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Derivation of Sawada-Kotera and Kaup-Kupershmidt equations KdV flow Equations from Derivative Nonlinear Schrödinger equation (DNLS)

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Abstract

The mathematical models of problems that arise in almost every branch of science are nonlinear equations of evolution (NLEE). In the past years, equations of formation have gained a significant place in applied mathematics. This study is about the multiple scales method, known as the perturbation method, for the derivative nonlinear Schrödinger (DNLS) equation. In this report, the multiple scales method was applied for the analysis of the derivative nonlinear Schrödinger (DNLS) equation. And (1 + 1) dimensional fifth-order nonlinear Korteweg-de Vries (fKdV) type equations were obtained. So, we have demonstrated the relationship between the KdV equations and the DNLS-type equation.

Keywords: Multiple scales method, Sawada-Kotera equation, Kaup-Kupershmidt equation, DNLS type equations.

Mathematical Subject Classification: 34D10, 34E13, 35Q53, 35Q55, 37K10

1. Introduction

Although the word wave generally refers to the shapes formed on the water surface in daily life, there are many areas where the wave finds its place. Nonlinear waves appear as partial differential equations that characterize wave propagation in many areas of physics such as dispersive wave equations, fluid mechanics, elasticity theory, nonlinear optics, and plasma physics. Nonlinear evolution

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equations serve as potent mathematical instruments for modeling intricate phenomena across diverse disciplines encompassing physics, engineering, and biol ogy [1]. By delving into the resolution of these equations, researchers can glean profound insights into the fundamental mechanisms and dynamics underpinning these phenomena. Since data on their exact solutions facilitates the confirmation of numerical solvers and supports in decisiveness analysis of solutions, the analytical study of these NLEE is significant. This not only helps to better understand the solutions and also helps us to understand the phenomenon they describe [2–4].

The following nonlinear Schrödinger-type equation:

$$iq_{\tau} + q_{\xi\xi} + i\alpha(q|q|^2)_{\xi} + 2\delta q|q|^2 = 0, \tag{1}$$

the equation including the nonlinear dispersion; for temporal pulses in optical fibers, q is the field amplitude, τ is the propagating distance, ξ is the time measured in a frame moving with the group velocity, δ and α are constants representing the relative magnitudes of the nonlinear dispersion term and the nonlinear term, is suggested to define the short pulse propagation in long singlemode optical fibers, taking into account the connatural characteristic of the asymmetric output pulse spectrum [5–8]. Since Eq. (1) is reduced to the conventional nonlinear Schrödinger equation (NLS), this equation is known as the modified nonlinear Schrödinger (MNLS) equation (see, for example, [9]), or mixed NLS-DNLS equation (see, for example, [10]):

$$iq_{\tau} + q_{\xi\xi} + 2\delta q|q|^2 = 0, \tag{2}$$

when $\alpha = 0$ and the derivative is nonlinear Schrödinger equation (DNLS):

$$iq_{\tau} + q_{\xi\xi} + i\alpha(q|q|^2)_{\xi} = 0, \tag{3}$$

when $\delta = 0$ which occurs in different physical context, [11], represents well known integrable systems.

The stability of solutions which are alongside the integrability of such systems peculiar to such equations is thought to be owing to a fine equilibrium among their linear dispersive and nonlinear collapsing terms. And so, in some physical conditions, demanding the addition of higher nonlinear terms to (2) or (3) [12–17], this equilibrium is seemingly lost and the system not become integrable in a way that never allows for analytical solutions [18–20]. The DNLS equation (3) was proposed by Register and Ruderman to describe nonlinear Alfven waves in the plasma field of physics [21, 22]. In the vanishing boundary case, the equation was solved by inverse scattering transform (IST) [23], or other approximation ways [24–26]. Also, its complete integrability was demonstrated by Kundu throughout the r-s matrix formalism [27]. In the non-vanishing boundary case, the DNLS equation (3) has been discussed by the researchers under ordinary spectral parameter conditions [28, 29]. It then appears in the multi-valued problem of the square root.

Obtaining exact or approximate solutions of nonlinear partial differential equations in mathematics and physics remains a significant problem since new approaches are required to identify their exact or approximate solutions. Because the majority of novel nonlinear equations lack an exact analytical solution, numerical methods have been widely employed to address them. Recently, many authors have used various methods to examine solutions of nonlinear partial differential equations due to the existence of analytical methods for nonlinear equations [30–32]. Moreover, mathematicians have been heavily involved in researching Partial Differential Equations and Fractional Partial Differential Equations to investigate numerical, analytical, travelling, and soliton solutions. Among these investigations, solitons in the setting of Nonlinear Fractional Partial Differential Equations have long captivated physicists and mathematics professionals alike [33–36].

