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Necessary and sufficient conditions for functions defined by binomial distribution to be in a general class of analytic functions

Tariq Al-Hawary, Basem Aref Frasin, Jamal Salah

[1]Department of Applied Science, Ajloun College, Al Balqa Applied University, Ajloun 26816. Jordan; [2]Faculty of Science, Department of Mathematics, Al al-Bayt University, P.O. Box 130095 Mafraq, Jordan; [3]College of Applied and Health Sciences, A'Sharqiyah University, P.O. Box No. 42, Post Code No. 400 Ibra, Sultanate of Oman.

Abstract

In this paper, a comprehensive class $\Theta_\Psi(E_1, E_2, E_3)$ of univalent functions is examined in light of the recent findings on binomial distribution. Additionally, we provide some requirements for the functions $B(\lambda, \gamma, \varsigma)$, $B(\lambda, \varpi, \varsigma)b$ and the operator $L(\lambda, \gamma, \varsigma)$ defined by the binomial distribution to be in these subclasses. This study will motivate the authors to discover new sufficient requirements, not only for binomial distributions but also for numerous other special functions. Additionally, we summarize many previous studies.

Key words and phrases: Analytic, univalent, binomial distribution, Poisson distribution, Bernoulli distribution.

Mathematics Subject Classification (2010): 30C45

1. Introduction and Preliminaries

The binomial distribution is one of the most significant distributions in probability and statistics distribution that can be used as a model for a number of real-world issues. It has applications in the domains of biology, health, social sciences, quality control, finance, and the results of surveys

Email addresses: tariq_amh@bau.edu.jo (Tariq Al-Hawary); bafrasin@aabu.edu.jo (Basem Aref Frasin); damous73@yahoo.com (Jamal Salah)

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or experiments with binary responses. The first people to deduce special examples of it were Pascal (1679) and Bernoulli (1713).

The number of successes in a τ-sized sample that is drawn with replacement from a λ-sized population is commonly modeled using the binomial distribution. The distribution that results is hypergeometric rather than binomial if the sampling is done without replacement because the draws are not independent. But when λ is significantly greater than τ, the binomial distribution is still a good estimate and is frequently utilized.

The binomial distribution $g(\lambda, \varpi)$ for any random variable is given by

$$g(\lambda,\varpi)=\Pr(Y=\tau)=\binom{\lambda}{\tau}\varpi^{\tau}(1-\varpi)^{\lambda-\tau}\equiv\frac{\lambda!}{(\lambda-\tau)!\,\tau!}\varpi^{\tau}(1-\varpi)^{\lambda-\tau},\tau=0,1,2,\cdots,\lambda,$$

when $\lambda>\tau$, then $g(\lambda, \varpi) = 0$.

When ϖ is small and λ is big ($\lambda\varpi$ is moderate), the Poisson distribution and the binomial distribution are connected. Similarly, when $\lambda = 1$, the Bernoulli distribution and the binomial distribution are related.

Let the class of analytic and univalent functions Λ defined in the open disk $\Xi = \{\varsigma \in \mathbb{C}: |\varsigma| < 1\}$ and given by

$$b(\varsigma)=\varsigma+\sum_{\tau=2}^{\infty}\alpha_{\tau}\varsigma^{\tau}. \tag{1.1}$$

Also, denote by Λ^- the subclass of Λ of functions of the form

$$b(\varsigma)=\varsigma-\sum_{\tau=2}^{\infty}\alpha_{\tau}\varsigma^{\tau},\ \ \alpha_{\tau}\ge 0,\ \ \varsigma\in\Xi. \tag{1.2}$$

Recently, Nazeer et. al [16] introduced the following power series whose coefficients are probabilities of the binomial distribution:

$$\Pi(\lambda,\varpi,\varsigma)=\varsigma+\sum_{\tau=2}^{\infty}\frac{(\lambda-1)!}{(\lambda-\tau)!(\tau-1)!}\varpi^{\tau-1}(1-\varpi)^{\lambda-\tau}\varsigma^{\tau},\qquad \varsigma\in\Xi.$$

