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On certain fixed point theorems for F-contractions in 2-metric spaces and their application

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Abstract

This research investigates F-contractions in 2-metric spaces and proves multiple fixed point theorems related to these contractions. The study sheds light on the existence and uniqueness of fixed points. The findings enhance the overall comprehension of fixed point theory and its applications, providing useful resources for further exploration in mathematical analysis and associated areas.

Key words and phrases: F-contraction, Fixed point, 2-metric spaces(2-m.s)

Mathematics Subject Classification (2020): 47H10, 54H25.

1. Introduction

Fixed point theory offers powerful tools for solving equations and understanding the behavior of various mathematical systems. The application of fixed point theory is extensively recognized as a crucial method for solving multiple problems in nonlinear and applied mathematical analysis.

In 1922 [1], Banach first investigate a fixed point theorem in metric space and it is well known as Banach Fixed Point Theorem. It provides fundamental tools for understanding when and how functions have fixed points. It finds broad application in fields such as mathematics, economics, and engineering. After that many researchers of this field have generalized that theorem in various ways [28, 29]. After Banach, Kannan [27] generarized that theorem. After Kannan, Chaterjea [25] generalized that fixed point theorem. In 1973, Hardy and Rogers [26] have also generalized the fixed point

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theorem of Reich. Since then, numerous researchers have utilized various types of metric spaces to derive new fixed-point results [21, 22, 31–33].

Gahler [3] introduced the idea of a 2-metric space, which generalizes the concept of a metric space. In a 2-metric space, the 2-metric function measures the area of a triangle, whereas the standard metric function evaluates the length of a line segment. Notably, a 2-metric space differs topologically from a metric space. Several researchers have demonstrated the fixed point theorem within the context of 2-metric spaces [4–7, 10, 24].

Definition 1.1: *A 2-m.s is defined by a set* Ω *and a function* $\zeta : \Omega\times\Omega\times\Omega \to R$ *that adheres to the following conditions for any* $\varphi,\pi,\mathfrak{h},\mu \in \Omega$*:*

- $\zeta(\varphi,\varphi,\mu)=0$,
- Non-negativity: $\zeta(\varphi,\pi,\mu)\neq 0$ if φ,π,μ are distinct,
- Symmetry: $\zeta(\varphi,\pi,\mu)=\zeta(\pi,\mu,\varphi)=\zeta(\varphi,\mu,\pi)=\ldots$,
- Rectangular Inequality: $\zeta(\varphi,\pi,\mu)\leq\zeta(\varphi,\pi,\mathfrak{h})+\zeta(\varphi,\mathfrak{h},\mu)+\zeta(\mathfrak{h},\pi,\mu)$.

2. Main Results

The F- contraction idea was first presented by Wardowski [8], and it generalizes the Banach fixed point theorem. Additionally, Wardowski and Dung [9] further extended this notion to an F-weak contraction, leading to fixed point results. In 2020, Dinanath Barman [2] demonstrated common fixed point theorems on a complete 2-metric space by using the T-Hardy Rogers Type Contraction condition and F-Contraction.

Konrawut Khammahawong [12] established fixed point theorems and provided instances for a generalized Roger Hardy-type F-contraction in metric-like spaces in 2017. Additionally, applications involving second-order differential equations and fractional differential equations have been demonstrated.

In 2016 [13], Hossein Piri established fixed point theorems concerning generalized F-Suzuki-contraction mappings within the entirety of b-metric spaces. Building on this topic, Ovidiu Popescu further advanced the subject by presenting two fixed-point theorems related to F-contractions within complete metric spaces [11]. In 2014, Minak et al. [34] obtained result for generalized F-contractions including Ciric type generalized F-contraction and almost F-contraction on complete metric space.

Furthermore, several writers [14–20, 30] used F-contraction mapping in different metric spaces as an example of the fixed point theorem in their respective works.

Definition 2.1: [2] *Consider a 2-m.s denoted by* (Ω,ζ) *and let* Γ *be a self-mapping within this m.s. We define* Γ *to be an* F*-contraction if there exists a constant* $\tau>0$ *such that for all* $\varphi,\pi,\mu\in\Omega$*:*

$$\zeta(\Gamma\varphi,\Gamma\pi,\mu)>0 \Rightarrow \tau+F(\zeta(\Gamma\varphi,\Gamma\pi,\mu))\leq F\zeta(\varphi,\pi,\mu) \tag{1}$$

where F satisfies the following properties:

- F is increasing strictly;
- $\lim_{n\to\infty}\alpha_n=0$ iff $\lim_{n\to\infty}F(\alpha_n)=-\infty$ for each sequence $\{\alpha_n\}\subset R^+$;
- for $0<k<1, \lim_{\alpha\to 0}\alpha^k F(\alpha)=0$.

