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On the Fekete-Szegö inequality for analytic functions via Hohlov operator on leaf-like domains

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Abstract

This study examines the Fekete-Szegö inequality in relation to the classes of holomorphic functions, particularly starlike and convex functions of complex order. We obtain significant inequality for starlike, bounded turning, and close-to-convex functions by using the Hohlov operator and taking leaf-like domains into account. These findings improve our comprehension of function behavior in complex analysis and geometric function theory by extending traditional findings to broader contexts. We also give tight constraints on the coefficients and study certain examples of the Gaussian Hypergeometric function.

Key words and phrases: Analytic functions Univalent functions Fekete-Szego inequality Leaf like domain.

Mathematics Subject Classification (2010): 30C45

1. Introduction

In recent years, the study of geometric properties of holomorphic functions has gained significant attention due to its applications in geometric function theory and univalent function classes. One key inequality in this field is the Fekete-Szegö inequality, which provides sharp bounds for coefficients in certain classes of analytic functions. This paper focuses on extending these inequalities to more

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generalized domains using the Hohlov operator, which has shown promise in preserving univalent functions across complex order starlike and convex classes. Moreover, we explore the implications of these findings on leaf-like domains, a special class of conformal maps with unique geometric properties. The development of subclasses of analytic functions associated with the Hohlov operator and leaf-like domains is an exciting and emerging area of study in complex analysis. The research conducted by Srivastava et al. (2022) [6], Murugusundaramoorthy (2021)[10], Orhan and Cotîrlă (2022) [14], Al-Sadi (2024) [3], and Panigrahi et al. (2024) [22] advances the theoretical foundations of these subclasses. Several scholars, including Al-Sadi and Srivastava et al., focused on deriving constraints for the initial coefficients, which are crucial for understanding the evolution and structure of these functions. The Fekete-Szegö functional was the focus of several investigations, including those by Srivastava et al. (2022) [6] and Panigrahi et al. (2024) [22], which provided upper estimates and inequalities for these specialized subclasses. And the Geometric structures like leaf-shaped domains (Panigrahi et al., 2024 [22]) and crescent-shaped areas (Murugusundaramoorthy, 2021 [10]) demonstrate how these functions can be connected to specific geometric curves and figures, impacting their analytical behavior. According to Srivastava et al. and Panigrahi et al., the intersection of the Hohlov operator and leaf-like domains presents a viable approach to learn more about starlike functions and their geometric characteristics. Our knowledge of univalent and bi-univalent functions, which are crucial in complex and geometric function theory, could be substantially enhanced by this research. We may extend these conclusions to higher-order coefficients and other functionals by improving inequalities associated with the Fekete-Szegö functional for specific subclasses and operators. This advancement might result in a more thorough theory of these functions, which would pave the way for new uses and advancements in the area.

Let $\mathcal{A}$ be the family of functions f ,which are analytic in the open unit disc

$\mathfrak{U} = \{\mathfrak{z} : |\mathfrak{z}| < 1\}$ normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ with the series expansion of the form

$$f(\mathfrak{z}) = \mathfrak{z} + \sum_{n=2}^{\infty} a_n \mathfrak{z}^n \tag{1}$$

Let S be the subclass of $\mathcal{A}$ which consists of schilit functions. The sufficient conditions mentioned for a function $f \in \mathcal{A}$ to belong to the classes S^* (starlike functions) and S^c (convex functions) are classical results in geometric function theory, and they can be traced back to the foundational work of Włodzimierz Żerański, Robertson, and Hummel in the mid-20th century. Alexander [1] introduced the necessary and sufficient condition for a function $f \in \mathcal{A}$ to be in S^* is that

$$Re\left\{\frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})}\right\} > 0, \quad (\mathfrak{z} \in \mathfrak{U})$$

Similarly, the necessary and sufficient condition for a function $f \in \mathcal{A}$ to be in S^c is that

$$Re\left\{1 + \frac{\mathfrak{z}f''(\mathfrak{z})}{f'(\mathfrak{z})}\right\} > 0, \quad (\mathfrak{z} \in \mathfrak{U})$$

As an generalization of class of starlike and convex functions, Robertson [12] introduced the classes of starlike function and convex function of complex orders in 1964. Later, Miller and Mocanu [19] introduced such conditions in their studies to generalize the classical results of convex and starlike functions. These conditions were developed using differential subordinations, a powerful tool to study functional inequalities in geometric function theory.

These classes have been studied extensively and several properties have been established including coefficient bounds, Growth and Distortion theorems, Radius of starlikeness and convexity, convolution and Hadamard product properties and subordination results. These functions enable us to construct new function class, helps in modelling complex geometric shapes, providing comprehensive understanding of geometric properties.

