Results in Nonlinear Analysis 8 (2025) No. 1, 41–59
https://doi.org/10.31838/rna/2025.08.01.006
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

Qualitative and numerical analysis to a time-fractional Stefan convection-diffusive model using Riemann-Liouville and Caputo operators

Habeeb A. Aal-Rkhais, Raeed A. Al-Khafaji, Ahmed H. Kamil

Department of Mathematics, University of Thi-Qar, Thi-Qar, Iraq.

This paper introduces a model of the fractional Stefan Problem(SP) to linear convection and diffusion forces. We assess the memory effect in the studied model. Rescaling technique is used to reduce the fractional PDEs under some restrictions. The fractional derivative method, particularly Riemann-Liouville and Caputo derivatives, is used to derive the given model. The self-similar solutions and the interface function are estimated to the time fractional SP model including boundary and Stefan conditions. In addition, we estimate an approximated numerical solution the time-fractional convection-diffusion equation by applying the Sumudu decomposition method (SDM). The proposed method depends on applying the Sumudu transform of the Caputo fractional derivative operator and then using the fractional integral of Riemann-Liouville. Furthermore, a special solution to this model is valid and applicable.

Key words and phrases: Caputo derivative, Convection- Diffusion equation Riemann-Liouville derivative self-similar solution Stefan problem, Sumudu Transform Time-fractional derivatives.

Mathematics Subject Classification (2010): 35R11, 46F12, 65R10

1. Introduction

A partial differential equation that describes the heat dispersion in a phase-changing material is called a Stefan Problem(SP). A significant divergence of the Stefan Problem between pressure-driven liquid infiltration into a gas-filled Hele-Shaw cell and the porous medium is discussed in [1, 2].

Email addresses: habeebk@utq.edu.iq (Habeeb Abed Kadhim Aal-Rkhais); raeed.abdulhadi.salih@utq.edu.iq (Raeed Abdul Hadi Al-Khafaji); ahmedhameed1992@utq.edu.iq (Ahmed Hameed Kamil)

Received October 12, 2024; Accepted November 28, 2024; Online December 30, 2024

For the SPs mentioned above, there are two different ways to find appropriate governing equations. The liquid fraction shows a leap discontinuity across the melt interface in the conventional method, [3], also known as the sharp-interface model. In the second method, a diffusive interface with a finite thickness separates the solid and liquid. The enthalpy technique [1, 4] and the phase-field approaches [5–7] are instances of this latter approach. The change in liquid fraction that occurs over a temperature range and implicitly defines an interface thickness is assumed to be the enthalpy method. Here, the subject of concern is how the introduction of memory into a system modifies the phrasing of problem. The focus should be on determining whether the diffusive interface approach to the sharp interface remains true in the presence of a vanishing it's thickness when memory is taken into consideration in the flux formulation, by looking at the derivation of both models of a Stefan melting problem. Tracking the solid-liquid interface while a pure solid melts at the phase transition temperature is the quintessential example of a SP. An originally isolated domain Ω is involved in an appropriate geometry.

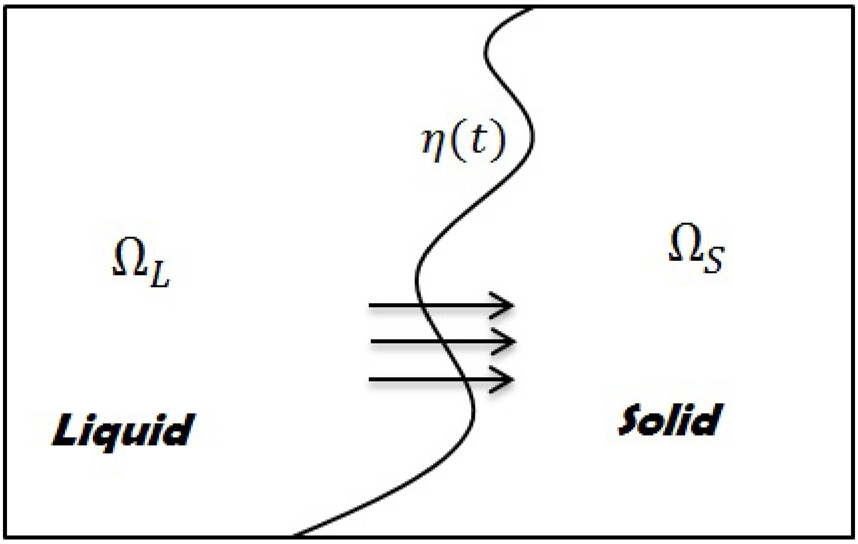

Figure 1: The model of melting problem with the (diffusive–sharp) interface function.

As this process proceeds in a physical setting, the liquid phase Ω_L and the solid phase Ω_S will be separated by a well-defined moving interface $\eta(t)$. The rate at which the solid can accumulate the latent heat L needed for melting governs the motion of this interface. According to the sharp-interface model, the position and speed of the contact are directly linked to the latent heat memory, which is lumped. In this research, we aim to investigate the process of medium-phase transition, where non-local in-time effects are observed in the diffusion with advection force. Our motivation came from the studies in [8–10], in which we assume that the time-fractional Riemann-Liouville derivative of the temperature gradient represents the non-locality in time and the advective-diffusive flux

$$F(x,t) = \partial^{1-\alpha}\left(\lambda u(x,t) - u_x(x,t)\right) \tag{1}$$

where $\lambda = \beta\,\Gamma(\alpha)$ is the plasma velocity. We estimate and derive the time-fractional SP for convection-diffusion equation based on the assumption in Eq.(1). The fractional Caputo derivative is used in place of the time derivative to characterize the sharp-interphase model that was produced in [7]. This problem has been attempted to be solved on multiple occasions in the past few years. The existence of weak solutions in non-cylindrical domains with defined bounds was demonstrated by the authors in [10]. In [11] the Hopf lemma was proved with appropriate regularity assumptions. It's also important to note the publication [12] where the unique solution to this issue was discovered. In [13], the convergence of the diffusive interface does not approach to the sharp results. This prompts researchers to look closely at Stefan's fractal model of time, see [14–16]. Other alternative formulations of time-fractional SP have been considered in [17, 18]. Under the least regularity assumptions, we develop in this study the sharp results with non-local flux provided by Eq.(1). The problems involving phase change over the time are typically nonlinear in nature, so there is considerable interest in finding analytical solutions that found in [19], but we also get more boundary conditions. Finally, we discover a self-similarity solutions to the problem that did not considered carefully in [20]. Let us consider the domain $\Omega = (0,l)$ in $\mathbb{R}$. Let $t = 0$, and Ω can be divided into $\Omega_L = (0, x_0)$ which is a liquid domain and $\Omega_S = (x_0,l)$ is a solid domain. The enthalpy function $E = \rho cu + \rho H\phi$ is defined from [8, 9], where $u = (x,t)$ is the temperature at point $x \in \Omega$ and time t,ϕ represents the latent heat, ρ is the density, c is the volumetric specific heat and H is the latent heat of fusion. To assume that ϕ has the following formula

$$\phi = \begin{cases} 1 & \text{in } \Omega_L, \\ 0 & \text{in } \Omega_S. \end{cases} \tag{2}$$

$F(x,t)$ is defined as the flux at $x \in \Omega$ and time t. Let $V = (a,b) \subseteq \Omega$ and define the energy conservation has the formula

$$\frac{\partial}{\partial t}\int_V E(x,t)dx = F(a,t) - F(b,t) \text{ for all } t > 0. \tag{3}$$

From the Mean Value Theorem [21], the equation Eq.(3) becomes

$$u_t = -\beta u_x + u_{xx} \quad x \in \Omega - \{\eta(t)\},\ t > 0. \tag{4}$$

and

$$\eta'(t) = \beta u(\eta(t),t) - u_x^-(\eta(t),t), \quad t > 0. \tag{5}$$

with $\eta = \eta(t)$ represents an interface function such that

$$\beta u(\eta(t),t) - u_x^-(\eta(t),t) = \lim_{\epsilon \to 0^+}\left[\beta u(\eta(t)-\epsilon,t) - u_x(\eta(t)-\epsilon,t)\right].$$

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-\zeta)^{\alpha-1} f(\zeta)d\zeta. \tag{6}$$

The Riemann-Liouville fractional derivative(RLFD) and Caputo fractional derivative (CFD) are introduced respectively as follows

$$\partial_a^\alpha f(x) = \frac{d}{dx} I_a^{1-\alpha} f(x), D_a^\alpha f(x) = \frac{d}{dx} I_a^{1-\alpha}\left[f(x) - f(0)\right].$$

From [21, 22], Assuming that the RLFD with regard to the time variable provides the flux for $\alpha \in (0,1)$. In Figure (1), $\eta(t)$ represents the phase interface, decomposing Ω on the liquid and solid parts. Let $V_S(t) = V \cap \Omega_S(t)$, and $V_L(t) = V \cap \Omega_L(t)$; then, (3) takes the form

$$\frac{d}{dt}\int_{V_L(t)}(u(x,t)+1)dx + \frac{d}{dt}\int_{V_S(t)} u(x,t)dx = \partial^{1-\alpha}\left(\lambda u(a,t) - u_x(a,t)\right) - \partial^{1-\alpha}\left(\lambda u(b,t) - u_x(b,t)\right), \tag{7}$$

To integrate w.r.t. time and assume that the gradient is bounded, we get

$$\int_{V_L(t)}(u(x,t)+1)dx - \int_{V_L(0)}(u(x,0)+1)dx + \int_{V_S(t)} u(x,t)dx - \int_{V_S(0)} u(x,t)dx$$
$$= \partial^{1-\alpha}\left(\lambda\left(u(a,t) - u(b,t)\right)\right) - \partial^{1-\alpha}\left(u_x(a,t) - u_x(b,t)\right). \tag{8}$$

The total enthalpy is added to the starting enthalpy plus the time average of the variations in local fluxes at the endpoints of V. Using the diffusive flux supplied by Eq.(1), we obtain the fractional SP from the stationary case Eq.(3). We must impose specific regularity constraints on the temperature function $u = (x,t)$ and the phase interface $\eta(t)$ in order to perform so rigorously. Initially, we take t^* to be positive.

$$\eta(t) \in AC[0,t^*], \quad u_x(x,\cdot) \in L^\infty(D^x), \quad \forall x \in \Omega$$
$$u_x(\cdot,t) \in AC[0,\eta(t)-\epsilon], \quad \text{for every } \epsilon > 0 \text{ and } t \in (0,t^*) \text{(Hyp.1)}$$
$$u_t(\cdot,t) \in L^1(0,\eta(t)), \quad \text{for every } t \in (0,t^*)$$

where we denote $\Omega_{(\eta,t^*)} = \{(x,t): 0 < x < \eta(t),\ t \in (0,t^*)\}$, and $D^x = \{t : (x,t) \in \Omega_{(\eta,t^*)}\}$

The initial-boundary conditions for the SP in standard setting are considered as follows $u(x,0) = u_0(x) \geq 0$, and $u(0,t) = u_D(t) \geq 0$ or $u_x(0,t) = u_N(t) \leq 0$.