Consider the series

$$B(\lambda,\varpi,\varsigma)=2\varsigma-\Pi(\lambda,\varpi,\varsigma)=\varsigma-\sum_{\tau=2}^{\infty}\frac{(\lambda-1)!}{(\lambda-\tau)!(\tau-1)!}\varpi^{\tau-1}(1-\varpi)^{\lambda-\tau}\varsigma^{\tau},\qquad \varsigma\in\Xi, \tag{1.3}$$

and making use of the convolution (*), let the linear operator $B(\lambda, \varpi, \varsigma)b : \Lambda^- \to \Lambda^-$ be defined as:

$$B(\lambda,\varpi,\varsigma)b=B(\lambda,\varpi,\varsigma)*b(\varsigma)=\varsigma-\sum_{\tau=2}^{\infty}\frac{(\lambda-1)!}{(\lambda-\tau)!(\tau-1)!}\varpi^{\tau-1}(1-\varpi)^{\lambda-\tau}\alpha_{\tau}\varsigma^{\tau},\qquad \varsigma\in\Xi.$$

The aims of this paper is to introduce the following comprehensive subclass $\Theta_{\Psi}(E_1, E_2, E_3)$ of the class Λ^-, that extend a number of earlier subclasses of analytic function defined in Ξ.

Definition 1.1. ([12]) A function $b \in \Lambda^-$ is belongs to the class $\Theta_{\Psi}(E_1, E_2, E_3)$ if and only if

$$\sum_{\tau=2}^{\infty}\left(E_1\tau^2+E_2\tau+E_3\right)\left|\alpha_{\tau}\right|\le\Psi, \tag{1.4}$$

where $E_1, E_2, E_3 \in \mathbb{R}$ and $\Psi > 0$.

By suitably specializing the real constants E_1, E_2, E_3 and Ψ in Definition 1.1 we get various subclasses of analytic functions with negative coefficients were taken into consideration in a number of works. For examples, we offer the subsequent subclasses:

(1) A function $b \in \mathrm{C}(\mu, \kappa) - \Theta_{\cos\mu-\kappa}(0, 2, -\cos\mu - \kappa)$ if and only if

$$\sum_{\tau=2}^{\infty}\left(2\tau - \cos\mu - \kappa\right)\left|\alpha_\tau\right| \le \cos\mu - \kappa.$$

(2) A function $b \in \mathrm{U}(\mu, \kappa) - \Theta_{\cos\mu-\kappa}(0, 2, -\cos\mu - \kappa, 0)$ if and only if

$$\sum_{\tau=2}^{\infty}\tau\left(2\tau - \cos\mu - \kappa\right)\left|\alpha_\tau\right| \le \cos\mu - \kappa.$$

(3) A function $b \in \wp^{*}_{\zeta_3}(\zeta_1,\zeta_2) \equiv \Theta_{1-\zeta_1}(0,\zeta_2+1,-\zeta_3(\zeta_1+\zeta))$ if and only if

$$\sum_{\tau=2}^{\infty}\left(\tau(\zeta_2+1) - \zeta_3(\zeta_1+\zeta_2)\right)\left|\alpha_\tau\right| \le 1-\zeta_1.$$

(4) A function $b \in \mathcal{G}^{*}_{\zeta_3}(\zeta_1,\zeta_2) \equiv \Theta_{1-\zeta_1}(\zeta_2+1,-\zeta_3(\zeta_1+\zeta_2),0)$ if and only if

$$\sum_{\tau=2}^{\infty}\tau\left(\tau(\zeta_2+1) - \zeta_3(\zeta_1+\zeta_2)\right)\left|\alpha_\tau\right| \le 1-\zeta_1.$$

The classes $\mathrm{C}(\mu, \kappa)$ and $\mathrm{U}(\mu, \kappa)$ were introduced and studied in [17] whereas the classes $\wp^{*}_{\zeta_3}(\zeta_1,\zeta_2)$ and $\mathcal{G}^{*}_{\zeta_3}(\zeta_1,\zeta_2)$ were introduced and studied in [14].

Remark 1.2. The class $\Theta_\Psi(E_1, E_2, E_3)$ leads to various subclasses of analytic functions with negative coefficients presented and examined by a number of authors, such as the subclasses, $\mathrm{UCT}(\alpha, \beta)$ and $\mathcal{P}\mathrm{T}(\alpha, \beta)$ ([4]), $\mathrm{T}(\gamma, ,\delta)$ and $C(\gamma,\delta)$ ([1]), $S^*(\alpha, \beta)$ and $\mathcal{C}^*(\alpha, \beta)$ ([9]), $\mathcal{P}^*_\lambda(\alpha)$ and $\mathcal{Q}^*_\lambda(\alpha)$ ([2]) and other subclasses.