Theorem 2.2: *Consider* (Ω,ζ) *be a complete 2-m.s and let* $\Gamma:\Omega\to\Omega$ *as an* F*-contraction. Suppose there exists a positive constant* $\tau>0$ *such that*

$$\zeta(\Gamma\varphi,\Gamma\pi,\mu)>0 \Rightarrow \tau+F(\zeta(\Gamma\varphi,\Gamma\pi,\mu))\leq F(\mathfrak{L}(\zeta(\varphi,\pi,\mu)))$$

where,

$$\mathfrak{L}(\zeta(\varphi,\pi,\mu)) = max\left\{\zeta(\varphi,\pi,\mu),\zeta(\varphi,\Gamma\varphi,\mu),\zeta(\pi,\Gamma\pi,\mu),\frac{\zeta(\varphi,\Gamma\pi,\mu)+\zeta(\pi,\Gamma\varphi,\mu)}{2}\right\}$$

for all $\varphi,\pi,\mu \in \Omega$.

Then Γ possesses a unique fixed point $\varphi^* \in \Omega$ and the sequence $\{\varphi_n\}(=\{\Gamma^n\varphi\})$ converges to φ^* for each $\varphi \in \Omega$.

Proof. If there exists an $n \geq 1$ such that

$$\zeta(\varphi_n,\Gamma\varphi_n,\mu) = 0$$

Thus, φ_n represents a fixed point of Γ. So, we can assume

$$\zeta(\Gamma\varphi_{n-1},\Gamma\varphi_n,\mu) = \zeta(\varphi_n,\Gamma\varphi_n,\mu) > 0 \;\; for\ all\ n \geq 1.$$

To begin, we aim to prove that,

$$\begin{aligned}&\lim_{n\to\infty}\zeta(\varphi_n,\Gamma\varphi_n,\mu) = 0\\ &\tau + F(\zeta(\Gamma\varphi_{n-1},\Gamma\varphi_n,\mu)) \leq F(\mathfrak{L}(\zeta(\varphi_{n-1},\varphi_n,\mu)))\end{aligned} \tag{2}$$

where

$$\begin{aligned}\mathfrak{L}(\zeta(\varphi_{n-1},\varphi_n,\mu)) &= max\left\{\begin{array}{l}\zeta(\varphi_{n-1},\varphi_n,\mu),\zeta(\varphi_{n-1},\Gamma\varphi_{n-1},\mu),\zeta(\varphi_n,\Gamma\varphi_n,\mu),\\ \dfrac{\zeta(\varphi_{n-1},\Gamma\varphi_n,\mu)+\zeta(\varphi_n,\Gamma\varphi_{n-1},\mu)}{2}\end{array}\right\}\\ &= max\{\zeta(\varphi_{n-1},\varphi_n,\mu),\zeta(\varphi_n,\Gamma\varphi_n,\mu)\}.\end{aligned}$$

If $\mathfrak{L}(\zeta(\varphi_{n-1},\varphi_n,\mu)) = \zeta(\varphi_n,\Gamma\varphi_n,\mu)$, then the inequality (2) implies that

$$\begin{aligned}\tau + F(\zeta(\Gamma\varphi_{n-1},\Gamma\varphi_n,\mu)) &\leq F(\zeta(\varphi_n,\Gamma\varphi_n,\mu))\\ F(\zeta(\Gamma\varphi_{n-1},\Gamma\varphi_n,\mu)) &\leq F(\zeta(\varphi_n,\Gamma\varphi_n,\mu)) - \tau.\end{aligned}$$

However, this contradict $\tau > 0$. Therefore, we have

$$\mathfrak{L}(\varphi_{n-1},\varphi_n,\mu) = \zeta(\varphi_{n-1},\varphi_n,\mu)$$

and the inequality (2) yields

$$F(\zeta(\Gamma\varphi_{n-1},\Gamma\varphi_n,\mu)) \leq F(\zeta(\varphi_{n-1},\varphi_n,\mu)) - \tau$$

continuing this process, we get

$$\begin{aligned}F(\zeta(\Gamma\varphi_{n-1},\Gamma\varphi_n,\mu)) &\leq F(\zeta(\varphi_{n-1},\varphi_n,\mu)) - \tau\\ &\leq F(\zeta(\varphi_{n-2},\varphi_{n-1},\mu)) - 2\tau \ldots\\ &\leq F(\zeta(\varphi,\varphi_1,\mu)) - n\tau.\end{aligned}$$