In particular, if $u_0, u_D \equiv 0$, or $u_0, u_N \equiv 0$ then $u \equiv 0$. Otherwise, we expect

$$\eta'(t) > 0 \text{ (Hyp.2)}$$

That denotes the solid state melting. We observe that the temperature in the solid disappears when we take the one-phase SP into consideration. Thus, the flux is provided by the formula and is nonzero only in the liquid portion of the domain, that is, in $\Omega_{(\eta,t^*)}$.

$$F(x,t) = \begin{cases} \partial^{1-\alpha}_{\eta^{-1}(x)} \left(\lambda u(x,t) - u_x(x,t)\right) & \text{for } (x,t) \in \Omega_{(\eta,t^*)}, \\ 0 & \text{for } (x,t) \notin \Omega_{(\eta,t^*)}. \end{cases} \tag{9}$$

Equations (10) and (2-3) imply that

$$\frac{d}{dt}\int_{V_L(t)} \left(u(x,t)+1\right) dx = F(a,t) - F(b,t). \tag{10}$$

To get an advantage of the last regularity assumptions, we consider

$$\eta'(t) \in L^\infty_{\text{loc}}([0,t^*]) \text{ and } D^\alpha_{\eta^{-1}(x)} u(\cdot,t) \in L^1(0,\eta(t)) \text{ for } t \in (0,t^*), \text{ (Hyp.3)}$$

where,

$$u(x,t) = \frac{d}{dt} I^{1-\alpha}_a \left[u(x,t) - u(\eta^{-1}(x),t)\right]$$

Now, we will derive the SP which represents the first result of this study.

2. Derivation the Fractional SP

The mathematical equation that describes the diffusion difficulties has attracted the attention of numerous academics over the years. Cherniha and Serov [23] were giving the non-linear diffusion equations a fresh analysis and accurate solution. New modification equations were derived by Kuske and Mileniski [24] for the hexagon-style in reaction-diffusion systems. These systems exhibit more non-linearities than Smith-Hohenberg models or Rayleigh-Bernard convection. Matano et al. [25] investigated the interaction and diffusion equations using the spatially heterogeneous interaction term. If this reaction's term coefficient is far higher than the dispersion coefficient, the strong interface between two separate phases will be visible. They demonstrated that the motion equation for this interface includes a drift term even though drift was absent from the original diffusion equations. The researchers in [26] investigated the uniqueness and existence of the solution to the self-similarity of diffusion equation. A study was performed to examine fast gas flow models by heating various materials using a microwave and by porous media. A nonlinear diffusion-advection-reaction model is being investigated in [27] and [5, 6]. Fractional derivatives offer more accurate representations of real-world issues than integer-order derivatives. Because of their numerous uses in science, fractional PDEs have proven to be a useful tool for describing the diffusion processes [27], viscoelasticity and electrical phenomena [28]. The Fractional PDE was discovered that, when taken along the time scaling limit, fractional time derivatives typically appear as infinitesimal generators of the time evolution. Therefore, the need to clarify the ideas of equilibrium, stability states, and temporal evolution at the long-term limit justifies the significance of looking into fractional equations. To find the approximate solutions, a variety of effective approaches have been presented for solving fractional partial differential equations. The double Laplace formulas for partial fractional derivatives were developed by the authors in [29], and they can be used to solve a fractional heat equation under specific initial and boundary conditions. They use the fractional complex transform approach in [30] to study the transport equations in fractal porous media.

The Fractional Stefan Problem for the Diffusion Equation is a mathematical model that extends the classical Stefan problem, incorporating fractional calculus. The classical Stefan problem involves the phase change, like ice melting into water, and is governed by a diffusion equation with a moving boundary where the phase change occurs. The key concepts of the SP includes Classical SP that involves the heat equation, a type of diffusion equation, which models heat transfer in a medium. The moving boundary, known as the Stefan boundary, separates different phases (e.g., solid and liquid). The position of this boundary is part of the solution. The problem is typically defined by partial differential equations with boundary conditions that include a latent heat term. Another concept is fractional calculus that extends the concept of integer-order derivatives and integrals to non-integer (fractional) orders. This allows for more accurate modeling of processes that exhibit memory or hereditary properties. In the context of diffusion, the fractional derivative can describe anomalous diffusion where the rate of diffusion varies differently from the classical (integer-order) case. Moreover, the concept of Fractional SP combines the classical Stefan problem with fractional derivatives, leading to a fractional diffusion equation. The time fractional derivative is often a Caputo derivative or a Riemann-Liouville derivative, which generalizes the time evolution in the diffusion process. This fractional approach allows modeling of more complex phase-change processes that may not follow standard diffusion laws, capturing effects like non-locality and long-term memory. The boundary conditions and the equation for the moving boundary (Stefan condition) are modified to incorporate the fractional nature of the diffusion process. The applications of Fractional Stefan Problem is relevant in areas such as the freezing modeling and thawing of soils or the solidification of magma, describing phase transitions in materials with complex thermal properties, and modeling tissue freezing and thawing. Now, let us consider the sharp SP with the boundary condition $u\left(\eta(t),t\right)=0$, and under the assumptions (Hyp.1)-(Hyp.2), the flux is given by Eq.(1) and Eq.(3) that indicates to the following equation

$$D^{\alpha}_{\eta^{-1}(x)}u-u_{xx}+\lambda u_x=\begin{cases}0, & x<\eta(0),\\ -\dfrac{\left(t-\eta^{-1}(x)\right)^{-\alpha}}{\Gamma(1-\alpha)}, & x\in\left(\eta(0),\eta(t)\right).\end{cases}$$

for almost every $(x,t)\in\Omega_{\eta,t^*}$, where

$$D^{\alpha}_{\eta^{-1}(x)}u(x,t)=\begin{cases}\dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_0^t(t-\tau)^{-\alpha}\frac{d}{d\tau}u(x,\tau)d\tau, & \text{for } x\le\eta(0),\\ \dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_{\eta^{-1}(x)}^t(t-\tau)^{-\alpha}\frac{d}{d\tau}u(x,\tau)d\tau, & \text{for } x>\eta(0).\end{cases}$$

Additionally, u and η are functions associated by the formula

$$\eta'(t)=\lim_{a\to\eta(t)}\left[\beta\frac{d}{dt}\int_{\eta^{-1}(a)}^t(t-\tau)^{\alpha-1}u(a,\tau)d\tau-\frac{1}{\Gamma(\alpha)}\frac{d}{dt}\int_{\eta^{-1}(a)}^t(t-\tau)^{\alpha-1}u_x(a,\tau)d\tau\right]$$

Moreover, if (Hyp.3) is satisfied, then we the following boundary condition

$$\lim_{\epsilon\to0^+}\left[\lambda u\left(\eta(t)-\epsilon,t\right)-u_x\left(\eta(t)-\epsilon,t\right)\right]=0 \tag{11}$$

It is important to note that [13] examined the fractional SP with the flux provided by the Riemann-Liouville derivative. Nevertheless, the authors were able to derive the subsequent set of equations

$$D^{\alpha}u(x,t)-u_{xx}(x,t)+\lambda u_x(x,t)=0 \tag{12}$$

$$D^{\alpha}\eta(t)=\beta u\left(\eta(t),t\right)-u_x\left(\eta(t),t\right) \tag{13}$$

The second result of this work will be discussed by finding self-similar solutions to the time-fractional SP in the region

$$\Omega=\left\{(x,t):x\in\mathbb{R},\ t\in(0,\infty);\ 0<x<\eta(t)\right\}. \tag{14}$$

The left boundary condition will be imposed with the assumption η(0) = 0. We shall formulate the problem in the following theorem

Theorem 2.1. *Let assume that (Hyp.1)-(Hyp.2), and Eq.(3) with the flux given by Eq.(1), then the SP is formulated as follows*

$$D^{\alpha}_{\eta^{-1}(x)}u(x,t)=u_{xx}(x,t)-\lambda u_x(x,t)-\frac{(t-\eta^{-1}(x))^{-\alpha}}{\Gamma(1-\alpha)}\quad \text{in } \Omega. \tag{15}$$

$$u(0,t)=\gamma \tag{16}$$

$$u\left(\eta(t),t\right)=0 \tag{17}$$

$$\eta'(t)=\lim_{a\to\eta(t)}\left[\beta\frac{d}{dt}\int_{\eta^{-1}(a)}^{t}(t-\tau)^{\alpha-1}u(a,\tau)d\tau-\frac{1}{\Gamma(\alpha)}\frac{d}{dt}\int_{\eta^{-1}(a)}^{t}(t-\tau)^{\alpha-1}u_x(a,\tau)d\tau\right] \tag{18}$$