Recently, several researchers determined necessary and sufficient conditions using hypergeometric functions (see for example, [8]), generalized Bessel functions (see for example, [6]), Poisson distribution series (see for example, [3, 7]), Struve functions (see for example, [13]), normalized Wright functions (see for example, [11]) and Pascal distribution series (see for example, [5, 15]). In this paper, we determine conditions for the functions $B(\lambda, \varpi, \varsigma)$ and $B(\lambda, \varpi, \varsigma)b$ to be in the general class $\Theta_\Psi(E_1, E_2, E_3)$. Furthermore, we estimate certain inclusion relations between the class $Y^{p}3\ (p_1,p_2)$ and $\Theta_\Psi(E_1, E_2, E_3)$. Finally, we give conditions for an integral operator $L(\lambda,\varpi,\varsigma) = \int_0^{\varsigma}\frac{B(\lambda,\varpi,s)}{s}ds$ *to be* in the class $\Theta_\Psi(E_1, E_2, E_3)$.

2. Necessary and Sufficient Condition

In this section, we give a necessary an sufficient condition for the function $B(\lambda, \varpi, \varsigma)$ to be in the class $\Theta_\Psi(E_1, E_2, E_3)$.

Theorem 2.1. *The function $B(\lambda, \varpi, \varsigma)$ is belongs to the class $\Theta_\Psi(E_1, E_2, E_3)$ if and only if*

$$E_1\varpi^2(\lambda-1)(\lambda-2) + (3E_1+E_2)\varpi(\lambda-1) + (E_1+E_2+E_3)\digamma_0(\lambda,\varpi) \le \Psi$$

where

$$\mathcal{F}_0(\lambda,\varpi)=\sum_{\tau=1}^{\infty}\frac{(\lambda-1)!}{(\lambda-\tau-1)!\,\tau!}\varpi^{\tau}(1-\varpi)^{\lambda-\tau-1}. \tag{2.1}$$

Proof. From the inequality *(1.4)* and the function $B(\lambda,\varpi,\varsigma)=\varsigma-\sum_{\tau=2}^{\infty}\frac{(\lambda-1)!}{(\lambda-\tau)!(\tau-1)!}\varpi^{\tau-1}(1-\varpi)^{\lambda-\tau}\varsigma^{\tau}$ it suffices to show that

$$\sum_{\tau=2}^{\infty}\left(E_1\tau^2+E_2\tau+E_3\right)\frac{(\lambda-1)!}{(\lambda-\tau)!(\tau-1)!}\varpi^{\tau-1}(1-\varpi)^{\lambda-\tau}\le\Psi.$$

By setting

$$\begin{cases}\tau=(\tau-1)+1;\\ \tau^2=(\tau-1)(\tau-2)+3(\tau-1)+1,\end{cases} \tag{2.2}$$

we get

$$\begin{aligned}
&\sum_{\tau=2}^{\infty}\left(E_1\tau^2+E_2\tau+E_3\right)\frac{(\lambda-1)!}{(\lambda-\tau)!(\tau-1)!}\varpi^{\tau-1}(1-\varpi)^{\lambda-\tau}\\
&=\sum_{\tau=2}^{\infty}\left\{E_1(\tau-1)(\tau-2)+(3E_1+E_2)(\tau-1)+E_1+E_2+E_3\right\}\\
&\times\frac{(\lambda-1)!}{(\lambda-\tau)!(\tau-1)!}\varpi^{\tau-1}(1-\varpi)^{\lambda-\tau}.\\
&=E_1\sum_{\tau=3}^{\infty}\frac{(\lambda-1)!}{(\lambda-\tau)!(\tau-3)!}\varpi^{\tau-1}(1-\varpi)^{\lambda-\tau}\\
&+(3E_1+E_3)\sum_{\tau=2}^{\infty}\frac{(\lambda-1)!}{(\lambda-\tau)!(\tau-2)!}\varpi^{\tau-1}(1-\varpi)^{\lambda-\tau}\\
&+(E_1+E_2+E_3)\sum_{\tau=2}^{\infty}\frac{(\lambda-1)!}{(\lambda-\tau)!(\tau-1)!}\varpi^{\tau-1}(1-\varpi)^{\lambda-\tau}\\
&=E_1\sum_{\tau=0}^{\infty}\frac{(\lambda-1)!}{(\lambda-\tau-3)!\,\tau!}\varpi^{\tau+2}(1-\varpi)^{\lambda-\tau-3}\\
&+(3E_1+E_2)\sum_{\tau=0}^{\infty}\frac{(\lambda-1)!}{(\lambda-\tau-2)!\,\tau!}\varpi^{\tau+1}(1-\varpi)^{\lambda-\tau-2}\\
&+(E_1+E_2+E_3)\sum_{\tau=1}^{\infty}\frac{(\lambda-1)!}{(\lambda-\tau-1)!\,\tau!}\varpi^{\tau}(1-\varpi)^{\lambda-\tau-1}
\end{aligned}$$