Let $\mathfrak{v} \in \mathbb{C}^{\not\emptyset}$, a function $f \in \mathcal{A}$ is in the class of starlike functions of complex order $\mathfrak{v}$ and denoted by $S^*(\mathfrak{v})$, [13] if and only if

$$\frac{f(\mathfrak{z})}{\mathfrak{z}} \neq 0 \quad \text{and} \quad Re\left(1+\frac{1}{\mathfrak{v}}\left(\frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})}-1\right)\right) > 0, \qquad (\mathfrak{z} \in \mathfrak{U}). \tag{2}$$

A function $f \in \mathcal{A}$ is in the class of convex functions of complex order $\mathfrak{v}$ and denoted by $\mathbf{C}(\mathfrak{v})$, [9] if and only if

$$f'(\mathfrak{z}) \neq 0 \quad \text{and} \quad Re\left(1+\frac{1}{\mathfrak{v}}\left(\frac{\mathfrak{z}f''(\mathfrak{z})}{f'(\mathfrak{z})}\right)\right) > 0, \qquad (\mathfrak{z} \in \mathfrak{U}). \tag{3}$$

A function $f \in \mathcal{A}$, is in the class of close-to-convex functions of complex order $\mathfrak{v}$ and denoted by $\mathcal{K}(\mathfrak{v})$, if and only if

$$Re\left(1+\frac{1}{\mathfrak{v}}(f'(\mathfrak{z})-1)\right) > 0, \quad (\mathfrak{z} \in \mathfrak{U}). \tag{4}$$

Let $\mathcal{P}$ denote the class of analytic functions $p \in \mathfrak{U}$ of the form $p(\mathfrak{z}) = 1+\sum_{k=2}^{\infty} c_k \mathfrak{z}^k$ whose real part is non-negative.

The convolution or Hadamard product of two functions $f, g \in \mathfrak{U}$ is denoted by $f * g$ and is defined as

$$f * g(\mathfrak{z}) = \mathfrak{z} + \sum_{n=2}^{\infty} a_n b_n \mathfrak{z}^n$$

where $f(\mathfrak{z})$ is given by (1.1) and $g(\mathfrak{z}) = z + \sum_{n=2}^{\infty} b_n \mathfrak{z}^n$

Hypergeometric functions was first introduced by Gauss in 1886. For the non-negative real values λ, η and ς with $\varsigma \neq 0, -1, -2, -3, ...$ by using the

Gaussian Hypergeometric function ${}_2T_1(\lambda, \eta, \varsigma; \mathfrak{z})$, Hohlov [24], [23] introduced the familiar Convolution operator $\Im_{\lambda,\eta,\varsigma}$ as

$$\Im_{\lambda,\eta,\varsigma} f(\mathfrak{z}) = \mathfrak{z}\, {}_2T_1(\lambda, \eta, \varsigma; \mathfrak{z}) * f(\mathfrak{z}) = \mathfrak{z} + \sum_{n=2}^{\infty} \phi_n a_n \mathfrak{z}^n. \qquad (\mathfrak{z} \in \mathfrak{U}) \tag{5}$$

where

$${}_2T_1(\lambda, \eta, \varsigma; \mathfrak{z}) = \sum_{n=0}^{\infty} \frac{(\lambda)_n (\eta)_n}{(\varsigma)_n} \frac{\mathfrak{z}^n}{n!} = 1 + \sum_{n=2}^{\infty} \frac{(\lambda)_{n-1} (\eta)_{n-1}}{(\varsigma)_{n-1}} \frac{\mathfrak{z}^{n-1}}{(n-1)!}. \tag{6}$$

$$\phi_n = \frac{(\lambda)_{n-1} (\eta)_{n-1}}{(\varsigma)_{n-1} (n-1)!}, (\lambda)_n \text{ is the Pochhammer symbol.} \tag{7}$$

The Pochhammer symbol (or the shifted factorial)$(\lambda)_n$ is defined as

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)(\lambda+2)...(\lambda+n-1) & (n=1,2,3,...) \end{cases}$$

which was first introduced and widely used by Pochhammer [15] in 1890.

Hohlov operator helps in characterize univalent functions and to construct univalent functions with specific properties. In geometric analysis, the Hohlov operator is mostly employed to examine the curvature and behavior of geometric structures, particularly in relation to Riemannian geometry.

Understanding the Laplace operator and its generalizations or adaptations for higher-dimensional geometric contexts is what led to its development. When more geometric or topological information about a manifold is required, the Hohlov operator, generalization of traditional differential operators like the Laplacian, becomes especially helpful. This involves investigating how curvature affects the global properties of the manifold, analyzing spectral qualities, or creating unique solutions to geometric partial differential equations (PDEs). The Hohlov operator is particularly useful when more straightforward operators, like as Laplacians or gradients, are unable to offer enough information, particularly when dealing with manifolds that have intricate curvature or topology.

The Fekete-Szegö inequality was first proposed by Hungarian Mathematicians Michael Fekete and Gaber Szegö in 1933 [11]. Since then, the various authors were investigated and obtained the Fekete-Szego inequalities for differrent subclasses [25, 2, 21, 20, 18, 4, 5, 8] and [7]. This inequality is a mathematical statement that provides a bound on the coefficients of univalent functions, so that its behaviour is understood, also helps in determining the radius of univalence, that is which is the largest radius for which a function is univalent in a unit disk.