Proof. To apply the energy conservation principle for a subset V and time $t\in(0,t^*)$, and obtain the system of equations from Eq.(10), and We look at two scenarios:

- If $V\subseteq\left(0,\eta(0)\right),\forall t\in(0,t^*)$ from (Hyp.2) such that

$$\int_V\frac{d}{dt}u(x,t)dx=\lambda\partial^{1-\alpha}\left(u(a,t)-u(b,t)\right)-\partial^{1-\alpha}\left(u_x(a,t)-u_x(b,t)\right)$$

  The fractional integral $I^{(1-\alpha)}$ is applied w.r.t. the time variable using the assumption (Hyp.1), and since $V\subseteq\left(0,\eta(0)\right)$ is arbitrary, we get

$$D^{\alpha}u(x,t)-u_{xx}(x,t)+\lambda u_x(x,t)=0\ \text{for}(x,t)\in\left(0,\eta(0)\right)\times(0,t^*) \tag{19}$$

- If $V=(a,b)$, where $\eta(0)<a<\eta(t)<b$, then Eq.(10) and applying $u(\eta(t),t)=0$, we get

$$\int_a^{\eta(t)}\frac{d}{dt}u(x,t)dx+\eta'(t)=\beta\frac{d}{dt}\int_{\eta^{-1}(a)}^{t}(t-\tau)^{\alpha-1}u(a,\tau)d\tau-\frac{1}{\Gamma(\alpha)}\frac{d}{dt}\int_{\eta^{-1}(a)}^{t}(t-\tau)^{\alpha-1}u_x(a,\tau)d\tau \tag{20}$$

If $a\to\eta(t)$, and by (Hyp.1), then it becomes as follows

$$\eta'(t)=\lim_{a\to\eta(t)}\left[\beta\frac{d}{dt}\int_{\eta^{-1}(a)}^{t}(t-\tau)^{\alpha-1}u(a,\tau)d\tau-\frac{1}{\Gamma(\alpha)}\frac{d}{dt}\int_{\eta^{-1}(a)}^{t}(t-\tau)^{\alpha-1}u_x(a,\tau)d\tau\right]$$

From Eq.(6), we consider the operator $I^{1-\alpha}_{\eta^{-1}(a)}$ to both sides of Eq.(20), we have

$$\begin{aligned}\frac{1}{\Gamma(1-\alpha)}\int_{\eta^{-1}(a)}^{t}(t-\tau)^{-\alpha}\int_a^{\eta(t)}\frac{d}{d\tau}u(x,\tau)dxd\tau+\frac{1}{\Gamma(1-\alpha)}\int_{\eta^{-1}(a)}^{t}(t-\tau)^{-\alpha}u'(\tau)d\tau\\ =\frac{\beta}{\Gamma(1-\alpha)}\int_{\eta^{-1}(a)}^{t}(t-\tau)^{-\alpha}\frac{d}{d\tau}\int_{\eta^{-1}(a)}^{\tau}(\tau-p)^{\alpha-1}u(a,p)dpd\tau\\ -\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_{\eta^{-1}(a)}^{t}(t-\tau)^{-\alpha}\frac{d}{d\tau}\int_{\eta^{-1}(a)}^{\tau}(\tau-p)^{\alpha-1}u_x(a,p)dpd\tau\end{aligned} \tag{21}$$

From (Hyp.1) then $u_x(a,.)\in L^{\infty}(\Omega^{a})$ and, the right hand side of Eq.(21) that can be rewritten as follows

$$I^{1-\alpha}_{\eta^{-1}(a)}\beta\frac{d}{dt}\left[I^{\alpha}_{\eta^{-1}(a)}u(a,\cdot)(t)\right]-I^{1-\alpha}_{\eta^{-1}(a)}\frac{d}{dt}\left[I^{\alpha}_{\eta^{-1}(a)}u_x(a,\cdot)(t)\right]=\beta u(a,t)-u_x(a,t) \tag{22}$$

We apply the Fubini's technique to the first part of Eq.(21), to get

$$\int_a^{\eta(t)} D^{\alpha}_{\eta^{-1}(a)} u(x,t)dx + \frac{1}{\Gamma(1-\alpha)}\int_{\eta^{-1}(a)}^{t} (t-\tau)^{-\alpha}\eta'(t)d\tau = \beta u(a,t) - u_x(a,t) \tag{23}$$

Applying the substitution $\tau = \eta^{-1}(x)$, we get

$$\int_a^{\eta(t)} D^{\alpha}_{\eta^{-1}(x)} u(x,t)dx + \frac{1}{\Gamma(1-\alpha)}\int_{a}^{\eta(t)} (t-\eta^{-1}(x))^{-\alpha} dx = \beta u(a,t) - u_x(a,t) \tag{24}$$

We anticipate that $u_x(.,t)$ may exhibit single behavior at the phase transition point. As a result, we proceed with extreme caution. Assuming $\varepsilon > 0$ and $a < \eta(t) - \epsilon$, (Hyp.1) yields

$$-u_x(a,t) = \int_a^{\eta(t)-\epsilon} u_{xx}(x,t)dx - u_x(\eta(t)-\epsilon,t) \tag{25}$$

Adding Eq.(25) to Eq.(24), it becomes

$$\int_a^{\eta(t)-\epsilon}\left[D^{\alpha}_{\eta^{-1}(x)} u(x,t) - u_{xx}(x,t) + \lambda u_x(x,t) + \frac{\left(t-\eta^{-1}(x)\right)^{-\alpha}}{\Gamma(1-\alpha)}\right]dx$$

$$= -\int_{\eta(t)-\epsilon}^{\eta(t)}\left[D^{\alpha}_{\eta^{-1}} u(x,t)dx + \frac{\left(t-\eta^{-1}(x)\right)^{-\alpha}}{\Gamma(1-\alpha)}\right]dx + \lambda u\left(\eta(t)-\epsilon,t\right) - u_x\left(\eta(t)-\epsilon,t\right) \tag{26}$$

To select an arbitrary *á* where $\acute{a} \in (\eta(0), a)$. To calculate as in Eq.(26) for *á* instead of *a*, we get that

$$\int_a^{\eta(t)-\epsilon}\left[D^{\alpha}_{\eta^{-1}(x)} u(x,t) + \frac{(t-\eta^{-1}(x))^{-\alpha}}{\Gamma(1-\alpha)} - u_{xx}(x,t) + \lambda u_x(x,t)\right]dx$$

$$= \lambda u\left(\eta(t)-\epsilon,t\right) - u_x\left(\eta(t)-\epsilon,t\right) - \int_{\eta(t)-\epsilon}^{\eta(t)}\left[D^{\alpha}_{\eta^{-1}(x)} u(x,t) + \frac{(t-\eta^{-1}(x))^{-\alpha}}{\Gamma(1-\alpha)}\right]dx \tag{27}$$

To subtract Eq.(27) from Eq.(26), then we get

$$\int_a^{a}\left[D^{\alpha}_{\eta^{-1}(x)} u(x,t) - u_{xx}(x,t) + \lambda u_x(x,t) + \frac{(t-\eta^{-1}(x))^{-\alpha}}{\Gamma(1-\alpha)}\right]dx = 0 \tag{28}$$

for arbitrary $x \in \left(\eta(0), \eta(t)\right), a, \acute{a} \in \left(\eta(t), \eta(t)-\epsilon\right)$ then, we get

$$D^{\alpha}_{\eta^{-1}(x)} u(x,t)dx - u_{xx}(x,t) + \lambda u_x(x,t) + \frac{(t-\eta^{-1}(x))^{-\alpha}}{\Gamma(1-\alpha)} = 0. \tag{29}$$

It remains to show Eq.(11). From Eq.(27) and Eq.(29) we infer that

$$0 = \lambda u(\eta(t)-\epsilon,t) - u_x(\eta(t)-\epsilon,t) - \int_{\eta(t)-\epsilon}^{\eta(t)}\left[D^{\alpha}_{\eta^{-1}(x)} u(x,t)dx + \frac{(t-\eta^{-1}(x))^{-\alpha}}{\Gamma(1-\alpha)}\right]dx.$$

To apply (Hyp.3) we obtain that

$$\lim_{\epsilon\to 0^+}\int_{\eta(t)-\epsilon}^{\eta(t)} \left(t-\eta^{-1}(x)\right)^{-\alpha} dx = 0 \text{ and } \lim_{\epsilon\to 0^+}\int_{\eta(t)-\epsilon}^{\eta(t)} D^{\alpha}_{\eta^{-1}(x)} u(x,t)dx = 0. \tag{30}$$

Also, from using Eq.(30) and the proof of Theorem 2.1.

3. Estimating the Self-Similar formula

In this section, we will use the rescaling technique to get asymptotically self-similar. Also, proof of Theorem 3.1 will be considered.

