corollary's proof can be obtained by simply setting $\theta_2 = 0$ in the preceding theorem.

Theorem 2.3: Let (X, d) be a CCMS The function $\zeta : P \cup \{0\} \to P \cup \{0\}$ is defined as for each $\epsilon > 0, \int_0^{\epsilon}\zeta(t)dt \gg 0$ and the mapping $T : X \to CB(X)$ be MV map satisfy the condition

$$\int_0^{H(T\vartheta,T\omega)}\zeta(t)dt \le r\int_0^{\max\{d(\vartheta,\omega),d(\vartheta,T\vartheta),d(\omega,T\omega)\}}\zeta(t)dt$$

For all ϑ, ω ∈ X and $r \in [0,1)$. Afterwards T has a unique FP in X

Proof: for all $\vartheta_0 \in X, n \geq 1, \vartheta_1 \in T\vartheta_0$ and $\vartheta_{n+1} \in T\vartheta_n$

$$\int_0^{d(\vartheta_{n+1},\vartheta_n)} \zeta(t)dt \leq \int_0^{H(T\vartheta_n,T\vartheta_{n-1})} \zeta(t)dt \leq r\int_0^{\max\{d(\vartheta_n,\vartheta_{n-1}),d(\vartheta_n,T\vartheta_n),d(\vartheta_{n-1},T\vartheta_{n-1})\}} \zeta(t)dt$$
$$\leq r\int_0^{\max\{d(\vartheta_n,\vartheta_{n-1}),d(\vartheta_n,\vartheta_{n+1}),d(\vartheta_{n-1},\vartheta_n)\}} \zeta(t)dt \leq r\int_0^{d(\vartheta_n,\vartheta_{n-1})} \zeta(t)dt \leq r^n \int_0^{d(\vartheta_1,\vartheta_0)} \zeta(t)dt$$

For $n > m$ we have

$$\int_0^{d(\vartheta_n,\vartheta_m)} \zeta(t)dt \leq \int_0^{d(\vartheta_n,\vartheta_{n-1})+d(\vartheta_{n-1},\vartheta_{n-2})+\cdots+d(\vartheta_{m+1},\vartheta_m)} \zeta(t)dt \leq \int_0^{(\vartheta_n,\vartheta_{n-1})} \zeta(t)dtd + \int_0^{d(\vartheta_{n-1},\vartheta_{n-2})} \zeta(t)dt + \cdots + \int_0^{d(\vartheta_{m+1},\vartheta_m)} \zeta(t)dt$$
$$\leq [r^{n-1} + r^{n-2} + \cdots + r^m]\int_0^{d(\vartheta_1,\vartheta_0)} \zeta(t)dt \leq \tfrac{r^m}{(1-r)} \int_0^{d(\vartheta_1,\vartheta_0)} \zeta(t)dt$$

Let 0 is an interior of α be given, there exist $N_1 \in \mathbb{N}$ such that $\tfrac{r^m}{(1-r)}\int_0^{d(\vartheta_1,\vartheta_0)} \zeta(t)dt \ll c$ for all $m \geq N_1$ this implies $\int_0^{d(\vartheta_n,\vartheta_m)} \zeta(t)dt \ll c$.

For $n > m$, $\{\vartheta_n\}$ is a CS in (X, d) is a CCMS, there exists $\Delta_1 \in X$ such that $\vartheta_n \to \Delta_1$. There exist N_2 and $\int_0^{d(\vartheta_n,\Delta_1)} \zeta(t)dt \ll \frac{\alpha}{3}$, for all $n \geq N_2$. Hence for $n \geq N_2$ we have $\int_0^{d(\vartheta_n,\Delta_1)} \zeta(t)dt \ll \frac{\alpha}{3}$