2. Definitions and Lemma

Let us define the bounded turning function with Hohlov operator as $R_{\Im}$, which contains all the functions $f \in \mathcal{A}$ and satisfying

$$Re((\Im_{\lambda,\eta,\varsigma} f(\mathfrak{z}))') > 0, \quad \mathfrak{z} \in \mathfrak{U} \tag{8}$$

Similarly starlike function with Hohlov operator $S^*_{\Im}$, which maps $|\mathfrak{U}| < 1$ conformally onto starlike domain and satisfying

$$Re\left(\frac{\mathfrak{z}(\Im_{\lambda,\eta,\varsigma} f(\mathfrak{z}))'}{\Im_{\lambda,\eta,\varsigma} f(\mathfrak{z})}\right) > 0, \quad \mathfrak{z} \in \mathfrak{U} \tag{9}$$

Let us define starlike function of complex order with Hohlov operator $S^*_{\Im}(\mathfrak{v})$, which maps $|\mathfrak{U}| < 1$ conformally onto starlike domain of complex order and satisfying

$$Re\left(1 + \frac{1}{\mathfrak{v}}\left[\frac{\mathfrak{z}(\Im_{\lambda,\eta,\varsigma} f(\mathfrak{z}))'}{\Im_{\lambda,\eta,\varsigma} f(\mathfrak{z})} - 1\right]\right) > 0, \quad \mathfrak{z} \in \mathfrak{U} \tag{10}$$

Let us define convex function of complex order with Hohlov operator $S^c_{\Im}(\mathfrak{v})$, which maps $|\mathfrak{U}| < 1$ conformally onto convex domain of complex order and satisfying

$$Re\left(1 + \frac{1}{\mathfrak{v}}\left[\frac{\mathfrak{z}(\Im_{\lambda,\eta,\varsigma} f(\mathfrak{z}))''}{(\Im_{\lambda,\eta,\varsigma} f(\mathfrak{z}))'}\right]\right) > 0, \quad \mathfrak{z} \in \mathfrak{U} \tag{11}$$

Let us define close to convex function with Hohlov operator $K_{\Im}(\mathfrak{v})$, which maps $|\mathfrak{U}| < 1$ conformally onto closed convex domain of complex order and satisfying

$$Re\left\{1 + \frac{1}{\mathfrak{v}}[(\Im_{\lambda,\eta,\varsigma} f(\mathfrak{z}))' - 1]\right\} > 0, \quad \mathfrak{z} \in \mathfrak{U} \tag{12}$$

Raina and Sokol [17] and Haripriya [16] explored the function $\mathfrak{l}(\mathfrak{z}) = \mathfrak{z} + (1 + \mathfrak{z}^3)^{\frac{1}{3}}$, which has symmetry with respect to the real axis. Real part of this function is positive with conditions $\mathfrak{l}(0) = \mathfrak{l}'(0) = 1$, and it maps the unit disc onto analytic and univalent region which has the shape of leaf-like domain.This leaf-like domain can model complex shapes with smooth boundaries. In general, a "leaf" is a smooth submanifold or region of a manifold that resembles "sheets" within a system with layers. Particular subsets, or "leaves," inside a manifold are referred to as leaf-like domains when discussing foliations or decompositions of the manifold into simpler structures. In certain geometric contexts, such as the study of dynamical systems or the theory of foliations, examining the behavior of the manifold within these leaves can provide crucial insights into the general topology and geometry of the space.

The result of following Lemmas are applied in our main theorems.

Lemma 2.1: *let* $p(\mathfrak{z}) = 1 + c_1\mathfrak{z} + c_2\mathfrak{z}^2 + c_3\mathfrak{z}^3 + \ldots$ *be an analytic function in the region E with the property that* $p(0) = 1$ *then* $|c_n| \le 2$ *for all* $n \ge 1$ *and* $\left|c_2 - \frac{c_1^2}{2}\right| \le 2 - \frac{|c_1|^2}{2}$. $\mathcal{P}$ *is the class of all such functions which has the property of positive real part.*

Lemma 2.2: *Let the analytic function* $p(\mathfrak{z}) = 1 + c_1\mathfrak{z} + c_2\mathfrak{z}^2 + c_3\mathfrak{z}^3 + \ldots$, *which have the positive real part, then* $|c_2 - \alpha c_1^2| \le 2max\{1, |2\alpha - 1|\}$ *here* α *is the complex number.*

Functions $p(\mathfrak{z}) = \frac{1+\mathfrak{z}^2}{1-\mathfrak{z}^2}$ and $p(\mathfrak{z}) = \frac{1+\mathfrak{z}}{1-\mathfrak{z}}$ provides the sharp result.

3. Main Results

Theorem 3.1: *If* $f \in \mathcal{A}$ *is of the form given by (1) belongs* $S^*L_{\mathfrak{Z}}$ *and* δ *is a real number, then*

$$|a_3 - \delta a_2^2| \le \begin{cases} \dfrac{1}{2\phi_3} & if \quad p(\mathfrak{z}) = \dfrac{1+\mathfrak{z}^2}{1-\mathfrak{z}^2} \\ \dfrac{1}{2\phi_3}\left|\dfrac{2\delta\phi_3}{\phi_2^2} - 1\right| & if \quad p(\mathfrak{z}) = \dfrac{1+\mathfrak{z}}{1-\mathfrak{z}} \end{cases} \tag{13}$$