Lemma 3.1. *Let u(x,t) be a solution of the SP* (15)–(18). *This solution is asymptotically equal to the solution of the following problem*

$$D^{\alpha}_{\eta^{-1}(x)}u(x,t)-u_{xx}(x,t)+\frac{(t-\eta^{-1}(x))^{-\alpha}}{\Gamma(1-\alpha)}=0 \text{ in } \Omega. \tag{31}$$

$$u\left(\eta(t),t\right)=0 \tag{32}$$

$$u(0,t)=\gamma \tag{33}$$

$$\eta'(t)=-\lim_{a\to\eta(t)}\frac{d}{dt}\left[\frac{1}{\Gamma(\alpha)}\int_{\eta^{-1}(a)}^{t}u_x(a,\tau)(t-\tau)^{\alpha-1}d\tau\right] \tag{34}$$

Proof. Now, we use the recalling technique to reduce the SP (15)–(18), and calculate it by using rescaling function $u_k(x,t)$ and then $\lim_{k\to\infty}u_k(x,t)=u(x,t)$ which solves

$$D^{\alpha}_{\eta^{-1}(x)}u(x,t)-u_{xx}(x,t)+\frac{(t-\eta^{-1}(x))^{-\alpha}}{\Gamma(1-\alpha)}=0.$$

This equation satisfies the diffusive flux where $\eta(0)<a<\eta(t)<b,$ we obtain Stefan's condition

$$\frac{d}{dt}\left[\int_a^{\eta(t)}\left(u(x,t)+1\right)dx\right]=\partial^{1-\alpha}\left(u_x(a,t)\right)$$

$$=-\frac{1}{\Gamma(\alpha)}\frac{d}{dt}\int_{\eta^{-1}(a)}^{t}(t-\tau)^{\alpha-1}u_x(a,\tau)d\tau.$$

To calculate the left hand side and apply $u(\eta(t),t)=0$, we get

$$\int_a^{\eta(t)}\frac{d}{dt}u(x,t)dx+\eta'(t)=-\frac{1}{\Gamma(\alpha)}\frac{d}{dt}\int_{\eta^{-1}(a)}^{t}(t-\tau)^{\alpha-1}u_x(a,\tau)$$

If $a\to\eta(t)$, then by (Hyp.1), we obtain that

$$\eta'(t)=-\frac{1}{\Gamma(\alpha)}\lim_{a\to\eta(t)}\frac{d}{dt}\left[\int_{\eta^{-1}(a)}^{t}(t-\tau)^{\alpha-1}u_x(a,\tau)d\tau\right]$$

Therefore, the problem(31)-(34) is satisfied.

Lemma 3.2. *If $f(\zeta)$ represents a self-similarity formula, then the interface function exists and*

$$\eta(t)=\zeta_1 t^{\frac{\alpha}{2}} \tag{35}$$

for some positive ζ_1. If we denote

$$\zeta_0=\zeta_1^{-\frac{2}{\alpha}} \tag{36}$$

then we may write

$$\eta^{-1}(x)=\zeta_0 x^{\frac{2}{\alpha}} \tag{37}$$

let u_k be rescaling function as previous assumption in proof of lemma 3.1. The functions u and u_k solve the same equation. From Eq.(37), *we conclude the functional equation $s(kx)=k^{\frac{2}{\alpha}}s(x)$ and*

$$s'(kx)=\lim_{k\to1}\frac{s(x)\left(1-k^{\frac{2}{\alpha}}\right)}{x\left(1-k\right)}=\frac{2s(x)}{\alpha x}$$

and it yields the limit form as $k\to1$, and implies $s(x)=\zeta x^{\frac{2}{\alpha}}$ and for more details in [9,10].

Theorem 3.3. *Let the SP*(31)-(32), (34) *have a solution that has interface function η in* Eq.(35). *The regularity of u is supposed, $\forall t>0$ and some $h>1$, assuming $u_x(.,t)\in L^1\left(0,\eta(t)\right), u_{xx}(.,t)\in L^1\left(\eta(t)/h,\eta(t)\right)$. Then, the function f is defined as follows*

$$f(\zeta) := u(x,t) \text{ and } u(1,\zeta) = f\left(tx^{-\frac{2}{\alpha}}\right) \tag{38}$$

satisfies $\acute{f} \in \mathrm{AC}\left(\left[\zeta_0, h^{\frac{2}{\alpha}}\zeta_0\right]\right)$, $f \in C^2\left(\zeta_0, h^{\frac{2}{\alpha}}\zeta_0\right)$, *and for* $\zeta \in \left(\zeta_0, h^{\frac{2}{\alpha}}\zeta_0\right)$ *there hold*

$$\left(\frac{2}{\alpha}\right)^2 \zeta^2 f''(\zeta) + \left[\left(\frac{2}{\alpha}\right)^2 + \frac{2}{\alpha}\right]\zeta f'(\zeta) - \frac{(\zeta - c_0)^{-\alpha}}{\Gamma(1-\alpha)} = \frac{1}{\Gamma(1-\alpha)}\int_{\zeta_0}^{\zeta}(\zeta - p)^{-\alpha} f'(p)dp \tag{39}$$

$$f(\zeta_0) = 0 \tag{40}$$

$$\left(\frac{\alpha}{2}\right)^2 \zeta_0^{-2}\Gamma(\alpha) = \lim_{b\to\zeta_0}\frac{d}{db}\left[\int_{\zeta_0}^{b}(b-p)^{\alpha-1} f'(p)dp\right], \tag{41}$$

Both the Eq.(39) *and the regularity of f imply*

$$\lim_{\zeta\to\zeta_0}\left(\frac{2}{\alpha}\right)^2\left[(\zeta-\zeta_0)^{\alpha} f''(\zeta)\right] = \frac{\zeta_0^{-2}}{\Gamma(1-\alpha)}. \tag{42}$$

and Eq.(41), *implies that*

$$f'(\zeta_0) = 0. \tag{43}$$

Proof. From the existence and uniqueness of the solution u that solves the SP (31)–(34), then rescaling function u_k is represented solution of the SP with,

$$\zeta = tx^{-2/\alpha}, \tag{44}$$

and the shape function f is defined in Eq.(38). The transformation from the predicted regularity properties of u to the properties of f is given in theorem 2.1. Furthermore, we will demonstrate that Eq.(34) implies the vanishing of the derivative of f at point ζ_0 by rewriting the criteria Eqs.(31), (32), and (34) in terms of f. We apply Eq.(38) to the Eq.(31) and do some calculation. Also, we assume that $p = \tau x^{-2/\alpha}$, then we get

$$D^{\alpha}_{\eta^{-1}(x)} u(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_{\zeta_0 x^{2/\alpha}}^{t}(t-\tau)^{-\alpha} u_{\tau}(x,\tau)d\tau$$

from change variable from τ to ρ

$$D^{\alpha}_{\eta^{-1}(x)} u(x,t) = \frac{x^{-2}}{\Gamma(1-\alpha)}\int_{\zeta_0}^{tx^{-2/\alpha}}\left(tx^{-2/\alpha} - p\right)^{-\alpha}\acute{f}(p)\, dp.$$

Using these findings in equation Eq.(31) and the value of η provided by Eq.(35), where $\zeta = tx^{-2/\alpha}$, then we get

$$\frac{1}{\Gamma(1-\alpha)}\int_{\zeta_0}^{\zeta}(\zeta - p)^{-\alpha} f'(p)dp = \left(\frac{2}{\alpha}\right)^2 \zeta^2 f''(\zeta) + \left[\left(\frac{2}{\alpha}\right)^2 + \frac{2}{\alpha}\right]\zeta f'(\zeta) - \frac{(\zeta - \zeta_0)^{-\alpha}}{\Gamma(1-\alpha)}$$

To demonstrate that Eq.(40) holds, then $u(\eta(t),t) = u\left(\zeta_1 t^{\alpha/2}, t\right) = 0.$ By using Eqs.(36) and (43), we shall now demonstrate the regularity results

$$\int_{\zeta_0}^{\infty}|f'(\zeta)|d\zeta < \infty. \tag{45}$$

From Eq.(45) we obtain in the similar way that

$$\int_{\zeta_0}^{(h^{2/\alpha})\zeta_0}|f''(\zeta)|d\zeta < \infty. \tag{46}$$

Then the condition $f' \in AC\left(\left[\zeta_0, h^{2/\alpha}\right]\right)$ is satisfied from estimating Eq.(45) and Eq.(46), and $f \in C^2\left(\zeta_0, h^{2/\alpha}\zeta_0\right)$ for the absolute continuity of f' in Eq.(39). The suggested regularity result is obtained.

To rewrite the Eq.(34) that pointers to Eq.(41). To fix $a \in \left(\eta(t)/h, \eta(t)\right)$ and substitute $p = \tau a^{(-2/\alpha)}$. Also, we use the integration by part and then according to the continuity of second derivatives to f in $\left(\zeta_0, h^{2/\alpha}\zeta_0\right)$, and we use the same calculation as in [10] and since f' is absolutely continuous and $I^{1-\alpha}_{\zeta_0}\partial^{1-\alpha}_{\zeta_0} f' = f'$.

$$\left[\left(\frac{\alpha}{2\zeta_0}\right)^2 - \epsilon\right]\frac{(x-\zeta_0)^{1-\alpha}}{\Gamma(2-\alpha)} \le f'(x) \le \left[\left(\frac{\alpha}{2\zeta_0}\right)^2 + \epsilon\right]\frac{(x-\zeta_0)^{1-\alpha}}{\Gamma(2-\alpha)}$$

hence for every $x \in (\zeta_0, x_0)$, it implies $\lim_{x\to\zeta_0} \frac{f'(x)}{(x-\zeta_0)^{1-\alpha}} = \left(\frac{\alpha}{2}\right)^2 \frac{\zeta_0^{-2}}{\Gamma(2-\alpha)}$. Thus, in particularly $f'(\zeta_0) = 0$, that makes the proof of theorem 2.1, is done.

It should be noted that the opposite is also true. From regressive calculations, the following result will be considered.