It is renowned that a multiscale analysis of the Korteweg-de Vries (KdV) equation (and indeed a wide diversity of equations) gives rise to the nonlinear Schrödinger (NLS) equation for modulated amplitude [37–41]. Zakharov and Kuznetsov demonstrated that analyzing the Schrödinger spectral problem on multiscale takes to the Zakharov-Shabat problem for the NLS equation. It is generally understood that doing a multi-scale analysis on the Korteweg-de Vries (KdV) equation and many other equations yields the nonlinear Schrödinger (NLS) equation for modulated amplitudes [37]. It was shown by Zakharov and Kuznetsov that multiscale analysis of the Schrödinger spectral problem leads to the Zakharov-Shabat problem for the NLS equation. With this analysis, it has been shown that there is a deeper relationship between the integrable equations, not only at the level of the equations, but also at the level of the linear spectral problems. In this article, we derive the KdV flow equations from the DNLS equation by applying the multi-scale method. This is an important derivation as it is derived from the DNLS equation. When we compare our derivative with the KdV-DNLS derivative, the equations for coefficients of all orders in the epsilon do not contain secular terms. Therefore, there is no freedom in choosing the coefficient and the expansion is uniquely determined. The derivation of this hierarchy is not a simple case of algorithms, but basically relies on two facts. First, different time flows must commute; i.e.: $q_{t_it_j} = q_{t_jt_i}$. Second, In each order in epsilon, the coefficient equations contain secular terms. Eliminating the secular terms requires q_{t_i} to have a certain high symmetry (flow) of the hierarchy, and this way all coefficients of expansion are fixed and no arbitrariness is left. This analysis has revealed a deeper connection between integrable equations, not just at the level of the equations but also at the level of the linear spectral problems. In recent years, Maccari has applied a similar approach to nonlinear evolutionary equations and their Lax pairs [42–44].

The paper has been organized in the following manner. In Section 2, we expressed shortly the fifth order KdV equaton (KdV5) flow equations and in section 3, we derive (KK equation) and (SK equation) as well as KdV flow equations by applying the method to the DNLS equation. Last section is given conclusions. We extensively used Reduce to calculate our results in the paper.

2. Background Materials

In this section, we provide some background information on the most well-known fifth-order KdV equations as well as the multi-scale method.

2.1. The fifth order Korteweg-de Vries Equation (fKdV) Flow Equations:

The best-known fifth-order KdV equations look like this

$$u_t = \omega u_{xxxxx} + \alpha u_{xxx} + \beta u_x u_{xx} + \gamma u^2 u_x. \tag{4}$$

where α, β, γ and ω are arbitrary nonzero and real parameters, and $u = u(x, t)$ is a smooth enough function. Since the parameters α, β, γ and ω are arbitrary and take different values, this will greatly change the properties of the fKdV equation (4). Changing the actual values of the parameters allows you to generate many different variations of the fKdV equation. The fKdV equations, which are widely used in nonlinear optics and quantum physics, are an important mathematical model. Characteristic examples are commonly utilized in many domains, including plasma physics, quantum field theory, solid-state physics, and fluid physics [45, 54].

There are some significant special cases of the Eq. (4), which are :

Kaup-Kupershmidt (KK) equation [49–53]

$$ut = u_{xxxxx} + 10uu_{xxx} + 25u_x u_{xx} + 20u^2 u_x, \tag{5}$$

Sawada-Kotera (SK) Eq. [46, 55]

$$ut = u_{xxxxx} + 5uu_{xxx} + 5u_x u_{xx} + 5u^2 u_x, \tag{6}$$

Caudrey-Dodd-Gibbon (CDG) Eq.

$$ut = u_{xxxxx} + 30u_{xxx} + 30u_x u_{xx} + 180u^2 u_x, \tag{7}$$

Lax equation [45]

$$ut = u_{xxxxx} + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x, \tag{8}$$

Ito equation [47, 48]

$$ut = u_{xxxxx} + 3u_{xxx} + 6u_x u_{xx} + 2u^2 u_x, \tag{9}$$

The equation (4) has varying characteristics as the values of α, β, and γ change. For example, the KK equation with $\alpha = 10$, $\beta = 25$, and $\gamma = 20$, which is known to be integrable [51], and has bilinear representations [51, 53], but the obvious form of the N-soliton solutions is unknown. Another example is the SK equation where $\alpha = \beta = \gamma = 5$, and the Lax equation with $\alpha = 10$, $\beta = 20$, and $\gamma = 30$, are both fully integrable. These equations have an endless set of conserved densities and N-soliton solutions. The last equation is the Ito equation, with $\alpha = 3$, $\beta = 6$, and $\gamma = 2$, which cannot be fully integrated but has limited of special conserved densities [48].