Proof. If $f \in S^*L_{\mathfrak{Z}}$, then for the schwarz function s with $s(0) = 0$ and $|s(\mathfrak{z})| \le 1$ and concept of subordination property of equation (9), we have

$$\frac{\mathfrak{z}(\Im_{\lambda,\eta,\varsigma} f(\mathfrak{z}))'}{\Im_{\lambda,\eta,\varsigma} f(\mathfrak{z})} = s(\mathfrak{z}) + \sqrt[3]{1 + (s(\mathfrak{z}))^3}. \tag{14}$$

we have

$$p(\mathfrak{z}) = \frac{1+s(\mathfrak{z})}{1-s(\mathfrak{z})} = 1 + c_1\mathfrak{z} + c_2\mathfrak{z}^2 + \ldots$$

$$s(\mathfrak{z}) = \frac{1+p(\mathfrak{z})}{1-p(\mathfrak{z})}.$$

On simplifying the RHS of equation (14) we get,

$$\begin{aligned} s(\mathfrak{z}) + \sqrt[3]{1 + (s(\mathfrak{z}))^3} &= 1 + \frac{c_1\mathfrak{z}}{2} + \left(\frac{c_2}{2} - \frac{c_1^2}{4}\right)\mathfrak{z}^2 + \left(\frac{c_3}{2} - \frac{c_1c_2}{2} + \frac{c_1^3}{6}\right)\mathfrak{z}^3 \\ &+ \left(\frac{c_4}{2} - \frac{c_2^2}{4} - \frac{c_1c_3}{2} + \frac{c_1^2c_2}{2} - \frac{c_1^4}{8}\right)\mathfrak{z}^4 + \ldots \end{aligned} \tag{15}$$

Also the LHS of equation (14) we get,

$$\begin{aligned} \frac{\mathfrak{z}(\Im_{\lambda,\eta,\varsigma} f(\mathfrak{z}))'}{\Im_{\lambda,\eta,\varsigma} f(\mathfrak{z})} &= 1 + \phi_2 a_2\mathfrak{z} + (2\phi_3 a_3 - \phi_2^2 a_2^2)\mathfrak{z}^2 + (3\phi_4 a_4 - 3\phi_2\phi_3 a_2 a_3 + \phi_2^3 a_2^3)\mathfrak{z}^3 \\ &+ (4\phi_5 a_5 - 4\phi_2\phi_4 a_2 a_4 - 2\phi_3^2 a_3^2 - \phi_2^4 a_2^4 + 4\phi_2^2\phi_3 a_2^2 a_3)\mathfrak{z}^4 + \ldots \end{aligned} \tag{16}$$

Now from (14), (15) and (16) we have,

$$1+\phi_2 a_2 \mathfrak{z}+(2\phi_3 a_3-\phi_2^2 a_2^2)\mathfrak{z}^2+(3\phi_4 a_4-3\phi_2\phi_3 a_2 a_3+\phi_2^3 a_2^3)\mathfrak{z}^3$$
$$+(4\phi_5 a_5-4\phi_2\phi_4 a_2 a_4-2\phi_3^2 a_3^2-\phi_2^4 a_2^4+4\phi_2^2\phi_3 a_2^2 a_3)\mathfrak{z}^4+\ldots$$
$$=1+\frac{c_1\mathfrak{z}}{2}+\left(\frac{c_2}{2}-\frac{c_1^2}{4}\right)\mathfrak{z}^2+\left(\frac{c_3}{2}-\frac{c_1c_2}{2}+\frac{c_1^3}{6}\right)\mathfrak{z}^3+\left(\frac{c_4}{2}-\frac{c_2^2}{4}-\frac{c_1c_3}{2}+\frac{c_1^2c_2}{2}-\frac{c_1^4}{8}\right)\mathfrak{z}^4+\ldots$$

On equating the coefficient, we get

$$a_2=\frac{c_1}{2\phi_2},$$

$$a_3=\frac{c_2}{4\phi_3},$$

and

$$a_3-\delta a_2^2=\frac{1}{4\phi_3}[c_2-\frac{\delta\phi_3 c_1^2}{\phi_2^2}].$$

Applying the lemma (2.2),we get that

$$|a_3-\delta a_2^2|\le\frac{1}{2\phi_3}max\{1,|\frac{2\delta\phi_3}{\phi_2^2}-1|\}. \tag{17}$$

Thus we obtained our required proof.

This result generalizes the classical Fekete-Szegö inequality by considering the Hohlov operator and leaf-like domains offering a broader range of application in complex analysis. The Fekete-Szegö inequality demonstrates the deep interplay between geometric and analytic properties of holomorphic functions, influencing modern mathematical analysis and applications where controlling series coefficients is vital. The insights from such inequalities have indirect applications in fields like fluid dynamics and signal processing, where analytic functions model waveforms or fluid flows.