4. Numerical Analysis by the SDM

Numerous authors have studied fractional order PDEs in recent years using a variety of techniques, including the variational iteration method(VIM), the homotopy perturbation method and the Laplace transform(LT), Laplace homotopy perturbation method and the homotopy analysis technique, see [34–36]. We apply the Sumudu decomposition method(SDM) to fractional derivatives, and use it to solve initial value fractional differential equations in [37]. The authors of [35, 37] examined a number of Sumudu transform(ST) properties using this knowledge to create simple and effective methods for handling ordinary and partial differential equations. There is clearly a desire to learn more about this change and apply it to a variety of mathematical and physical research challenges. The ST, which is linear and bilateral with scale and unit preserving properties [36], can be utilized to solve a wide range of difference and differential equation problems without the need for a new frequency domain. We use the ST to solve the generic form of a time-Fractional diffusion equation

$$D_t^\alpha u(x,t) = u_{xx}(x,t) - \lambda u_x(x,t), \text{ for } n-1 < \alpha \le n \tag{47}$$

where D_t^α is the fractional Caputo derivative(FCD) of order α respect on the time variable. and $I_0^{(1-\alpha)}$ is the fractional integral of Riemann-Liouville in the time variable of order $1 - \alpha$, defined for every summable function u as

$$I_0^\alpha u(x,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} u(x,\tau)d\tau \tag{48}$$

Then the FCD of order α respect on the time variable is defined as follows

$$D_t^\alpha u(x,t) = I_0^{1-\alpha} u_t(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha} u_t(x,\tau)d\tau \tag{49}$$

The Mittag-Leffler function(MLF) is an essential function that is widely used in fractional calculus. The MLF solves non-integer order DEs in the same way as the exponential does for integer order DEs. The exponential function is actually a very particular variation among an infinite number of this function that appears to be used by everyone. Eq.(47) provides the conventional definition of Mittag-Leffler.

$$\mathbb{E}_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+1)}, \ \alpha > 0 \tag{50}$$

The exponential function corresponds to $\alpha = 1$,

$$\mathbb{E}_1(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k+1)} = e^t \tag{51}$$

Moreover, it is typical to express the MLF as two arguments, α and β, so that

$$\mathbb{E}_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0,\ \beta > 0 \tag{52}$$

The ST is defined over the set of function

$$\mathbb{A} = \left\{ u(x,t) \ \middle|\ \exists C, \mu_1, \mu_2 > 0,\ |u(x,t)| < Ce^{|t|/\mu_j} \text{ if } t \in (-1)^j \times [0,\infty) \right\}$$

by the following formula

$$\mathbb{S}\left[u(x,t)\right] = \int_0^{\infty} \frac{1}{v} u(x,t) e^{-t/v}\, dt, \quad v \in \left(-\mu_1, \mu_2\right) \tag{53}$$

In Belgacem et al. [36], demonstrated that the ST represents the theoretical dual to the LT. As a result, one should be able to compete with it on a large scale for solving of the problem. Many of the ST's unique qualities are discussed and summarized in [36]. Some other features of the ST are considered as the following:

(1) $\mathbb{S}[1] = 1$,

(2) $\mathbb{S}\left[\frac{t^m}{\Gamma(m+1)}\right] = u^m,\ m > 0$

(3) $\mathbb{S}\left[u(x,t) \mp v(x,t)\right] = \mathbb{S}\left[u(x,t)\right] \mp \mathbb{S}\left[v(x,t)\right]$

The ST of the fractional derivative presented by using Caputo operator as follows

$$\mathbb{S}\left[D_t^{\alpha} u(x,t)\right] = p^{-\alpha} \mathbb{S}\left[u(x,t)\right] - \sum_{k=0}^{n-1} p^{k-\alpha} u^{(k)}(x,0), \quad (n-1 < \alpha \le n) \tag{54}$$

5. Mathematical Analysis of Sumudu Decomposition Method

The linear time-Fractional diffusion-convection equation(1) is introduced with an initial data

$$u(x,0) = \phi(x) \tag{55}$$

Apply the SDM to Eq.(47), then it becomes

$$\mathbb{S}\left[D_t^{\alpha} u(x,t)\right] = \mathbb{S}\left[u_{xx}(x,t) - \lambda u_x(x,t)\right], \quad t \in (0, t^*)$$

$$p^{-\alpha} \mathbb{S}[u(x,t)] - \sum_{k=0}^{n-1} p^{k-\alpha} u^{(k)}(x,0) = \mathbb{S}\left[u_{xx}(x,t) - \lambda u_x(x,t)\right]$$

$$p^{-\alpha} \mathbb{S}[u(x,t)] = \sum_{k=0}^{n-1} p^{k-\alpha} u^{(k)}(x,0) + \mathbb{S}\left[u_{xx}(x,t) - \lambda u_x(x,t)\right]$$

Then, we obtain

$$\mathbb{S}\left[u(x,t)\right] = p^{\alpha} \sum_{k=0}^{n-1} p^{k-\alpha} u^{(k)}(x,0) + p^{\alpha} \mathbb{S}\left[u_{xx}(x,t) - \lambda u_x(x,t)\right] \tag{56}$$

Using the convolution theorem and the inverse ST on both sides of Eq.(56), we can now obtain

$$u(x,t) = \mathbb{S}^{-1}\left(p^{\alpha} \sum_{k=0}^{n-1} p^{k-\alpha} u^{(k)}(x,0) + p^{\alpha} \mathbb{S}\left[u_{xx}(x,t) - \lambda u_x(x,t)\right] \right)$$

$$u(x,t) = \mathbb{S}^{-1}\left(p^{\alpha} \sum_{k=0}^{n-1} p^{k-\alpha} u^{(k)}(x,0) \right) + \mathbb{S}^{-1}\left(p^{\alpha} \mathbb{S}\left[u_{xx}(x,t) - \lambda u_x(x,t)\right] \right)$$

It can be written in the form, and $\lambda = 1$

$$u(x,t) = \mathbb{S}^{-1}\left(p^{\alpha} \sum_{k=0}^{n-1} p^{k-\alpha} u^{(k)}(x,0) \right) + \mathbb{S}^{-1}\left(\frac{1}{p^{\alpha}} \mathbb{S}\left[u_{xx}(x,t) - u_x(x,t)\right] \right) \tag{57}$$

Now, describing the solution as an infinite series provided below

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t) \tag{58}$$

Substitution of equation (58) in Eq.(57) leads to

$$\sum_{n=0}^{\infty}u_n(x,t)=\mathbb{S}^{-1}\left(p^{\alpha}\sum_{k=0}^{n-1}p^{k-\alpha}\varphi^{(k)}(x)\right)+\mathbb{S}^{-1}\left(\frac{1}{p^{-\alpha}}\mathbb{S}\left[\sum_{n=0}^{\infty}(u_n)_{xx}-\sum_{n=0}^{\infty}(u_n)_x\right]\right) \tag{59}$$

By comparing both sides of the Eq. (59), we get

$$\begin{aligned}
u_0(x,t)&=\phi(x),\\
u_1(x,t)&=\mathbb{S}^{-1}\left(\frac{1}{p^{-\alpha}}\mathbb{S}\left(\phi_{xx}-\phi_x\right)\right),\\
u_2(x,t)&=\mathbb{S}^{-1}\left(\frac{1}{p^{-\alpha}}\mathbb{S}\left((u_1)_{xx}-(u_1)_x\right)\right),\\
&\vdots\\
u_{n+1}(x,t)&=\mathbb{S}^{-1}\left(\frac{1}{p^{-\alpha}}\mathbb{S}\left((u_n)_{xx}-(u_n)_x\right)\right).
\end{aligned}$$

Finally, we approximate the analytical solution $u(x,t)$ by truncated series:

$$u(x,t)=\lim_{n\to\infty}\phi_n(x,t) \tag{60}$$

where $\phi_n(x,t)=\sum_{k=0}^{n-1}u_k(x,t)$ is the sequence of partial sums to the series (59).

6. Numerical Experiments

Example 6.1. *Solve this one-dimensional linear inhomogeneous fractional diffusion equation(FDE):*

$$\begin{aligned}D_t^{\alpha}u(x,t)&=u_{xx}(x,t),\\ u(x,0)&=\sin(x),\end{aligned}\quad n-1<\alpha\le n \tag{61}$$

From equation (59), *we apply SDM*

$$\begin{aligned}
u_0(x,t)&=\sin(x)\\
u_1(x,t)&=S^{-1}\left(\frac{1}{p^{-\alpha}}\mathbb{S}\left((u_0)_{xx}\right)\right)=-\frac{\sin(x)t^{\alpha}}{\Gamma(\alpha+1)},\\
u_2(x,t)&=S^{-1}\left(\frac{1}{p^{-\alpha}}\mathbb{S}\left((u_1)_{xx}\right)\right)=\frac{\sin(x)t^{2\alpha}}{\Gamma(2\alpha+1)},\\
u_3(x,t)&=S^{-1}\left(\frac{1}{p^{-\alpha}}\mathbb{S}\left((u_2)_{xx}\right)\right)=-\frac{\sin(x)t^{3\alpha}}{\Gamma(3\alpha+1)},\\
&\vdots\\
u_n(x,t)&=S^{-1}\left(\frac{1}{p^{-\alpha}}\mathbb{S}\left((u_{n-1})_{xx}\right)\right)=(-1)^n\frac{\sin(x)t^{n\alpha}}{\Gamma(n\alpha+1)}.
\end{aligned}$$

From the sequence of partial sums, we get

$$\phi_n(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+\cdots+u_n(x,t)$$

and by substituting the above solutions, it becomes

$$\phi_n(x,t) = \sin(x)\left[-\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \cdots + \frac{(-t^{\alpha})^n}{\Gamma(n\alpha+1)}\right] \tag{62}$$

Thus, making use of the MLF definition in a single parameter, the problem's solution is provided by

$$\lim_{n\to\infty}\phi_n(x,t) = \sin(x)\sum_{k=0}^{\infty}\frac{(-t^{\alpha})^k}{\Gamma(k\alpha+1)}$$

Then from equation(60)

$$u(x,t) = \sin(x)E_{(\alpha,1)}(-t^{\alpha}). \tag{63}$$

The Eq. (63) *for* α = 1 *is approximate to the form* $u(x,t) = \sin(x)e^{-t}$ *which is the exact solution of* Eq. (61) *for* α = 1.