Thus,

$$\lim_{n\to\infty}F(\zeta(\Gamma\varphi_{n-1},\Gamma\varphi_n,\mu)) = -\infty$$

This, along with $F \in \mathcal{F}$ and Lemma 3.2 in [23], gives

$$\lim_{n\to\infty}\zeta(\varphi_n,\Gamma\varphi_n,\mu) = 0 \tag{3}$$

We'll prove that the sequence $\{\varphi_n\}$ satisfies the Cauchy sequence. If this is not the case, then $\exists\ \varepsilon > 0$ and the sequence $\{\alpha(n)\}$ and $\{\beta(n)\}$ of $\mathbb{N}$ such that for all $\alpha(n) > \beta(n) > n$,

$$\zeta(\varphi_{\alpha(n)}, \varphi_{\beta(n)}, \mu) \geq \varepsilon,\ \zeta(\varphi_{\alpha(n)-1}, \varphi_{\beta(n)}, \mu) < \varepsilon \quad \text{for all } n \geq 1$$

$$\begin{aligned} \varepsilon &\leq \zeta(\varphi_{\alpha(n)}, \varphi_{\beta(n)}, \mu) \\ &\leq \zeta(\varphi_{\alpha(n)}, \varphi_{\beta(n)}, \varphi_{\alpha(n)-1}) + \zeta(\varphi_{\alpha(n)}, \varphi_{\alpha(n)-1}, \mu) + \zeta(\varphi_{\alpha(n)-1}, \varphi_{\beta(n)}, \mu) \\ &= \zeta(\Gamma\varphi_{\alpha(n)-1}, \varphi_{\alpha(n)-1}, \mu) + \varepsilon \end{aligned} \tag{4}$$

Now, combining this with equation (4), we have

$$\lim_{n\to\infty} \zeta(\varphi_{\alpha(n)}, \varphi_{\beta(n)}, \mu) = \varepsilon \tag{5}$$

However, according to equation (4), there exists n_0 $(n \leq n_0$)such that

$$\zeta(\varphi_{\alpha(n)}, \Gamma\varphi_{\alpha(n)}, \mu) < \frac{\varepsilon}{4} \ \ and \ \ \zeta(\varphi_{\beta(n)}, \Gamma\varphi_{\beta(n)}, \mu) < \frac{\varepsilon}{4} \tag{6}$$

Next, to prove that,

$$\zeta(\varphi_{\alpha(n)}, \varphi_{\beta(n)}, \mu) = \zeta(\varphi_{\alpha(n)+1}, \varphi_{\beta(n)+1}, \mu) > 0 \tag{7}$$

for all $n \geq n_0$. If not, there exists $m(\geq n_0)$ so that

$$\zeta(\varphi_{\alpha(m)+1}, \varphi_{\beta(m)+1}, \mu) = 0 \tag{8}$$

Equation (4), (5), and (8) imply that

$$\begin{aligned} \varepsilon &\leq \zeta(\varphi_{\alpha(m)}, \varphi_{\beta(m)}, \mu) \\ &\leq \zeta(\varphi_{\alpha(n)}, \varphi_{\beta(n)}, \varphi_{\alpha(m)+1}) + \zeta(\varphi_{\alpha(n)}, \varphi_{\alpha(m)+1}, \mu) + \zeta(\varphi_{\alpha(m)+1}, \varphi_{\beta(m)}, \mu) \\ &\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2} \end{aligned}$$