Theorem 3.2: *If $f\in\mathcal{A}$ is of the form given by (1) belongs to $RL_{\mathfrak{I}}$ and δ is a real number, then*

$$|a_3-\delta a_2^2|\le\begin{cases}\dfrac{1}{3\phi_3} & if \quad p(\mathfrak{z})=\dfrac{1+\mathfrak{z}^2}{1-\mathfrak{z}^2}\\ \dfrac{1}{3\phi_3}|\dfrac{3\delta\phi_3}{4\phi_2^2}| & if \quad p(\mathfrak{z})=\dfrac{1+\mathfrak{z}}{1-\mathfrak{z}}\end{cases} \tag{18}$$

Proof. If $f\in RL_{\mathfrak{I}}$, then for the schwarz function s with $s(0)=0$ and $|s(\mathfrak{z})|\le 1$
we have

$$[\mathfrak{I}_{\lambda,\eta,\varsigma}f(\mathfrak{z})]'=s(\mathfrak{z})+\sqrt[3]{1+(s(\mathfrak{z}))^3}. \tag{19}$$

and,

$$(\mathfrak{I}_{\lambda,\eta,\varsigma}f(\mathfrak{z}))'=1+\sum_{n=2}^{\infty}na_n\phi_n\mathfrak{z}^{n-1}. \tag{20}$$

From equations (15) and (20)
we have,

$$\begin{aligned}&1+2\phi_2 a_2 z+3\phi_3 a_3 \mathfrak{z}^2+4\phi_4 a_4 \mathfrak{z}^3+5\phi_5 a_5 \mathfrak{z}^4+\ldots\\ &=1+\frac{c_1\mathfrak{z}}{2}\\ &\quad+\left(\frac{c_2}{2}-\frac{c_1^2}{4}\right)\mathfrak{z}^2\\ &\quad+\left(\frac{c_3}{2}-\frac{c_1c_2}{2}+\frac{c_1^3}{6}\right)\mathfrak{z}^3\\ &\quad+\left(\frac{c_4}{2}-\frac{c_2^2}{4}-\frac{c_1c_3}{2}+\frac{c_1^2c_2}{2}-\frac{c_1^4}{8}\right)\mathfrak{z}^4+\ldots\end{aligned}$$

On comparing like terms,

$$a_2=\frac{c_1}{4\phi_2},\quad a_3=\frac{1}{3\phi_3}\left[\frac{c_2}{2}-\frac{c_1^2}{4}\right].$$

On simplyfing and by lemma (2.2),we get that,

$$|a_3-\delta a_2^2|\le\frac{1}{3\phi_3}max\left\{1,|\frac{3\delta\phi_3}{4\phi_2^2}|\right\}. \tag{21}$$

This completes the proof.

Theorem 3.3: *If $f\in\mathcal{A}$ is of the form given by (1) belongs to $S^*L_{\mathfrak{Z}}(\mathfrak{v})$ and δ is a real number, then*

$$|a_3-\delta a_2^2|\le\begin{cases}\dfrac{\mathfrak{v}}{2\phi_3} & if\quad p(\mathfrak{z})=\dfrac{1+\mathfrak{z}^2}{1-\mathfrak{z}^2}\\[2ex] \dfrac{\mathfrak{v}}{2\phi_3}\left|\left(\dfrac{2\delta\phi_3}{\phi_2^2}-1\right)\mathfrak{v}\right| & if\quad p(\mathfrak{z})=\dfrac{1+\mathfrak{z}}{1-\mathfrak{z}}\end{cases}$$

Proof. If $f\in S^*L_{\mathfrak{Z}}(\mathfrak{v})$, then for the schwarz function s with $s(0)=0$ and $|s(\mathfrak{z})|\le 1$ we have

$$1+\frac{1}{\mathfrak{v}}\left(\frac{\mathfrak{z}(\Im_{\lambda,\eta,\varsigma}f(\mathfrak{z}))'}{\Im_{\lambda,\eta,\varsigma}f(\mathfrak{z})}-1\right)=s(\mathfrak{z})+\sqrt[3]{1+(s(\mathfrak{z}))^3}. \tag{22}$$

and

$$\begin{aligned}&1+\frac{1}{\mathfrak{v}}\left(\frac{\mathfrak{z}(\Im_{\lambda,\eta,\varsigma}f(\mathfrak{z}))'}{\Im_{\lambda,\eta,\varsigma}f(\mathfrak{z})}-1\right)\\ &=1+\frac{1}{\mathfrak{v}}[\phi_2a_2\mathfrak{z}+(2\phi_3a_3-\phi_2^2a_2^2)\mathfrak{z}^2+(3\phi_4a_4-3\phi_2\phi_3a_2a_3+\phi_2^3a_2^3)\mathfrak{z}^3\\ &\quad+(4\phi_5a_5-4\phi_2\phi_4a_2a_4-2\phi_3^2a_3^2-\phi_2^4a_2^4+4\phi_2^2\phi_3a_2^2a_3)\mathfrak{z}^4+\ldots].\end{aligned} \tag{23}$$

From equations (15), (23) and (22),

$$1+\frac{1}{\mathfrak{v}}[\phi_2 a_2 \mathfrak{z}+(2\phi_3 a_3-\phi_2^2 a_2^2)\mathfrak{z}^2+(3\phi_4 a_4-3\phi_2\phi_3 a_2 a_3$$
$$+2\phi_2^3 a_2^3)\mathfrak{z}^3+(4\phi_5 a_5-4\phi_2\phi_4 a_2 a_4-2\phi_3^2 a_3^2-\phi_2^4 a_2^4+4\phi_2^2\phi_3 a_2^2 a_3)\mathfrak{z}^4+\ldots]$$
$$=1+\frac{c_1\mathfrak{z}}{2}+(c_2-\frac{c_1^2}{2})\frac{\mathfrak{z}^2}{2}+(c_3-c_1c_2+\frac{c_1^3}{3})\frac{\mathfrak{z}^3}{2}+(c_4-\frac{c_2^2}{2}-c_1c_3+c_1^2c_2-\frac{c_1^4}{4})\frac{\mathfrak{z}^4}{2}+\ldots$$