Example 6.2. *Solve this one-dimensional linear inhomogeneous fractional PDE:*

$$\begin{aligned} &D_t^{\alpha}u(x,t) = u_{xx}(x,t) + 2u_x(x,t),\\ &u(x,0) = e^{-x}, \end{aligned} \tag{64}$$

From equation (59), *we apply SDM*

$$\begin{aligned}
&u_0(x,t) = e^{-x},\\
&u_1(x,t) = \mathbb{S}^{-1}\left(\frac{1}{p^{-\alpha}}\mathbb{S}\left((u_0)_{xx} + 2(u_0)_x\right)\right) = -\frac{e^{-x}t^{\alpha}}{\Gamma(\alpha+1)},\\
&u_2(x,t) = \mathbb{S}^{-1}\left(\frac{1}{p^{-\alpha}}\mathbb{S}\left((u_1)_{xx} + 2(u_1)_x\right)\right) = \frac{e^{-x}t^{2\alpha}}{\Gamma(2\alpha+1)},\\
&u_3(x,t) = \mathbb{S}^{-1}\left(\frac{1}{p^{-\alpha}}\mathbb{S}\left((u_2)_{xx} + 2(u_2)_x\right)\right) = -\frac{e^{-x}t^{3\alpha}}{\Gamma(3\alpha+1)},\\
&\quad\vdots\\
&u_n(x,t) = \mathbb{S}^{-1}\left(\frac{1}{p^{-\alpha}}\mathbb{S}\left((u_{n-1})_{xx} + 2(u_{n-1})_x\right)\right) = (-1)^n\frac{e^{-x}t^{n\alpha}}{\Gamma(n\alpha+1)}.
\end{aligned}$$

From the sequence of partial sums, we get

$$\phi_n(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots + u_n(x,t)$$

and by substituting the above solutions, it becomes

$$\phi_n(x,t) = e^{-x}\left[-\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \cdots + \frac{(-t^{\alpha})^n}{\Gamma(n\alpha+1)}\right]. \tag{65}$$

Thus, making use of the MLF definition in a single parameter, the problem's solution is provided by

$$\lim_{n\to\infty}\phi_n(x,t) = e^{-x}\sum_{k=0}^{\infty}\frac{(-t^{\alpha})^k}{\Gamma(k\alpha+1)}$$

Then from equation (60)

$$u(x,t) = e^{-x}E_{\alpha,1}(-t^{\alpha}). \tag{66}$$

The Eq. (66) *for* α = 1 *is approximate to the form* $u(x,t) = e^{-(x+t)}$ *which is the exact solution of* Eq. (64) *for* α = 1.

Example 6.3. *The solution this one-dimensional linear inhomogeneous fractional diffusion equation(FDE):*

$$D_t^{\alpha} u(x,t) = u_{xx}(x,t) - u(x,t),$$
$$u(x,0) = u_0(x) = e^{-x} + x, \tag{67}$$

From equation (59), *we apply SDM*

$$u_0(x,t) = e^{-x} + x,$$
$$u_1(x,t) = \mathbb{S}^{-1}\left(\frac{1}{p^{-\alpha}}\mathbb{S}\left((u_0)_{xx} - u_0\right)\right) = -\frac{xt^{\alpha}}{\Gamma(\alpha+1)},$$
$$u_2(x,t) = \mathbb{S}^{-1}\left(\frac{1}{p^{-\alpha}}\mathbb{S}\left((u_1)_{xx} - u_1\right)\right) = \frac{xt^{2\alpha}}{\Gamma(2\alpha+1)},$$
$$u_3(x,t) = \mathbb{S}^{-1}\left(\frac{1}{p^{-\alpha}}\mathbb{S}\left((u_2)_{xx} - u_2\right)\right) = -\frac{xt^{3\alpha}}{\Gamma(3\alpha+1)},$$
$$\vdots$$
$$u_n(x,t) = \mathbb{S}^{-1}\left(\frac{1}{p^{-\alpha}}\mathbb{S}\left((u_{n-1})_{xx} - u_{n-1}\right)\right) = \frac{x(-t^{\alpha})^n}{\Gamma(n\alpha+1)}.$$

From the sequence of partial sums

$$\phi_n(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots + u_n(x,t)$$

and by substituting the above solutions, it becomes

$$\phi_n(x,t) = e^{-x} + x - \frac{xt^{\alpha}}{\Gamma(\alpha+1)} + \frac{xt^{2\alpha}}{\Gamma(2\alpha+1)} - \cdots + \frac{x(-t^{\alpha})^n}{\Gamma(n\alpha+1)} \tag{68}$$

Thus, making use of the MLF definition in a single parameter, the problem's solution is provided by

$$\lim_{n\to\infty}\phi_n(x,t) = e^{-x} + x\sum_{k=0}^{\infty}\frac{(-t^{\alpha})^k}{\Gamma(k\alpha+1)}$$

Then from equation (68)

$$u(x,t) = e^{-x} + xE_{\alpha,1}(-t^{\alpha}). \tag{69}$$

The Eq. (69) *for* $\alpha = 1$ *is approximate to the form* $u(x,t) = e^{-x} + xe^{-t}$ *which is the exact solution of* Eq. (67) *for* $\alpha = 1$.

7. Numerical Results and Discussion

The numerical solution of the Cauchy problem(CP) represented the time-fractional diffusion-advection equation (1) with the initial condition $u_0(x) = \phi(x)$, can be described and illustrated in above three application examples. The numerical values in Figure 2 represents the approximate solution $u(x,t)$ of the CP(17) in Example 6.1. The numerical solution approximated to the exact solution u_{Exact} given by equation (19) through different values of t and x = 1, 0.9, 0.8 and 0.7; with values α = 1,0.9,0.8 and 0.7 for Example 6.1. In Figure 2, we observe that the approximate solution for Example 6.1 decreases when t increases and for fixed values of x. In Figure 4, the approximate solution $u(x,t)$ of the CP(17) in Example 6.1 show the graph through different values of x,t when α = 1,0.9,0.8 and 0.7; respectively. Therefore, both graphs in Figure 3 and Figure 4 come close to the exact solution u_{Exact} for $0 < \alpha \le 1$. Now, the approximate solution $u(x,t)$ of the CP(20) to Example 6.2 is described in Figure 5. The numerical values approach to the exact solution u_{Exact} given by the equation (22) through different values of t and x = 0.75,1.5,2.25,3.00; with values α = 1,0.9,0.8 and 0.7; for Example 6.2. Figure 5, we observe that the approximate solution for Example 6.2 decreases when t increases for fixed values of x, for

$0 < \alpha \leq 1$. In Figure 6, the approximate solution $u(x,t)$ of the CP(20) in Example 6.2 show the graph of the approximate solution among different values of x,t when $\alpha = 1,0.9,0.8$ and 0.7; respectively. Similarly, we discuss the approximate solution $u(x,t)$ of the CP(23) to Example 6.3 in two figures. In Figure 7, the numerical solution is illustrated through different values of t and $x = 0.75,1.5,2.25,3.00$; with values $\alpha = 1,0.9,0.8$ and 0.7; for Example 6.3. In this case, we observe that the approximate solution decreases when t increases for fixed values of x. Figure 7 shows the approximate solution $u(x,t)$ of the CP(23) in Example 6.3, approach to the exact solution u_{Exact} given by the equation (25) among different values of x,t when $\alpha = 1,0.9,0.8$ and 0.7; respectively. We can determine from the preceding reasoning and the numerical answers that the absolute error is extremely tiny, indicating that the suggested SDM is very successful in providing the analytical solutions for the time-fractional diffusion-convection equation with ease and without the need for any assumptions.

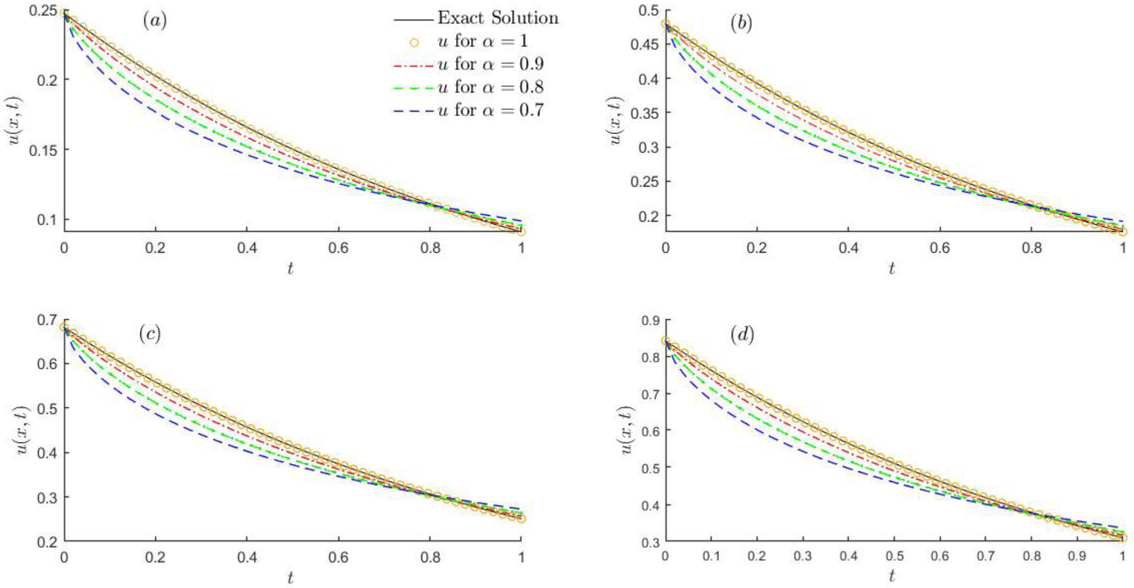

Figure 2: For Example 6.1, (a) $x = 0.25$; (b) $x = 0.50$; (c) $x = 0.75$; (d) $x = 1.0$; and among different values of t when $\alpha = 0.7, 0.8, 0.9, 1.0$; it shows the graphs of the approximated solutions.