$$\begin{aligned}
&=E_1\varpi(\lambda-1)(\lambda-2)\sum_{\tau=0}^{\infty}\frac{(\lambda-3)!}{(\lambda-\tau-3)!\,\tau!}\varpi^{\tau}(1-\varpi)^{\lambda-\tau-3}\\
&+(3E_1+E_2)\varpi(\lambda-1)\sum_{\tau=0}^{\infty}\frac{(\lambda-2)!}{(\lambda-\tau-2)!\,\tau!}\varpi^{\tau}(1-\varpi)^{\lambda-\tau-2}\\
&+(3E_1+E_2+E_3)\sum_{\tau=1}^{\infty}\frac{(\lambda-1)!}{(\lambda-\tau-1)!\,\tau!}\varpi^{\tau}(1-\varpi)^{\lambda-\tau-1}\\
&=E_1\varpi^2(\lambda-1)(\lambda-2)+(3E_1+E_2)\varpi(\lambda-1)+(E_1+E_2+E_3)\mathcal{F}_0(\lambda,\varpi)\le\Psi.
\end{aligned}$$

3. Inclusion Properties

For $\rho_1 \in (0, 1]$, $\rho_2 < 1$ and $\rho_3 \in (\mathrm{C} - \{0\}$, a function $b \in \Lambda$ is belongs to the class $\mathcal{G}^{\rho_3}(\rho_1,\rho_2)$ if it satisfies

$$\left|\frac{(1-\rho_1)\dfrac{b(\varsigma)}{\varsigma}+\rho_1 b'(\varsigma)-1}{2\rho_3(1-\rho_2)+(1-\rho_1)\dfrac{b(\varsigma)}{\varsigma}+\rho_1 b'(\varsigma)-1}\right|<1,\ \ |\varsigma|<1.$$

The class $\mathcal{G}^{\rho_3}(\rho_1,\rho_2)$ was introduced by Swaminathan [18].

Lemma 3.1. [18] *If* $b \in \mathcal{G}^{\rho_3}(\rho_1,\rho_2)$ *of the form (1.1), then*

$$|\alpha_\tau| \le \frac{2|\rho_3|(1-\rho_2)}{\rho_1(\tau-1)+1},\ \tau \in \mathbb{N}-\{1\}.$$

Theorem 3.2. *Let* $b \in \mathcal{G}^{\rho_3}(\rho_1,\rho_2)$*, and the inequality*

$$\varpi E_1(\lambda-1)+(E_1+E_2)\mathcal{F}_0(\lambda,\varpi) \le \frac{\rho_1\Psi}{2|\rho_3|(1-\rho_2)}$$

is satisfied where $\mathcal{F}_0(\lambda, \varpi)$ *is given by (2.1), then* $B(\lambda, \varpi, \varsigma)b \in \Theta_\Psi(E_1, E_2, 0)$.

Proof. Let $b \in \mathcal{G}^{\rho_3}(\rho_1,\rho_2)$, by the inequality (1.4), it suffices to show that

$$\sum_{\tau=2}^{\infty}\left(E_1\tau^2+E_2\tau\right)\frac{(\lambda-1)!}{(\lambda-\tau)!(\tau-1)!}\varpi^{\tau-1}(1-\varpi)^{\lambda-\tau}|\alpha_\tau| \le \Psi.$$

By Lemma 3.1

$$\sum_{\tau=2}^{\infty}\left(E_1\tau^2+E_2\tau\right)\frac{(\lambda-1)!}{(\lambda-\tau)!(\tau-1)!}\varpi^{\tau-1}(1-\varpi)^{\lambda-\tau}|\alpha_\tau|$$
$$\le 2|\rho_3|(1-\rho_2)\sum_{\tau=2}^{\infty}\frac{\left(E_1\tau^2+E_2\tau\right)(\lambda-1)!}{\left(\rho_1(\tau-1)+1\right)(\lambda-\tau)!(\tau-1)!}\varpi^{\tau-1}(1-\varpi)^{\lambda-\tau}.$$