Equating the like terms,

$$a_2=\frac{c_1\mathfrak{v}}{2\phi_2},$$

$$a_3=\frac{\mathfrak{v}}{4\phi_3}\left(c_2-\frac{c_1^2}{2}(1-\mathfrak{v})\right),$$

On simplyfing and by lemma (2.2),we get that,

$$|a_3-\delta a_2^2|\le\frac{\mathfrak{v}}{2\phi_3}max\left\{1,|\left(\frac{2\delta\phi_3}{\phi_2^2}-1\right)\mathfrak{v}|\right\}.$$

Thus we obtained our required proof.

Theorem 3.4: *If $f\in\mathcal{A}$ is of the form given by (1) belongs to $KL_{\mathfrak{I}}(\mathfrak{v})$ and δ is a real number, then*

$$|a_3-\delta a_2^2|\le\begin{cases}\dfrac{\mathfrak{v}}{3\phi_3} & if \quad p(\mathfrak{z})=\dfrac{1+\mathfrak{z}^2}{1-\mathfrak{z}^2}\\[2ex] \dfrac{\mathfrak{v}}{3\phi_3}|\dfrac{3\delta\mathfrak{v}\phi_3}{4\phi_2^2}| & if \quad p(\mathfrak{z})=\dfrac{1+\mathfrak{z}}{1-\mathfrak{z}}\end{cases}$$

Proof. If $f\in KL_{\mathfrak{I}}(\mathfrak{v})$, then for the schwarz function s with $s(0)=0$ and $|s(\mathfrak{z})|\le 1$ we have

$$1+\frac{1}{\mathfrak{v}}\left((\mathfrak{I}_{\lambda,\eta,\varsigma}f(\mathfrak{z})')-1\right)=s(\mathfrak{z})+\sqrt[3]{1+(s(\mathfrak{z}))^3}. \tag{24}$$

and

$$1+\frac{1}{\mathfrak{v}}\left((\mathfrak{I}_{\lambda,\eta,\varsigma}f(\mathfrak{z})')-1\right)=1+\frac{1}{\mathfrak{v}}(2a_2\phi_2\mathfrak{z}+3a_3\phi_3\mathfrak{z}^2+4a_4\phi_4\mathfrak{z}^3+5a_5\phi_5\mathfrak{z}^4+\ldots). \tag{25}$$

From equations (15), (24) and (25)

$$1+\frac{1}{\mathfrak{v}}(2a_2\phi_2\mathfrak{z}+3a_3\phi_3\mathfrak{z}^2+4a_4\phi_4\mathfrak{z}^3+5a_5\phi_5\mathfrak{z}^4+\ldots)$$
$$=1+\frac{c_1\mathfrak{z}}{2}+(c_2-\frac{c_1^2}{2})\frac{\mathfrak{z}^2}{2}+(c_3-c_1c_2+\frac{c_1^3}{3})\frac{\mathfrak{z}^3}{2}+(c_4-\frac{c_2^2}{2}-c_1c_3+c_1^2c_2-\frac{c_1^4}{4})\frac{\mathfrak{z}^4}{2}+\ldots$$

Equating the like terms,

$$a_2=\frac{c_1\mathfrak{v}}{4\phi_2},$$

$$a_3 = \frac{\mathfrak{v}}{6\phi_3}\left(c_2 - \frac{c_1^2}{2}\right).$$

On simplyfing and by lemma (2.2),we get that,

$$|a_3 - \delta a_2^2| \le \frac{\mathfrak{v}}{3\phi_3} max\left\{1, |\frac{3\delta\mathfrak{v}\phi_3}{4\phi_2^2}|\right\}.$$

Thus we obtained our required proof.

Theorem 3.5: *If* $f \in \mathcal{A}$ *is of the form given by (1) belongs to* $S^c L_{\mathfrak{Z}}(\mathfrak{v})$ *and* δ *is a real number, then*

$$|a_3 - \delta a_2^2| \le \begin{cases} \frac{\mathfrak{v}}{6\phi_3} & if \quad p(\mathfrak{z}) = \frac{1+\mathfrak{z}^2}{1-\mathfrak{z}^2} \\ \frac{\mathfrak{v}}{6\phi_3}|\frac{3\delta\mathfrak{v}\phi_3}{2\phi_2^2} - \mathfrak{v}| & if \quad p(\mathfrak{z}) = \frac{1+\mathfrak{z}}{1-\mathfrak{z}} \end{cases}$$