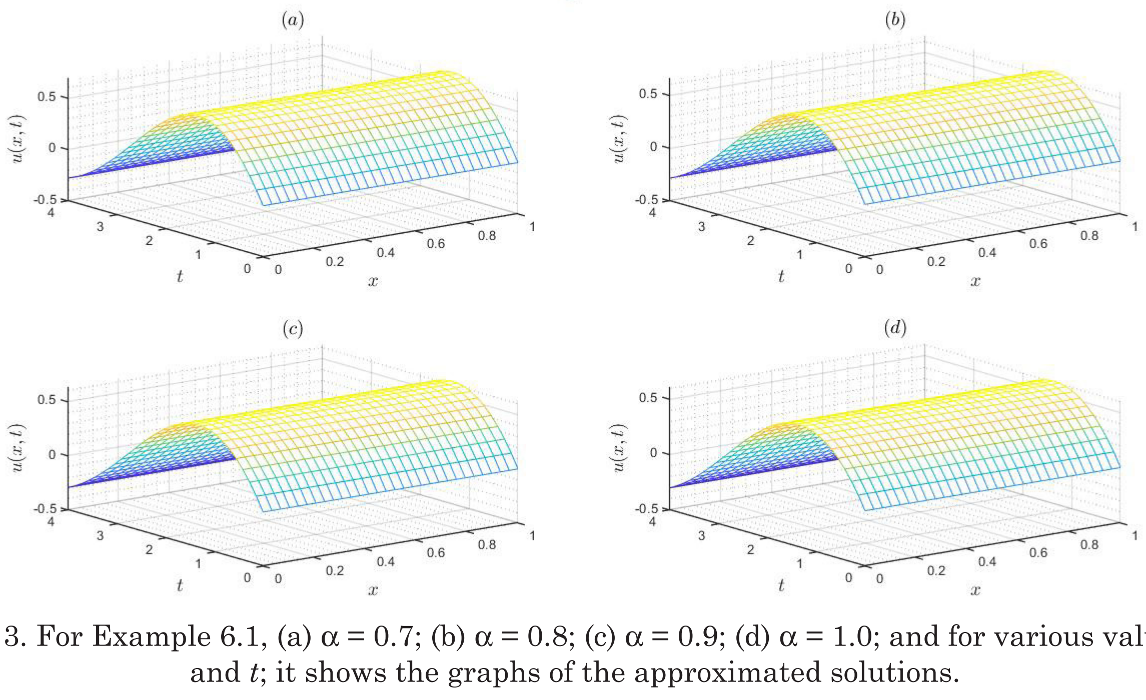

Figure 3. For Example 6.1, (a) $\alpha = 0.7$; (b) $\alpha = 0.8$; (c) $\alpha = 0.9$; (d) $\alpha = 1.0$; and for various values of x and t; it shows the graphs of the approximated solutions.

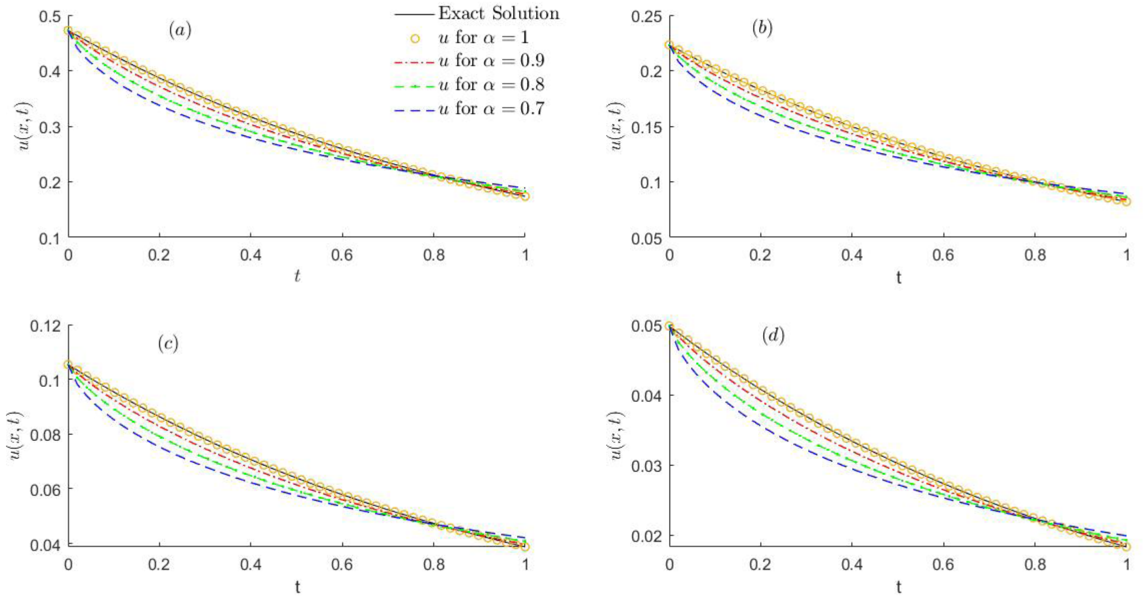

Figure 4: For Example 6.2(a) x =0.75; (b) x = 1.50; (c) x = 2.25; (d) x = 3.00; and among different values of t when α = 0.7, 0.8, 0.9, 1.0; it shows the graphs of the approximated solutions.

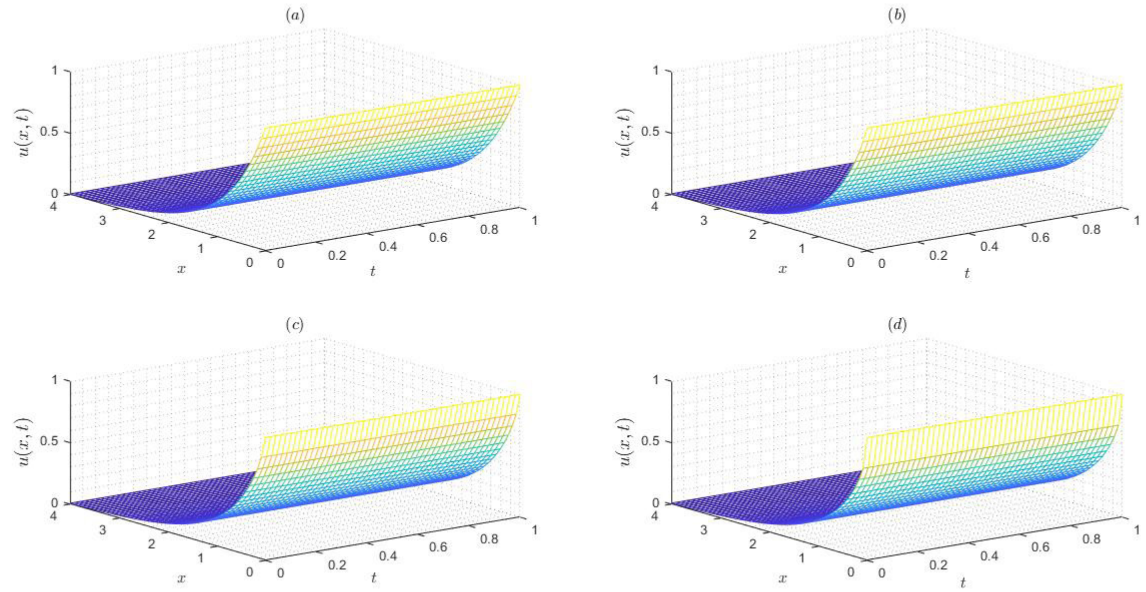

Figure 5: For Example 6.2, for (a) α = 1.0; (b) α = 0.9; (c) α = 0.8; (d) α = 0.7; and for various values of x and t; it shows the graphs of the approximated solutions.

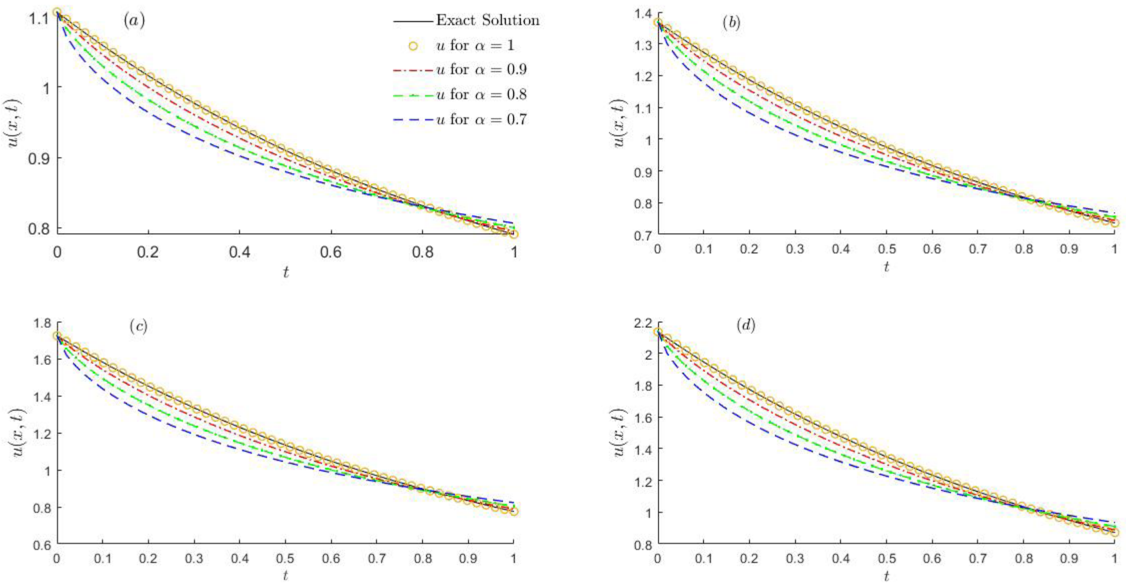

Figure 6: For Example 6.3 (a) $x = 0.75$; (b) $x = 1.50$; (c) $x = 2.25$; (d) $x = 3.00$; and among different values of t when $\alpha = 0.7, 0.8, 0.9, 1.0$; it shows the graphs of the approximated solutions.