Moreover, since $\rho_1(\tau-1)+1 \ge \tau\rho_1$, we have

$$\sum_{\tau=2}^{\infty}\left(E_1\tau^2+E_2\tau\right)\frac{(\lambda-1)!}{(\lambda-\tau)!(\tau-1)!}\varpi^{\tau-1}(1-\varpi)^{\lambda-\tau}|\alpha_\tau|$$
$$\le \frac{2|\rho_3|(1-\rho_2)}{\rho_1}\sum_{\tau=2}^{\infty}\frac{\left(E_1\tau+E_2\right)(\lambda-1)!}{(\lambda-\tau)!(\tau-1)!}\varpi^{\tau-1}(1-\varpi)^{\lambda-\tau}$$
$$\equiv \frac{2|\rho_3|(1-\rho_2)}{\rho_1}\sum_{\tau=2}^{\infty}(E_1(\tau-1)+E_1+E_2)\frac{(\lambda-1)!}{(\lambda-\tau)!(\tau-1)!}\varpi^{\tau-1}(1-\varpi)^{\lambda-\tau}$$
$$=\frac{2|\rho_3|(1-\rho_2)}{\rho_1}\left\{E_1\sum_{\tau=2}^{\infty}\frac{(\lambda-1)!}{(\lambda-\tau)!(\tau-2)!}\varpi^{\tau-1}(1-\varpi)^{\lambda-\tau}\right.$$
$$\left.+(E_1+E_2)\sum_{\tau=2}^{\infty}\frac{(\lambda-1)!}{\%(\lambda-\tau)!(\tau-1)!}\varpi^{\tau-1}(1-\varpi)^{\lambda-\tau}\right\}$$

$$=\frac{2|\rho_3|(1-\rho_2)}{\%\rho_1}\left\{E_1\sum_{\tau=0}^{\infty}\frac{(\lambda-1)!}{(\lambda-\tau-2)!\tau!}\varpi^{\tau+1}(1-\varpi)^{\lambda-\tau-2}\right.$$
$$\left.+(E_1+E_2)\sum_{\tau=1}^{\infty}\frac{(\lambda-1)!}{(\lambda-\tau-1)!\tau!}\varpi^{\tau}(1-\varpi)^{\lambda-\tau-1}\right\}$$
$$=\frac{2|\rho_3|(1-\rho_2)}{\rho_1}\left\{\varpi E_1(\lambda-1)\sum_{\tau=0}^{\infty}\frac{(\lambda-2)!}{(\lambda-\tau-2)!\tau!}\varpi^{\tau}(1-\varpi)^{\lambda-\tau-2}\right.$$
$$\left.+(E_1+E_2)\sum_{\tau=1}^{\infty}\frac{(\lambda-1)!}{\%(\lambda-\tau-1)!\tau!}\varpi^{\tau-1}(1-\varpi)^{\lambda-\tau-1}\right\}$$
$$=\frac{2|\rho_3|(1-\rho_2)}{\rho_1}\{\varpi E_1(\lambda-1)+(E_1+E_2)F_0(\lambda,\varpi)\}.$$

Since the final equation, which is bounded above by Ψ, the proof of Theorem 3.2 is completed.

4. An Integral Operator $L(\lambda, \varpi, \varsigma)$

Theorem 4.1. *Let the integral operator $L(\lambda, \varpi, \varsigma)$ is given by*

$$L(\lambda,\varpi,\varsigma)=\int_0^{\varsigma}\frac{B(\lambda,\varpi,s)}{s}ds, |\varsigma|<1, \tag{4.1}$$

then $L(\lambda, \varpi, \varsigma) \in \Theta_\Psi(E_1, E_2, 0)$ if and only if

$$\varpi E_1(\lambda-1)+(E_1+E_2)F_0(\lambda,\varpi)\leq\Psi$$

where $\mathcal{F}_0(\lambda, \varpi)$ is given by (2.1).