2.2. The Multiple Scales Method:

We derive the KdV flow equations from the DNLS equation (3) using the multiscale method developed by Zakharov and Kuznetsov [37].

Now let's consider the DNLS equation (3) and search for a solution by separating phase and amplitude as shown below:

$$q(\xi,\tau) = e^{i\theta(\xi,\tau)}\sqrt{N(\xi,\tau)},\ q^*(\xi,\tau) = e^{-i\theta(\xi,\tau)}\sqrt{N(\xi,\tau)} \tag{10}$$

Adding this default solution to the DNLS equation (3) and grouping the real and imaginary parts, respectively, we obtain the following partial differential equations:

$$\begin{aligned} N_\tau &= -2N_\xi\theta_\xi - 2\theta_{\xi\xi}N_\xi + 3\alpha N_\xi N, \\ \theta_\tau &= \frac{N_{\xi\xi}}{2N} - \frac{N_\xi^2}{4N^2} - \theta_\xi^2 + \alpha\theta_\xi N \end{aligned} \tag{11}$$

If we take $\theta(\xi,\tau)_\xi = V(\xi,\tau)$ then we find a particular case of (11) as a system:

$$\begin{aligned} N_\tau &= -2(NV)_\xi + 3\alpha N_\xi N, \\ V_\tau &= \left(\frac{N_{\xi\xi}}{2N} - \frac{N_\xi^2}{4N^2}\% - V^2 + \alpha VN\right)_\xi . \end{aligned} \tag{12}$$

Then we suppose the following series expansions for solutions (12)

$$\begin{aligned} \theta &= 2\tau + \sum_{n=1}^{\infty}\varepsilon^{2n-1}\theta_n(x,t_1,t_2,\ldots,t_n), \\ N &= 1 + \sum_{n=1}^{\infty}\varepsilon^{2n}N_n(x,t_1,t_2,\ldots,t_n), \\ V &= \sum_{n=1}^{\infty}\varepsilon^{2n}V_n(x,t_1,t_2,\ldots,t_n). \end{aligned} \tag{13}$$

We also define slow variables for the scaling parameter $\varepsilon > 0$ respectively as follows:

$$x = \varepsilon(\xi + 2\tau),\quad t_n = \varepsilon^{2n+1}\tau,\ n = 1,2,\ldots \tag{14}$$

Now we insert series expansions (13) with equations (14) into the system (12). Then we equate the coefficients in the ε powers individually to zero. Thus we get an infinite set of equations for N_n in the powers of ε for each n. By allowing $\varepsilon \to 0$ and zeroing the terms in the minimum powers of ε, we can obtain the following for the case $n \geq 1$:

(i) For the coefficients of ε^3, we find

$$\begin{aligned} &-3\alpha N_{1x} + 2V_{1x} = 0, \\ &2\alpha N_{1x} - 2\alpha V_{1x} = 0, \end{aligned} \tag{15}$$

(ii) For the coefficient of ε^5, we find

$$\begin{aligned} &-3\alpha N_{2x} - 3\alpha N_{1x}N_1 + 2N_{1x}V_1 + 2N_{1t_1} + 2V_{2x}\% + 2V_{1x}N_1 = 0, \\ &2\alpha N_{2x} - N_{1xxx} + 6\alpha N_{1x}N_1 - 2\alpha N_{1x}V_1 - 2\alpha V_{2x} \\ &-8\alpha V_{1x}N_1 + 4V_{1x}V_1 + 2V_{1t_1} = 0, \end{aligned} \tag{16}$$

(iii) For the coefficients of ε^7, we find

$$\begin{aligned} &-3\alpha N_{3x} - 3\alpha N_{2x}N_1 + 2N_{2x}V_1 + N_{2t_1} - 3\alpha N_{1x}N_2 + 2N_{1x}V_2 \\ &+N_{1t_2} + 2V_{3x} + 2V_{2x}N_1 + 2V_{1x}N_2 = 0, \\ &2\alpha N_{3x} - N_{2xxx} + 6\alpha N_{2x}N_1 - 2\alpha N_{2x}V_1 - 2N_{1xxx}N_1 + 2N_{1xx}N_{1x} \\ &+6\alpha N_{1x}N_2 + 6\alpha N_{1x}N_1^2 - 6\alpha N_{1x}N_1V_1 - 2\alpha N_{1x}V_2 - 8\alpha V_{2x}N_1 \\ &+4N_{2x}V_1 + 2V_{2t_1} - 8\alpha V_{1x}N_2 - 12\alpha V_{1x}N_1^{\ 2} + 12V_{1x}N_1V_1 \\ &+4V_1V_2 + 6V_{1t_1}N_1 + 2V_{1t_2} = 0. \end{aligned} \tag{17}$$