$$\int_0^{d(T\Delta_1,\Delta_1)} \zeta(t)dt \leq \int_0^{H(T\vartheta_n,T\Delta_1)} \zeta(t)dt + \int_0^{d(T\vartheta_n,\Delta_1)} \zeta(t)dt \leq r\int_0^{\max\{d(\vartheta_n,\Delta_1),d(\vartheta_n,T\vartheta_n),d(\Delta_1,T\Delta_1)\}} \zeta(t)dt$$
$$+ \int_0^{(d(\vartheta_{n+1},\Delta_1)} \zeta(t)dt \leq r\int_0^{\max\{d(\vartheta_n,\Delta_1),d(\vartheta_n,\vartheta_{n+1}),d(\Delta_1,T\Delta_1)\}} \zeta(t)dt + \int_0^{d(\vartheta_{n+1},\Delta_1)} \zeta(t)dt$$
$$\leq r\int_0^{\max\{d(\vartheta_n,\Delta_1),d(\vartheta_n,\Delta_1)+d(\Delta_1,\vartheta_{n+1}),d(\Delta_1,T\Delta_1)\}} \zeta(t)dt + \int_0^{d(\vartheta_{n+1},\Delta_1)} \zeta(t)dt\int_0^{d(T\Delta_1,\Delta_1)} \zeta(t)dt \ll \alpha$$

For all $n \geq N_2$, $\int_0^{d(T\Delta_1,\Delta_1)} \zeta(t)dt \ll \frac{\alpha}{m}$ for all $m \geq 1$, we get $\frac{\alpha}{m} - \int_0^{d(T\Delta_1,\Delta_1)} \zeta(t)dt \in P$ and $m \to \infty$ we get $\frac{\alpha}{m} \to 0$ and P is closed $\int_0^{-d(T\Delta_1,\Delta_1)} \zeta(t)dt \in P\, bu\int_0^{d(T\Delta_1,\Delta_1)} \zeta(t)dt \in P$

$\therefore \int_0^{d(T\Delta_1,\Delta_1)} \zeta(t)dt = 0$ and so $\Delta_1 \in T\Delta_1$.

Corollary 2.4: Let (X, d) be a CCMS, The function $\zeta : P \cup \{0\} \to P \cup \{0\}$ is defined as for each $\epsilon > 0, \int_0^{\epsilon} \zeta(t)dt \gg 0$ and the mapping $T : X \to CB(X)$ be MV map satisfy the condition

$$\int_0^{H(T\vartheta,T\omega)} \zeta(t)dt \leq k\int_0^{d(\vartheta,\omega)} \zeta(t)dt$$

For all ϑ, ω ∈ X and $k \in [0,1)$. Then T has a fixed point in X

Proof: The corollary's proof can be obtained by taking the maximum value of $d(\vartheta, \omega)$ from the prior theorem.

Theorem 2.5: Let (X, d) be a CCMS and P a NC with normal constant κ. The function $\zeta : P \cup \{0\} \to P \cup \{0\}$ is defined as for each $\epsilon > 0, \int_0^{\epsilon} \zeta(t)dt \gg 0$

Suppose the mapping $T : X \to CB(X)$ be MV map satisfy the condition

$$\int_0^{H(T\vartheta,T\omega)} \zeta(t)dt \leq r\int_0^{\max\{d(\vartheta,\omega),d(\vartheta,T\vartheta),d(\omega,T\omega),d(\vartheta,T\omega),d(T\vartheta,\omega)\}} \zeta(t)dt$$

For all ϑ, ω ∈ X and $r \in [0,1)$. Then T has a FP in X

Proof: for all $\vartheta_0 \in X, n \geq 1, \vartheta_1 \in T\vartheta_0, \vartheta_{n+1} \in T\vartheta_n$

$$\int_0^{d(\vartheta_{n+1},\vartheta_n)}\zeta(t)dt \le \int_0^{H(T\vartheta_n,T\vartheta_{n-1})}\zeta(t)dt \le r\int_0^{\max\{d(\vartheta_n,\vartheta_{n-1}),d(\vartheta_n,T\vartheta_n),d(\vartheta_{n-1},T\vartheta_{n-1}),d(\vartheta_n,T\vartheta_{n-1}),d(T\vartheta_n,\vartheta_{n-1})\}}\zeta(t)dt$$
$$\le r\int_0^{\max\{d(\vartheta_n,\vartheta_{n-1}),d(\vartheta_n,\vartheta_{n+1}),d(\vartheta_{n-1},\vartheta_n),d(\vartheta_n,\vartheta_n),d(\vartheta_{n+1},\vartheta_{n-1})\}}\zeta(t)dt$$
$$\le r\int_0^{\max\{d(\vartheta_n,\vartheta_{n-1}),d(\vartheta_n,\vartheta_{n+1}),d(\vartheta_{n+1},\vartheta_{n-1})\}}\zeta(t)dt \le r\int_0^{\max\{d(\vartheta_n,\vartheta_{n-1}),d(\vartheta_{n+1},\vartheta_{n-1})\}}\zeta(t)dt$$