Proof. Since

$$L(\lambda,\varpi,\varsigma)=\varsigma-\sum_{\tau=2}^{\infty}\frac{(\lambda-1)!}{\tau(\lambda-\tau)!(\tau-1)!}\varpi^{\tau-1}(1-\varpi)^{\lambda-\tau}\varsigma^{\tau},$$

then by the inequality *(1.4)* it suffices to show that

$$\sum_{\tau=2}^{\infty}\left(E_1\tau^2+E_2\tau\right)\frac{(\lambda-1)!}{\tau(\lambda-\tau)!(\tau-1)!}\varpi^{\tau-1}(1-\varpi)^{\lambda-\tau}$$
$$\equiv\sum_{\tau=2}^{\infty}\left(E_1\tau+E_2\right)\frac{(\lambda-1)!}{(\lambda-\tau)!(\tau-1)!}\varpi^{\tau-1}(1-\varpi)^{\lambda-\tau}$$
$$\leq\Psi.$$

Similar to proof Theorem 3.2, we get

$$\sum_{\tau=2}^{\infty}\left(E_1\tau^2+E_2\tau\right)\frac{(\lambda-1)!}{\tau(\lambda-\tau)!(\tau-1)!}\varpi^{\tau-1}(1-\varpi)^{\lambda-\tau}$$
$$\leq\varpi E_1(\lambda-1)+(E_1+E_2)F_0(\lambda,\varpi)\leq\Psi. \tag{4.2}$$

Theorem 4.2. *If the integral operator $L(\lambda, \varpi, \varsigma)$ is given by (4.1), then $L(\lambda, \varpi, \varsigma) \in \Theta_\Psi(E_1, E_2, E_3)$ if and only if*

$$E_1F_1(\lambda,\varpi)+(3E_1+E_2)F_2(\lambda,\varpi)+(E_1+E_2+E_3)F_3(\lambda,\varpi)\le\Psi.$$

where

$$F_1(\lambda,\varpi)=\sum_{\tau=0}^{\infty}\frac{(\lambda-1)!}{(\lambda-\tau-3)!(\tau+3)\tau!}\varpi^{\tau+2}(1-\varpi)^{\lambda-\tau-3},$$
$$F_2(\lambda,\varpi)=\sum_{\tau=0}^{\infty}\frac{\%(\lambda-1)!}{(\lambda-\tau-2)!(\tau+2)\tau!}\varpi^{\tau+1}(1-\varpi)^{\lambda-\tau-2} \tag{4.3}$$

and

$$F_3(\lambda,\varpi)=\sum_{\tau=2}^{\infty}\frac{(\lambda-1)!}{(\lambda-\tau)!\tau!}\varpi^{\tau-1}(1-\varpi)^{\lambda-\tau}. \tag{4.4}$$

Proof. Since

$$L(\lambda,\varpi,\varsigma)=\varsigma-\sum_{\tau=2}^{\infty}\frac{(\lambda-1)!}{\tau(\lambda-\tau)!(\tau-1)!}\varpi^{\tau-1}(1-\varpi)^{\lambda-\tau}\varsigma^{\tau},$$

then by the inequality (1.4) it suffices to show that

$$\sum_{\tau=2}^{\infty}\left(E_1\tau^2+E_2\tau+E_3\right)\frac{(\lambda-1)!}{\tau(\lambda-\tau)!(\tau-1)!}\varpi^{\tau-1}(1-\varpi)^{\lambda-\tau}$$
$$\equiv\sum_{\tau=2}^{\infty}\left(E_1\tau^2+E_2\tau+E_3\right)\frac{(\lambda-1)!}{(\lambda-\tau)!\tau!}\varpi^{\tau-1}(1-\varpi)^{\lambda-\tau}$$
$$\le\Psi.$$

So

$$\sum_{\tau=2}^{\infty}\left(E_1\tau^2+E_2\tau+E_3\right)\frac{(\lambda-1)!}{(\lambda-\tau)!\tau!}\varpi^{\tau-1}(1-\varpi)^{\lambda-\tau}$$
$$=E_1\sum_{\tau=3}^{\infty}\frac{(\lambda-1)!}{\tau(\lambda-\tau)!(\tau-3)!}\varpi^{\tau-1}(1-\varpi)^{\lambda-\tau}$$
$$+(3E_1+E_2)\sum_{\tau=2}^{\infty}\frac{(\lambda-1)!}{\tau(\lambda-\tau)!(\tau-2)!}\varpi^{\tau-1}(1-\varpi)^{\lambda-\tau}$$
$$+(E_1+E_2+E_3)\sum_{\tau=2}^{\infty}\frac{(\lambda-1)!}{(\lambda-\tau)!\tau!}\varpi^{\tau-1}(1-\varpi)^{\lambda-\tau}$$

$$=E_1\sum_{\tau=0}^{\infty}\frac{(\lambda-1)!}{(\lambda-\tau-3)!(\tau+3)\tau!}\varpi^{\tau+2}(1-\varpi)^{\lambda-\tau-3}$$
$$+(3E_1+E_2)\sum_{\tau=0}^{\infty}\frac{(\lambda-1)!}{(\lambda-\tau-2)!(\tau+2)\tau!}\varpi^{\tau+1}(1-\varpi)^{\lambda-\tau-2}$$
$$+(E_1+E_2+E_3)\sum_{\tau=2}^{\infty}\frac{(\lambda-1)!}{(\lambda-\tau)!\tau!}\varpi^{\tau-1}(1-\varpi)^{\lambda-\tau}\le\Psi.$$