(iv) For the coefficients of ε^9, we find:

$$\begin{aligned} &-3\alpha N_{4x} - 3\alpha N_{3x}N_1 + 2N_{3x}V_1 + N_{3t_1} - 3\alpha N_{2x}N_2 + 2N_{2x}V_2 + N_{2t_2} \\ &-3\alpha N_{1x}N_3 + 2N_{1x}V_3 + N_{1t_3} + 2V_{4x} + 2V_{3x}N_1 + 2V_{2x}\%N_2 + 2V_{1x}N_3 = 0, \\ &2\alpha N_{4x} - N_{3xxx} + 6\alpha N_{3x}N_1 - 2\alpha N_{3x}V_1 - 2N_{2xxx}N_1 + 2N_{2xx}N_{1x} \\ &+2N_{2x}N_{1xx} + 6\alpha N_{2x}N_2 + 6\alpha N_{2x}N_1^2 - 6\alpha N_{2x}N_1V_1 - 2\alpha N_{2x}V_2 \\ &-2N_{1xxx}N_2 - N_{1xxx}N_1^2 + 2N_{1xx}N_{1x}N_1 - N_{1x}^3 + 6\alpha N_{1x}N_3 \\ &+12\alpha N_{1x}N_2N_1 - 6\alpha N_{1x}N_2V_1 + 2\alpha N_{1x}N_1^{\ 3} - 6\alpha N_{1x}N_1^2V_1 \\ &-6\alpha N_{1x}N_1V_2 - 2\alpha N_{1x}V_3 - 2\alpha V_{4x} - 8\alpha V_{3x}N_1 + 4V_{3x}V_1 \\ &+2V_{3t_1} - 8\alpha V_{2x}N_2 + 12V_{2x}N_1V_1 - 12\alpha V_{2x}N_1^{\ 2} + 4V_{2x}V_2 \\ &+6V_{2t_1}N_1 + 2V_{2t_2} - 24\alpha V_{1x}N_2N_1 - 8\alpha V_{1x}N_3 + 12V_{1x}N_2V_1 \\ &-8\alpha V_{1x}N_1^3 + 12V_{1x}N_1^2V_1 + 12V_{1x}N_1V_2 + 4V_{1x}V_3 + 6V_{1t_1}N_2 \\ &+6V_{1t_1}N_1^2 + 6V_{1t_1}N_1 + 2V_{1t_3} = 0. \\ &\vdots \end{aligned} \tag{18}$$

and so on. Now taking $\alpha = \dfrac{2}{3}$ in (15), we find

$$N_1 = V_1, \tag{19}$$

by assuming integration constants as zero.

3. The Derivation of KdV Flow Equations

We now use (19) in the system (16) and take

$$N_2 = V_2 + \frac{1}{4}\left(\frac{3}{4}V_{1xx} + V_1^2\right), \tag{20}$$

so that we find the following equation

$$V_{1t_1} = \frac{3}{8}V_{1xxx} - V_1V_{1x}, \tag{21}$$

or making the transformation

$$t_1 \to \frac{3}{8}t_1, \quad V_1 \to -\frac{9}{4}u$$

we derive the well known KdV equation

$$u_{t1} = u_{xxx} + 6uu_x. \tag{22}$$

If we take in the equation (20) for V_2 as

$$V_2 = k_1V_1{}^2 + k_2V_{1xx}, \tag{23}$$

then add this to the equation (17), we get the equation

$$\begin{aligned} V_{1t_2} = \frac{1}{128}\Big(&256(N_{3x} - V_{3x}) - (96k_1 + 128k_2)V_{1xxx}V_1 - \\ &(48k_2 + 9)V_{1xxxxx} + (128k_2 + 72)V_{1xx}V_{1x} + (-512k_1 + 64)V_{1x}V^2\Big). \end{aligned} \tag{24}$$

Now chosing

$$N_3 = V_3 + \frac{1}{10240}(-567V_{1xxxx} - 1944V_{1xx}V_{1x} + 648V_{1x}{}^2 + 230V_1^3), \tag{25}$$

with

$$k_1 = \frac{9}{20}, \quad k_2 = -\frac{399}{640},$$

as a result from the equation (24),

$$V_{1t_2} = \frac{27}{512}(V_{1xxxxx} - \frac{16}{9}V_{1xxx}V_1 - \frac{32}{\%}9V_{1xx}V_{1x} + \frac{128}{135}V_{1x}V^2).$$

the equation is obtained and using an appropriate transformation for $t_2 \to \frac{27}{512}t_2$, $V_1 \to -\frac{45}{8}u$ we derive the Lax's fifth order KdV equation

$$u_{t_2} = \left(u_{xxxx} + 10uu_{xx} + 5u_x^2 + 10u^3\right)_x. \tag{26}$$

as t_2 KdV flow equation.