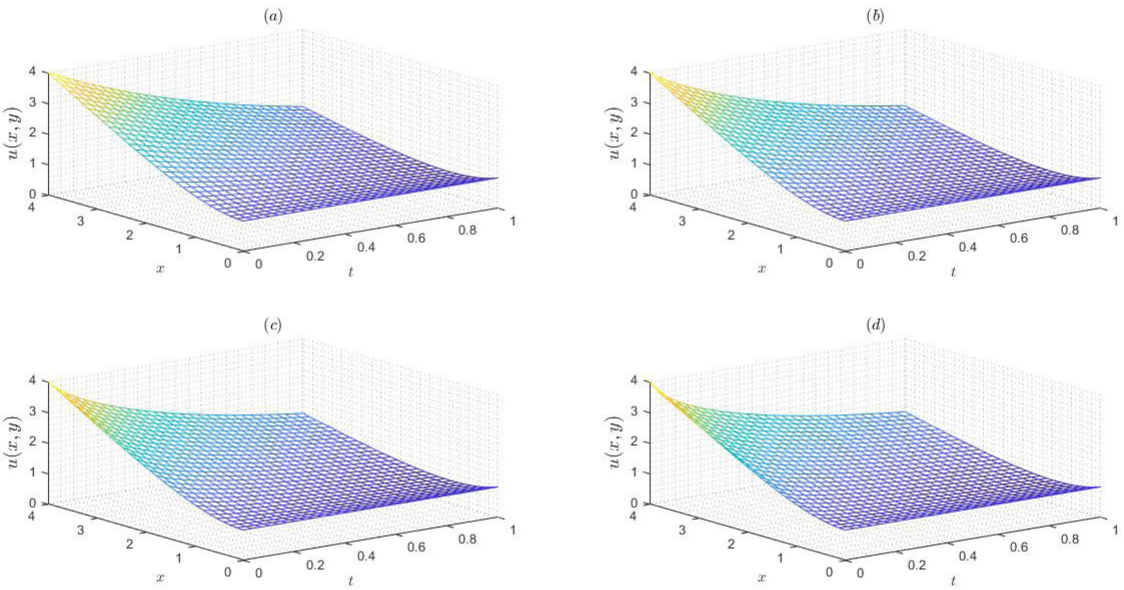

Figure 7: For Example 6.3, for (a) $\alpha = 1.0$; (b) $\alpha = 0.9$; (c) $\alpha = 0.8$; (d) $\alpha = 0.7$; and for various values of x and t; it shows the graphs of the approximated solutions.

8. Conclusion

Under three hypotheses, we derived the Fractional SP with boundary and Stefan conditions in Theorem 2.1. The fractional derivative method, particularly Riemann-Liouville and Caputo derivatives, was used to find approximated solutions of the SP under some restrictions. After that, the rescaling technique was used in Lemma 3.1 to satisfy the asymptotic solutions. The self-similarity solutions and the interface formula were obtained for the time fractional Stefan problem in Theorem 3.3. Finally, we estimated the numerical solutions for the time-fractional convection-diffusion equation considered by using the SDM. The fractional integral of Riemann-Liouville was used in conjunction with the Caputo fractional derivative operator to the suggested method. Therefore, the SDM is incredibly effective and powerful in numerical solutions for the time-fractional convection-diffusion equation. The computations and graphics were all done with the MATLAB software to prove the usefulness and validity of the proposed method, illustrative examples were provided.

References

[1] J. Crank, Free and moving boundary problems, *Journal of Fluid Mechanics*, Volume 158, September 1985, pp. 532– 533. https://doi.org/10.1017/S0022112085212750

[2] V. R. Voller, A Monte Carlo scheme for tracking filling fronts, *J. Comput. Phys.*, vol. 200, no. 2, pp. 399–411, 2004. https://doi.org/10.1016/j.jcp.2004.04.009

[3] Ezekoye, Ofodike A. "Conduction of heat in solids." *SFPE Handbook of Fire Protection Engineering* (2016): 25–52.

[4] Douglas, Jim, and Henry H. Rachford. "On the numerical solution of heat conduction problems in two and three space variables." *Transactions of the American Mathematical Society* 82(2) (1956): 421–439.

[5] W. J. Boettinger, J. A. Warren, C. Beckermann, and A. Karma, Phase-field simulation of solidification, *Annu. Rev. Mater. Res.*, vol. 32, no. 1, pp. 163–194, 2002.

[6] G. Caginalp, Stefan and Hele-Shaw type models as asymptotic limits of the phase-field equations, *Phys. Rev. A*, vol. 39, no. 11, p. 5887, 1989.

[7] Chen, Long-Qing, and Yuhong Zhao. "From classical thermodynamics to phase-field method." *Progress in Materials Science* 124 (2022): 100868.

[8] K. Ryszewska, A space-fractional Stefan problem, pp. 1–37, 2019, [Online]. Available: http://arxiv.org/abs/1910.00379

[9] A. Kubica, K. Ryszewska, A self-similar solution to time-fractional Stefan problem. *Mathematical Methods in the Applied Sciences*, 44(6), 4245–4275, 2021.

[10] A. Kubica, P. Rybka, and K. Ryszewska, Weak solutions of fractional differential equations in non cylindrical domains, *Nonlinear Anal. Real World Appl.*, vol. 36, pp. 154–182, 2017, doi: 10.1016/j.nonrwa.2017.01.005.

[11] S. D. Roscani, Hopf lemma for the fractional diffusion operator and its application to a fractional free-boundary problem, *J. Math. Anal. Appl.*, vol. 434, no. 1, pp. 125–135, 2016. https://doi.org/10.1016/j.jmaa.2015.08.070

[12] J. Liu and M. Xu, Some exact solutions to Stefan problems with fractional differential equations, *J. Math. Anal. Appl.*, vol. 351, no. 2, pp. 536–542, 2009. https://doi.org/10.1016/j.jmaa.2008.10.042

[13] A. N. Ceretani, A note on models for anomalous phase-change processes, *Fract. Calc. Appl. Anal.*, vol. 23, no. 1, pp. 167–182, 2020, https://doi.org/10.1515/fca-2020-0006

[14] S. D. Roscani, and V. R. Voller. "On an enthalpy formulation for a sharp-interface memory-flux Stefan problem." *Chaos, Solitons and Fractals* 181 (2024): 114679.

[15] S. Rogosin, and M. Dubatovskaya. "Fractional Stefan Problem: A Survey of the Recent Results." *Lobachevskii Journal of Mathematics* 44(8) (2023): 3535–3554.

[16] S. D. Roscani, D. A. Tarzia, and L. D. Venturato. "The similarity method and explicit solutions for the fractional space one-phase Stefan problems." *Fractional Calculus and Applied Analysis* 25(3) (2022): 995–1021.

[17] S. D. Roscani, N. D. Caruso, and D. A. Tarzia, Explicit solutions to fractional Stefan-like problems for Caputo and Riemann Liouville derivatives, *Commun. Nonlinear Sci. Numer. Simul.*, vol. 90, 2020. https://doi.org/10.1016/j.cnsns.2020.105361

[18] S. D. Roscani, Moving-boundary problems for the time-fractional diffusion equation, *Electron. J. Differ. Equations*, vol. 2017, no. 44, pp. 1–12, 2017.

[19] Chaurasiya, Vikas, Sunil Kumar Sharma, and Subrahamanyam Upadhyay. "Theoretical study of solidification phase change heat and mass transfer with thermal resistance and convection

subjected to a time-dependent boundary condition" *Thermal Science and Engineering Progress* 54 (2024): 102834.

[20] S. D. Roscani, N. D. Caruso, and D. A. Tarzia, Explicit solutions to fractional Stefan like problems for Caputo and Riemann Liouville derivatives, *Commun. Nonlinear Sci. Numer. Simul.*, vol. 90, 2020. https://doi.org/10.1016/j.cnsns.2020.105361

[21] T. M. Flett, 2742. a mean value theorem, *Math. Gaz.*, vol. 42, no. 339, pp. 38–39, 1958.

[22] V. R. Voller, F. Falcini, and R. Garra, l, Phys. Rev. E - Stat. *Nonlinear, Soft Matter Phys.*, vol. 87, no. 4, pp. 1–6, 2013, https://doi.org/10.1103/PhysRevE.87.042401

[23] Van Tien Nguyen. (2022). Note on the convergence of fractional conformable diffusion equation with linear source term. *Results in Nonlinear Analysis*, 5(3), 387–392. Retrieved from https://nonlinear-analysis.com/index.php/pub/ article/view/114

[24] R. Kuske and P. Milewski, Modulated two-dimensional patterns in reaction diffusion systems, *Eur. J. Appl. Math.*, vol. 10, no. 2, pp. 157–184, 1999.

[25] K. I. Nakamura, H. Matano, D. Hilhorst, and R. Schatzle, Singular limit of a reaction diffusion equation with a spatially inhomogeneous reaction term, *J. Stat. Phys.*, vol. 95, pp. 1165–1185, 1999.

[26] R. O. Ayeni and F. B. Agusto, On the existence and uniqueness of self-similar diffusion equation, *J. Nig. Ass. Math. Phys*, vol. 44, p. 183, 2000.

[27] Bendib, I., Batiha, I., Hioual, A., Anakira, N., Dalah, M., Ouannas, A. (2024). On a New Version of Grierer-Meinhardt Model Using Fractional Discrete Calculus. *Results in Nonlinear Analysis*, 7(2), 1–15. Retrieved from

[28] A. L. Gladkov, The Cauchy problem in classes of increasing functions for the equation of filtration with convection, *Sb. Math.*, vol. 186, no. 6, pp. 803–825, 1995.