5. Corollaries

Specializing the coefficients E_1, E_2, E_3 and Ψ in our Theorems, we obtain numerous results, some of them have been previously examined by numerous authors. For example:

Let $E_1 = 0$, $E_2 = 2$, $E_3 = -\cos\mu - \kappa$ and $\Psi = \cos\mu - \kappa$ in Theorem 2.1, we get the following corollary.

Corollary 5.1. *The function $B(\lambda, \varpi, \varsigma) \in \mathcal{C}(\mu, \kappa)$ if and only if*

$$2\varpi(\lambda-1)+(2-\cos\mu-\kappa)F_0(\lambda,\varpi)\le\cos\mu-\kappa.$$

Let $E_1 = 2$, $E_2 = -\cos\mu - \kappa$, $E_3 = 0$ and $\Psi = \cos\mu - \kappa$ in Theorem 2.1, we get the following corollary.

Corollary 5.2. *The function $B(\lambda, \varpi, \varsigma) \in \mathcal{U}(\mu, \kappa)$ if and only if*

$$2\varpi^2(\lambda-1)(\lambda-2)+(6-\cos\mu-\kappa)\varpi(\lambda-1)+(2-\cos\mu-\kappa)F_0(\lambda,\varpi)\le\cos\mu-\kappa.$$

Let $E_1 = 2$, $E_2 = -\cos\mu - \kappa$, $E_3 = 0$ and $\Psi = \cos\mu - \kappa$ in Theorem 3.2, we get the following corollary.

Corollary 5.3. *Let $b \in \mathcal{G}^{\rho_3}(\rho_1,\rho_2)$, and the inequality*

$$2\varpi(\lambda-1)+(2-\cos\mu-\kappa)F_0(\lambda,\varpi)\le\frac{\rho_1(\cos\mu-\kappa)}{2|\rho_3|(1-\rho_2)}$$

is satisfied, then $B(\lambda, \varpi, \varsigma)b \in \mathcal{U}(\mu, \kappa)$.

Let $E_1 = 2$, $E_2 = -\cos\mu - \kappa$, $E_3 = 0$ and $\Psi = \cos\mu - \kappa$ in Theorem 4.1, we get the following corollary.

Corollary 5.4. *If the integral operator $L(\lambda, \varpi, \varsigma)$ is given by (4.1), then $L(\lambda, \varpi, \varsigma) \in U(\mu, \kappa)$ if and only if*

$$2\varpi(\lambda-1)+(2-\cos\mu-\kappa)F_0(\lambda,\varpi)\le\cos\mu-\kappa$$

Let $E_1 = 0$, $E_2 = 2$, $E_3 = -\cos\mu - \kappa$ and $\Psi = \cos\mu - \kappa$ in Theorem 4.2, we get the following corollary.

Corollary 5.5. *If integral operator $L(\lambda, \varpi, \varsigma)$ is given by (4.1), then $L(\lambda, \varpi, \varsigma) \in C(\mu, \kappa)$ if and only if*

$$(6-\cos\mu-\kappa)F_2(\lambda,\varpi)+(2-\cos\mu-\kappa)F_3(\lambda,\varpi)\le\cos\mu-\kappa.$$

where $F_2(\lambda, \varpi)$ given by (4.3) and $F_3(\lambda, \varpi)$ given by (4.4).

Let $E_1 = 0$, $E_2 = \zeta_2 + 1$, $E_3 = -\zeta_3(\zeta_1 + \zeta_2)$ and $\Psi = 1 - \zeta_1$ in Theorem 2.1, we get the following corollary, which returns to [10, Theorem 2.1].