Proof. If $f \in S^c L_{\mathfrak{Z}}(\mathfrak{v})$, then for the schwarz function s with $s(0) = 0$ and $|s(\mathfrak{z})| \le 1$ we have

$$1 + \frac{1}{\mathfrak{v}}\left(\frac{\mathfrak{z}(\Im_{\lambda,\eta,\varsigma} f(\mathfrak{z}))''}{(\Im_{\lambda,\eta,\varsigma} f(\mathfrak{z}))'}\right) = s(\mathfrak{z}) + \sqrt[3]{1 + (s(\mathfrak{z}))^3}. \tag{26}$$

and

$$1 + \frac{1}{\mathfrak{v}}\left(\frac{\mathfrak{z}(\Im_{\lambda,\eta,\varsigma} f(\mathfrak{z}))''}{(\Im_{\lambda,\eta,\varsigma} f(\mathfrak{z}))'}\right) = 1 + \frac{1}{\mathfrak{v}}[2a_2\phi_2\mathfrak{z} + (6a_3\phi_3 - 4a_2^2\phi_2^2)\mathfrak{z}^2 + (12a_4\phi_4 - 18a_2a_3\phi_2\phi_3 + 8a_2^3\phi_2^3)\mathfrak{z}^3 + \ldots]. \tag{27}$$

From equations (15), (26) and (27) , we have

$$a_2 = \frac{c_1\mathfrak{v}}{4\phi_2},$$

$$a_3 = \frac{\mathfrak{v}}{6\phi_3}\left(\frac{c_2}{2} - \frac{c_1^2}{4}(1-\mathfrak{v})\right),$$

On simplyfing and by lemma (2.2),we get that,

$$|a_3 - \delta a_2^2| \le \frac{\mathfrak{v}}{6\phi_3} max\left\{1, |\frac{3\delta\mathfrak{v}\phi_3}{2\phi_2^2} - \mathfrak{v}|\right\}.$$

This completes the proof.

Remark 3.6: *Case(1): If* $p(\mathfrak{z}) = \frac{1+\mathfrak{z}^2}{1-\mathfrak{z}^2}$, *then in this case* $c_1 = c_3 = c_5 = \ldots = 0$ and $c_2 = c_4 = c_6 = \ldots = 2$.

case(2): If $p(\mathfrak{z}) = \frac{1+\mathfrak{z}}{1-\mathfrak{z}}$, then in this case $c_1 = c_2 = c_3 = c_4 = \ldots = 2$.

On taking account of these above cases we get the results of above theorems.

4. Special Cases

Remark 4.1: *If we take* $\lambda=\eta=1,\varsigma=2$ *then* $\phi_2=\frac{1}{2}$ *and* $\phi_3=\frac{1}{3}$, *so that in theorems (3.1), (3.2), (3.3), (3.4), (3.5) we find the corresponding results of Alexander operator* $\Im_{1,1,2}$.

Corollary 4.2: *Let* $\lambda=\eta=1,\varsigma=2$. *If* $f\in S^*L_{\Im}$, *then*

$$|a_3-\delta a_2^2|\le\begin{cases}\frac{3}{2} & if \quad p(\mathfrak{z})=\frac{1+\mathfrak{z}^2}{1-\mathfrak{z}^2}\\ \frac{3}{2}\left|\frac{8\delta}{3}-1\right| & if \quad p(\mathfrak{z})=\frac{1+\mathfrak{z}}{1-\mathfrak{z}}\end{cases}$$

Corollary 4.3: *Let* $\lambda=\eta=1,\varsigma=2$. *If* $f\in RL_{\Im}$, *then*

$$|a_3-\delta a_2^2|\le\begin{cases}1 & if \quad p(\mathfrak{z})=\frac{1+\mathfrak{z}^2}{1-\mathfrak{z}^2}\\ |\delta| & if \quad p(\mathfrak{z})=\frac{1+\mathfrak{z}}{1-\mathfrak{z}}\end{cases}$$

Corollary 4.4: *Let* $\lambda=\eta=1,\varsigma=2$. *If* $f\in S^*L_{\Im}(\mathfrak{v})$, *then*

$$|a_3-\delta a_2^2|\le\begin{cases}\frac{3\mathfrak{v}}{2} & if \quad p(\mathfrak{z})=\frac{1+\mathfrak{z}^2}{1-\mathfrak{z}^2}\\ \frac{3\mathfrak{v}}{2}\left|\left(\frac{8\delta}{3}-1\right)\mathfrak{v}\right| & if \quad p(\mathfrak{z})=\frac{1+\mathfrak{z}}{1-\mathfrak{z}}\end{cases}$$

Corollary 4.5: *Let* $\lambda=\eta=1,\varsigma=2$. *If* $f\in KL_{\Im}(\mathfrak{v})$, *then*

$$|a_3-\delta a_2^2|\le\begin{cases}\mathfrak{v} & if \quad p(\mathfrak{z})=\frac{1+\mathfrak{z}^2}{1-\mathfrak{z}^2}\\ \mathfrak{v}\,|\delta\mathfrak{v}| & if \quad p(\mathfrak{z})=\frac{1+\mathfrak{z}}{1-\mathfrak{z}}\end{cases}$$