3.1. The Derivation of Kaup-Kupershmidt equation:

If we take in the equation (20) for V_2 as (23) then add this to the equation (17), we get the equation (24). Now chosing equation (25)
with

$$k_1 = \frac{7}{15}, \quad k_2 = -\frac{213}{320},$$

as a result from the equation (24),

$$V_{1t_2} = \frac{27}{512}(V_{1xxxxx} - \frac{16}{9}V_{1xxx}V_1 - \frac{40}{9}9V_{1xx}V_{1x} + \frac{256}{405}V_{1x}V^2).$$

the equation is obtained and using an appropriate transformation for $t_2 \to \frac{27}{512}t_2$, $V_1 \to -\frac{45}{8}u$ we derive the Kaup-Kupershmidt equation

$$u_t = u_{xxxxx} + 10uu_{xxx} + 25u_xu_{xx} + 20u^2u_x, \tag{27}$$

as t_2 KdV flow equation.

3.2. The Derivation of Sawada-Kotera equation:

If we take in the equation (20) for V_2 as (23) then add this to the equation (17), we get the equation (24). Now chosing equation (25)
with

$$k_1 = \frac{7}{15}, \quad k_2 = -\frac{381}{640},$$

as a result from the equation (24),

$$V_{1t_2} = \frac{27}{512}(V_{1xxxxx} - \frac{16}{9}V_{1xxx}V_1 - \frac{16}{\%}9V_{1xx}V_{1x} + \frac{256}{405}V_{1x}V^2).$$

the equation is obtained and using an appropriate transformation for $t_2 \to \frac{27}{512}t_2$, $V_1 \to -\frac{45}{16}u$ we derive the Sawada-Kotera equation

$$u_t = u_{xxxxx} + 5uu_{xxx} + 5u_x u_{xx} + 5u^2 u_x, \tag{28}$$

as t_2 KdV flow equation.
We now insert (23) and (25) into the system (18) and choose

$$V_3 = k_3 V_{1xxxx} + k_4 V_{1xx} V_{1x} + k_5 V_{1x}^2 + k_6 V_1^3, \tag{29}$$

we finally get from the coefficients of ε^9, the seventh order KdV flow equation

$$\begin{aligned} u_{t_3} &= u_{xxxxxxx} + 14uu_{xxxxx} + 42u_x u_{xxxx} + 70u_{xx}u_{xxx} \\ &+70u^2 u_{xxx} + 280uu_x u_{xx} + 70u_x^3 + 140u^3 u_x \end{aligned} \tag{30}$$

where we take

$$k_1 = \frac{39}{112}, k_2 = \frac{3(\sqrt{7009} - 535)}{3584}, k_3 = -\frac{9(\sqrt{7009} + 494377)}{12845056},$$
$$k_4 == \frac{3(7\sqrt{7009} - 10048)}{25088}, k_5 = \frac{3(17\sqrt{7009} - 53993)}{75264}, k_6 = \frac{1137}{3136}$$

and make an appropriate transformation $t_3 \to \frac{27}{16384}t_3$. Generally, extending this proceeding the calculation as before, we obtain all equations of KdV flow equations

$$R\left[u\right] = \partial^2 + 4u + 2u_x \partial^{-1} \tag{31}$$

the KdV equation the hierarchy recursion operator and

$$u_{\tau_{2n+1}} = R^n\left[u\right]u_\xi = K_{2n+1}\left[u\right],\ n = 1,2,3,... \tag{32}$$

the infinite hierarchy of mutually commuting flows satisfies relations.

4. Conclusions

We have used a multiple scales method to provide a new derivation of the KdV flow equations from the DNLS equation. The equations for the coefficients at each order in epsilon, contain no secular terms in our derivation of KdV flow equations. Therefore no freedom is left in choosing coefficients at each order in epsilon and the expansion is uniquely determined. Thus, a relation exists between the DNLS and the KdV flow equations. In conclusion, the methodology presented in this paper shows significant potential for application to many nonlinear equations of evolution (NLEE) in mathematical physics. It opens up opportunities for further research and applications in various scientific and engineering disciplines.

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