Corollary 5.6. *The function $B(\lambda,\varpi,\varsigma)\in\wp^*_{\zeta_3}(\zeta_1,\zeta_2)$ if and only if*

$$(\zeta_2+1)\varpi(\lambda-1)+(\zeta_2-\zeta_3(\zeta_1+\zeta_2)+1)F_0(\lambda,\varpi)\le1-\zeta_1.$$

Let $E_1 = \zeta_2 + 1$, $E_2 = -\zeta_3(\zeta_1 + \zeta_2)$, $E_3 = 0$ and $\Psi = 1 - \zeta_1$ in Theorem 2.1, we get the following corollary, which returns to [10, Theorem 2.2].

Corollary 5.7. *The function $B(\lambda, \varpi, \varsigma) \in G^*_{\zeta_3}((\zeta_1 + \zeta_2)$ if and only if*

$$\begin{aligned}&(\zeta_2+1)\varpi^2(\lambda-1)(\lambda-2)+(3(\zeta_2+1)-\zeta_3(\zeta_1+\zeta_2))\varpi(\lambda-1)\\&+(\zeta_2-\zeta_3(\zeta_1+\zeta_2)+1)F_0(\lambda,\varpi)\\&\le1-\zeta_1.\end{aligned}$$

Let $E_1 = \zeta_2 + 1$, $E_2 = -\zeta_3(\zeta_1 + \zeta_2)$, $E_3 = 0$ and $\Psi = 1 - \zeta_1$ in Theorem 3.2, we get the following corollary, which returns to [10, Theorem 3.2].

Corollary 5.8. *Let $b \in \mathcal{G}^{\rho_3}(\rho_1, \rho_2)$. If*

$$\varpi(\lambda-1)(1-\varpi\kappa)+(1-\varpi)F_0(\lambda,\varpi) \le \frac{\rho_1(1-\zeta_1)}{2|\rho_3|(1-\rho_2)},$$

then $B(\lambda,\varpi,\varsigma)b \in G^*_{\zeta_3}(\zeta_1,\zeta_2)$.

Let $E_1 = \zeta_2 + 1$, $E_2 = -\zeta_3(\zeta_1 + \zeta_2)$, $E_3 = 0$ and $\Psi = 1 - \zeta_1$ in Theorem 4.1, we get the following corollary, which returns to [10, Theorem 4.1].

Corollary 5.9. *If the integral operator $L(\lambda, \varpi, \varsigma)$ is given by (4.1), then $L(\lambda, \varpi, \varsigma) \in G^*_{\zeta_3}(\zeta_1,\zeta_2)$ if and only if*

$$\varpi(1-\varpi\kappa)(\lambda-1) + (1-\varpi)F_0(\lambda, \varpi) \le 1-\zeta_1.$$

Let $E_1 = 0$, $E_2 = \zeta_2 + 1$, $E_3 = -\zeta_3(\zeta_1 + \zeta_2)$ and $\Psi = 1 - \zeta_1$ in Theorem 4.2, we get the following corollary.

Corollary 5.10. *If the integral operator $L(\lambda, \varpi, \varsigma)$ is given by (4.1), then $L(\lambda, \varpi, \varsigma) \in \wp^*_{\zeta_3}(\zeta_1,\zeta_2)$ if and only if*

$$(1-\varpi\kappa)F_2(\lambda, \varpi) + (1-\varpi)F_3(\lambda, \varpi) \le 1-\delta.$$

where $F_2(\lambda, \varpi)$ given by (4.3) and $F_3(\lambda, \varpi)$ given by (4.4).

Remark 5.11. If we put $E_1 = 0$, $E_2 = (1-,\gamma\delta)$, $E_3 = \delta(\gamma - 1)$, $\Psi = 1 - \delta$ and $E_1 = (1 - \gamma\delta)$, $E_2 = \delta(\gamma - 1)$, $E_3 = 0$, $\Psi = 1 - \delta$, respectively, we get the sufficient and necessary conditions for functions in the classes $T(\gamma, \delta)$ and $C(\gamma, \delta)$ obtained by Nazeer et. al [16].

6. Conclusions

By applying the binomial distribution, we determine the necessary and sufficient requirements for the functions $B(\lambda, \varpi, \varsigma)$, $B(\lambda, \varpi, \varsigma)b$ and the operator $L(\lambda, \varpi, \varsigma)$ to be in the inclusive subclass $\Theta_\Psi(E_1, E_2, E_3)$. Further, our main results can lead to several additional new results by suitably specializing the real constants E_1, E_2, E_3 and Ψ in other subclasses of analytic functions with negative coefficients introduced and examined by a number of writers, as mentioned in Remark *(1.2) and other classes.*

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