Case (i) If $\int_0^{d(\vartheta_{n+1},\vartheta_n)}\zeta(t)dt \le r\int_0^{d(\vartheta_n,\vartheta_{n-1})}\zeta(t)dt$ then we get,

$$\int_0^{d(\vartheta_{n+1},\vartheta_n)}\zeta(t)dt \le r^n\int_0^{d(\vartheta_1,\vartheta_0)}\zeta(t)dt$$

for $n > m$

$$\int_0^{d(\vartheta_n,\vartheta_m)}\zeta(t)dt \le \int_0^{d(\vartheta_n,\vartheta_{n-1})+d(\vartheta_{n-1},\vartheta_{n-2})+\cdots+d(\vartheta_{m+1},\vartheta_m)}\zeta(t)dt \le \int_0^{d(\vartheta_n,\vartheta_{n-1})}\zeta(t)dt+\int_0^{d(\vartheta_{n-1},\vartheta_{n-2})}\zeta(t)dt+\cdots+\int_0^{d(\vartheta_{m+1},\vartheta_m)}\zeta(t)dt$$
$$\le [r^{n-1}+r^{n-2}+\cdots+r^m]\int_0^{d(\vartheta_1,\vartheta_0)}\zeta(t)dt \le \frac{r^m}{(1-r)}\int_0^{d(\vartheta_1,\vartheta_0)}\zeta(t)dt$$

We get $\|d(\vartheta_n,\vartheta_m)\| \le K\frac{r^m}{(1-r)}\|d(\vartheta_1,\vartheta_0)\|$. $d(\vartheta_n,\vartheta_m)\to 0$ as n, $m\to\infty$.

Hence $\{\vartheta_n\}$ is a CS. Given X's completeness, $\Delta_1 \in X$ exists. such that $\vartheta_n \to \Delta_1$ as $n\to\infty$

$$\int_0^{d(T\Delta_1,\Delta_1)}\zeta(t)dt \le \int_0^{H(T\vartheta_n,T\Delta_1)}\zeta(t)dt+\int_0^{d(T\vartheta_n,\Delta_1)}\zeta(t)dt \le r\int_0^{\max\{d(\vartheta_n,\Delta_1),d(\vartheta_n,T\vartheta_n),d(\Delta_1,T\Delta_1),d(\vartheta_n,T\Delta_1),d(T\vartheta_n,\Delta_1)\}+d(\vartheta_{n+1},\Delta_1)}\zeta(t)dt$$
$$\le r\int_0^{\max\{d(\vartheta_n,\Delta_1),d(\vartheta_n,\vartheta_{n+1}),d(\Delta_1,T\Delta_1),d(\vartheta_n,T\Delta_1),d(\vartheta_{n+1},\Delta_1)\}+d(\vartheta_{n+1},\Delta_1)}\zeta(t)dt \le r\int_0^{d(\Delta_1,T\Delta_1)}\zeta(t)dt\int_0^{d(T\Delta_1,\Delta_1)}\zeta(t)dt = 0.$$

Hence $\Delta_1 \in T\Delta_1$

Case (ii) $\int_0^{d(\vartheta_{n+1},\vartheta_n)}\zeta(t)dt \le r\int_0^{d(\vartheta_{n+1},\vartheta_{n-1})}\zeta(t)dt$ then we get

$$\int_0^{d(\vartheta_{n+1},\vartheta_n)}\zeta(t)dt \le r\int_0^{[d(\vartheta_{n+1},\vartheta_n)+d(\vartheta_n,\vartheta_{n-1})]}\zeta(t)dt \le \frac{r}{1-r}\int_0^{[d(\vartheta_n,\vartheta_{n-1})]}\zeta(t)dt \le \delta_1\int_0^{[d(\vartheta_n,\vartheta_{n-1})]}\zeta(t)dt$$