Corollary 4.6: *Let* $\lambda=\eta=1,\varsigma=2$. *If* $f\in S^cL_{\Im}(\mathfrak{v})$, *then*

$$|a_3-\delta a_2^2|\le\begin{cases}\frac{\mathfrak{v}}{2} & if \quad p(\mathfrak{z})=\frac{1+\mathfrak{z}^2}{1-\mathfrak{z}^2}\\ \frac{\mathfrak{v}}{2}\,|2\delta\mathfrak{v}-\mathfrak{v}| & if \quad p(\mathfrak{z})=\frac{1+\mathfrak{z}}{1-\mathfrak{z}}\end{cases}$$

Remark 4.7: *If we take* $\lambda=1,\eta=2,\varsigma=3$ *then* $\phi_2=\frac{2}{3}$ *and* $\phi_3=\frac{1}{2}$, *so that in theorems (3.1), (3.2), (3.3), (3.4), (3.5) we find the corresponding results of Bernardi- Libera-Livingston operator* $\Im_{1,2,3}$.

Corollary 4.8: *Let* $\lambda = 1, \eta = 2, \varsigma = 3$. *If* $f \in S^* L_{\mathfrak{z}}$, *then*

$$|a_3 - \delta a_2^2| \le \begin{cases} 1 & \text{if} \quad p(\mathfrak{z}) = \dfrac{1+\mathfrak{z}^2}{1-\mathfrak{z}^2} \\ \left|\dfrac{9\delta}{4} - 1\right| & \text{if} \quad p(\mathfrak{z}) = \dfrac{1+\mathfrak{z}}{1-\mathfrak{z}} \end{cases}$$

Corollary 4.9: *Let* $\lambda = 1, \eta = 2, \varsigma = 3$. *If* $f \in RL_{\mathfrak{z}}$, *then*

$$|a_3 - \delta a_2^2| \le \begin{cases} \dfrac{2}{3} & \text{if} \quad p(\mathfrak{z}) = \dfrac{1+\mathfrak{z}^2}{1-\mathfrak{z}^2} \\ \dfrac{2}{3}\left|\dfrac{27\delta}{32}\right| & \text{if} \quad p(\mathfrak{z}) = \dfrac{1+\mathfrak{z}}{1-\mathfrak{z}} \end{cases}$$

Corollary 4.10: *Let* $\lambda = 1, \eta = 2, \varsigma = 3$. *If* $f \in S^* L_{\mathfrak{z}}(\mathfrak{v})$, *then*

$$|a_3 - \delta a_2^2| \le \begin{cases} \mathfrak{v} & \text{if} \quad p(\mathfrak{z}) = \dfrac{1+\mathfrak{z}^2}{1-\mathfrak{z}^2} \\ \mathfrak{v}\left|\left(\dfrac{9\delta}{4} - 1\right)\mathfrak{v}\right| & \text{if} \quad p(\mathfrak{z}) = \dfrac{1+\mathfrak{z}}{1-\mathfrak{z}} \end{cases}$$

Corollary 4.11: *Let* $\lambda = 1, \eta = 2, \varsigma = 3$. *If* $f \in KL_{\mathfrak{z}}(\mathfrak{v})$, *then*

$$|a_3 - \delta a_2^2| \le \begin{cases} \dfrac{2\mathfrak{v}}{3} & \text{if} \quad p(\mathfrak{z}) = \dfrac{1+\mathfrak{z}^2}{1-\mathfrak{z}^2} \\ \dfrac{2\mathfrak{v}}{3}\left|\dfrac{27\delta\mathfrak{v}}{32}\right| & \text{if} \quad p(\mathfrak{z}) = \dfrac{1+\mathfrak{z}}{1-\mathfrak{z}} \end{cases}$$

Corollary 4.12: *Let* $\lambda = 1, \eta = 2, \varsigma = 3$. *If* $f \in S^c L_{\mathfrak{z}}(\mathfrak{v})$, *then*

$$|a_3 - \delta a_2^2| \le \begin{cases} \dfrac{\mathfrak{v}}{3} & \text{if} \quad p(\mathfrak{z}) = \dfrac{1+\mathfrak{z}^2}{1-\mathfrak{z}^2} \\ \dfrac{\mathfrak{v}}{3}\left|\dfrac{27\delta\mathfrak{v}}{16} - \mathfrak{v}\right| & \text{if} \quad p(\mathfrak{z}) = \dfrac{1+\mathfrak{z}}{1-\mathfrak{z}} \end{cases}$$

5. Conclusion

In conclusion, by extending the Fekete-Szegö inequality to a wider class of holomorphic functions namely, starlike, bounded turning, and close-to-convex functions of complex order, this work advances our knowledge of the inequality. We have developed new inequalities that generalize classical conclusions by taking into account leaf-like domains and the Hohlov operator. Our comprehension of how these functions behave in complex analysis and geometric function theory is enhanced by these discoveries. Moreover, our work analyzes particular examples, such the Gaussian Hypergeometric function, and gives tight limitations on the coefficients, providing useful information for further study.

This study provides additional avenues for future research by applying the Fekete-Szegö inequality to these broader groups of functions. The applications of these findings in other branches of mathematics, such conformal mapping and geometric function theory, may be examined in future studies. Furthermore, it may be possible to investigate whether similar disparities apply to even more generalized operators or domains in future research, which could increase the study's reach and significance.

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