where $\delta_1 = \frac{r}{1-r} < 1$, For $n > m$

$$\int_0^{d(\vartheta_n,\vartheta_m)}\zeta(t)dt \le \int_0^{(d(\vartheta_n,\vartheta_{n-1})+d(\vartheta_{n-1},\vartheta_{n-2})+\cdots+d(\vartheta_{m+1},\vartheta_m)}\zeta(t)dt \le \int_0^{d(\vartheta_n,\vartheta_{n-1})}\zeta(t)dt+\int_0^{d(\vartheta_{n-1},\vartheta_{n-2})}\zeta(t)dt+\cdots+\int_0^{d(\vartheta_{m+1},\vartheta_m)}\zeta(t)dt$$
$$\le [\delta_1{}^{n-1}+\delta_1{}^{n-2}+\cdots+\delta_1{}^m]\int_0^{d(\vartheta_1,\vartheta_0)}\zeta(t)dt \le \frac{\delta_1{}^m}{(1-\delta_1)}\int_0^{d(\vartheta_1,\vartheta_0)}\zeta(t)dt$$

We get $\|d(\vartheta_n,\vartheta_m)\| \le \kappa\frac{h^m}{(1-h)}\|d(\vartheta_1,\vartheta_0)$. $d(\vartheta_n,\vartheta_m)\to 0$ as n, $m\to\infty$. Hence $\{\vartheta_n\}$ is a CS. Given X's completeness, $\Delta_1 \in X$ exists. such that $\vartheta_n \to \Delta_1$ as $n\to\infty$

$$\int_0^{d(T\Delta_1,\Delta_1)}\zeta(t)dt \le \int_0^{H(T\vartheta_n,T\Delta_1)}\zeta(t)dt+\int_0^{d(T\vartheta_n,\Delta_1)}\zeta(t)dt$$
$$\le r\int_0^{\max\{d(\vartheta_n,\Delta_1),d(\vartheta_n,T\vartheta_n),d(\Delta_1,T\Delta_1),d(\vartheta_n,T\Delta_1),d(T\vartheta_n,\Delta_1)\}}\zeta(t)dt+\int_0^{d(\vartheta_{n+1},\Delta_1)}\zeta(t)dt$$
$$\le r\int_0^{\max\{d(\vartheta_n,\Delta_1),d(\vartheta_n,\vartheta_{n+1}),d(\Delta_1,T\Delta_1),d(\vartheta_n,T\Delta_1),d(\vartheta_{n+1},\Delta_1)\}}\zeta(t)dt+\int_0^{d(\vartheta_{n+1},\Delta_1)}\zeta(t)dt$$
$$\le r\int_0^{d(\Delta_1,T\Delta_1)}\zeta(t)dt\int_0^{d(T\Delta_1,\Delta_1)}\zeta(t)dt = 0.$$

Hence $\Delta_1 \in T\Delta_1$

$$\int_0^{d(\Delta_1,\Delta_2)}\zeta(t)dt=\int_0^{H(T\Delta_1,T\Delta_2)}\zeta(t)dt\le r\int_0^{\max[d(\vartheta,\omega),d(\Delta_1,T\Delta_1),d(\Delta_2,T\Delta_2),d(\Delta_1,T\Delta_2),d(T\Delta_1,\Delta_2)]}\zeta(t)dt$$
$$\le r\int_0^{\max[d(\Delta_1,\Delta_2),d(\Delta_1,\Delta_1),d(\Delta_2,\Delta_2),d(\Delta_1,\Delta_2),d(\Delta_1,\Delta_2)]}\zeta(t)dt\le r\int_0^{[d(\Delta_1,\Delta_2)]}\zeta(t)dt$$

Because of the contradiction, T has a unique FP in X

Corollary 2.6: Let (X, d) be a CCMS and P a NC with normal constant κ. The function $\zeta : P\cup\{0\}\to P\cup\{0\}$ is defined as for each $\epsilon>0,\int_0^{\epsilon}\zeta(t)dt\gg 0$

Suppose the mapping $T : X\to CB(X)$ be MV map satisfy the condition

$$\int_0^{H(T\vartheta,T\omega)}\zeta(t)dt\le r\int_0^{\max\{d(\vartheta,\omega),d(\vartheta,T\vartheta),d(\omega,T\omega)\}}\zeta(t)dt$$

For all ϑ, $\omega\in X$ and $r\in[0,1)$. Then T has a FP in X

Proof: The corollary's proof is as follows right away, since

$$\int_0^{\max\{d(\vartheta,\omega),d(\vartheta,T\vartheta),d(\omega,T\omega)\}}\zeta(t)dt\le\int_0^{\max\{d(\vartheta,\omega),d(\vartheta,T\vartheta),d(\omega,T\omega),d(\vartheta,T\omega),d(T\vartheta,\omega)\}}\zeta(t)dt$$

3. Applications

Mathematical Biology: In biological systems, like population models or ecosystems, interactions between different species can be intricate and multifaceted. The study of species stability and persistence in dynamic ecosystems, where growth rates and interactions are controlled by various factors, is aided by fixed point results in CMS. Integral type contractions are useful in modeling how populations stable in the face of varying environmental conditions throughout time.

Optimization Problems: Many real-world optimization issues involve maximizing or decreasing a function, and they frequently have several workable solutions. Such systems, in which a single input (parameter set) corresponds to numerous outputs (best solutions), can be modelled using multivalued mappings. Cone metric spaces make it possible to extend vector space optimization, and integral type contractions aid in the establishment of convergence criteria, guaranteeing that algorithms for optimization provide solutions that meet practical limitations.

References

[1] H. Mohebi, "Topical functions and their properties in a class of ordered Banach spaces,in continuous Optimization", *Current Trends and Modern Applications, Part II, Springer*, pp.343–361, 2005.

[2] Seong Hoon Cho, Mi Sun Kim, "Fixed point theorems for general contractive multivalued mappings," *J.Appl.Math. Informatics* Vol.27, 343–350, 2009.

[3] Arslan Hojat Ansari, R. Krishnakumar, K. Dinesh, and D. Dhamodharan, "Fixed Point of Cyclic Contraction Mappings in Banach Spaces Via C-Class Function" *Int. J. Math. And Appl.*, Vol.6(1-E), pp.1085–1092, 2018.

[4] Huang Gaung, Zhang Xian, "Cone metric spaces and fixed point theorems of contractive mappings", *J.math.Anal.Appl.* Vol.332, pp.1468–1476, 2007.

[5] K. Dinesh, R. Krishnakumar, Nagaral Pandit Sanatammappa, "Some results in b-metric space", *Malaya Journal of Matematik*, Vol.9(1), pp.539–541, 2021.

[6] H. Mohebi, H. Sadeghi, A.M. Rubinov, "Best approximation in a class of normed spaces with star-shaped cone", *Numer. Funct.Anal.Optim.*Vol.27(3–4), pp.411–436, 2006.

[7] Sh. Rezapour, R. Hamlbarani, "Some notes on the Paper-Cone metric Spaces and fixed point theorems of contractive mappings," *J.Math.Anal.Appl.* Vol.345, pp.719–724, 2008.

[8] R. Krishnakumar, K. Dinesh, Arslan Hojat Ansari, "Fixed point Theorems of Multivalued Mappings in Cone Metric Spaces via Cone C-Class function" *Int. J. Sci. Res. in Mathematical and Statistical Sciences*, Vol. 5(4), Aug 2018.

[9] Siddamsetti, Swapna, Almngoshi, Hussein Z., K. Dinesh, Prashant, G. C., Bennet, M. Anto, Dadheech, Pankaj & Sengan, Sudhakar (2024) "Modular metric spaces : Some fixed-point theorems and application of secure dynamic routing for WSN", *Journal of Interdisciplinary Mathematics*, **27**:2, 393–401, https://doi.org/10.47974/JIM-1890.

[10] K. Dinesh, Kastriot Zoto, Adriana Topi, B Shoba, Doaa Rizk, "Some Fixed Point Theorems Using Φp Operator", *Communications on Applied Nonlinear Analysis*, Vol 32 No. 6s, 2025.

[11] Huang H, Zoto K, Mitrović ZD, Radenović S. "Fixed Point Results for Generalized F-Contractions in b-Metric-like Spaces" *Fractal and Fractional*. 2022; **6(5)**:272. https://doi.org/10.3390/fractalfract6050272.