However, this leads to a contradiction, confirming the validity of inequality (7). Consequently, based on (7) and the theorem's assumption, it can be deduced that for all $n \geq n_0$,

$$\tau + F(\zeta(\Gamma\varphi_{\alpha(n)}, \Gamma\varphi_{\beta(n)}, \mu)) \leq F(\mathfrak{L}(\varphi_{\alpha(n)}, \varphi_{\beta(n)}, \mu)) \tag{9}$$

where

$$\begin{aligned} \mathfrak{L}(\varphi_{\alpha(n)}, \varphi_{\beta(n)}, \mu) &= max \left\{ \begin{matrix} \zeta(\varphi_{\alpha(n)}, \varphi_{\beta(n)}, \mu), \zeta(\varphi_{\alpha(n)}, \Gamma\varphi_{\alpha(n)}, \mu), \zeta(\varphi_{\beta(n)}, \Gamma\varphi_{\beta(n)}, \mu), \\ \dfrac{\zeta(\varphi_{\alpha(n)}, \Gamma\varphi_{\beta(n)}, \mu) + \zeta(\varphi_{\beta(n)}, \Gamma\mathfrak{f}_{\alpha(n)}, \mu)}{2} \end{matrix} \right\} \\ &= max \left\{ \begin{matrix} \zeta(\varphi_{\alpha(n)}, \Gamma\varphi_{\alpha(n)}, \mu), \zeta(\Gamma\varphi_{\beta(n)}, \Gamma\varphi_{\alpha(n)}, \mu), \zeta(\varphi_{\beta(n)}, \Gamma\varphi_{\beta(n)}, \mu), \\ \dfrac{\zeta(\varphi_{\alpha(n)}, \Gamma\varphi_{\alpha(n)}, \mu) + \zeta(\varphi_{\beta(n)}, \Gamma\varphi_{\beta(n)}, \mu)}{2} \end{matrix} \right\} \end{aligned} \tag{10}$$

Substituting (10) into (9) and letting $n \to \infty$, we have

$$\tau + \lim_{n\to\infty} F(\zeta(\Gamma\varphi_{\alpha(n)}, \Gamma\varphi_{\beta(n)}, \mu)) \leq \lim_{n\to\infty} F(\zeta(\Gamma\varphi_{\alpha(n)}, \Gamma\varphi_{\beta(n)}, \mu))$$

This is a contraction. Hence the sequence $\{\varphi_n\}$ is a Cauchy sequence.
The completeness of Ω ensure that there exist $\varphi^* \in \Omega$ such that $\{\varphi_n\}$ converges to φ^*.

We demonstrate that φ^* is the fixed point of Γ. If not, we can assume that $\zeta(\varphi^*,\Gamma\varphi^*,\mu) > 0$. By applying the hypothesis of the theorem, we acquire

$$\tau + F(\zeta(\Gamma\varphi_n,\Gamma\varphi^*,\mu)) \leq F(\mathfrak{L}(\varphi_n,\varphi^*,\mu)) \tag{11}$$

where

$$\mathfrak{L}(\varphi_n,\varphi^*,\mu) = max\left\{\begin{matrix}\zeta(\varphi_n,\varphi^*,\mu),\zeta(\varphi_n,\Gamma\varphi_n,\mu),\zeta(\varphi^*,\Gamma\varphi^*,\mu), \\ \dfrac{\zeta(\varphi_n,\Gamma\varphi^*,\mu)+\zeta(\varphi^*,\Gamma\varphi_n,\mu)}{2}\end{matrix}\right\}$$

Letting $n \to \infty$, then $\mathfrak{L}(\varphi_n,\varphi^*,\mu) \to \zeta(\varphi^*,\Gamma\varphi^*,\mu)$. Hence, from (11),

$$\tau + F(\zeta(\varphi^*,\Gamma\varphi^*,\mu)) \leq F(\zeta(\varphi^*,\Gamma\varphi^*,\mu))$$

This represents a contraction for $\tau > 0$. Thus φ^* is a fixed point of Γ.
Next, we aim to demonstrate the uniqueness of the fixed point of Γ.
Let φ_1^*,φ_2^* are fixed points of Γ. Suppose $\varphi_1^* \neq \varphi_2^*$ then $\Gamma\varphi_1^* \neq \Gamma\varphi_2^*$.

$$\begin{aligned}\tau + F(\zeta(\Gamma\varphi_1^*,\Gamma\varphi_2^*,\mu)) &\leq F(\mathfrak{L}(\varphi_1^*,\varphi_2^*,\mu)) \\ \mathfrak{L}(\varphi_1^*,\varphi_2^*,\mu) &= max\left\{\begin{matrix}\zeta(\varphi_1^*,\varphi_2^*,\mu),\zeta(\varphi_1^*,\Gamma\varphi_1^*,\mu),\zeta(\varphi_2^*,\Gamma\varphi_2^*,\mu), \\ \dfrac{\zeta(\varphi_1^*,\Gamma\varphi_2^*,\mu)+\zeta(\varphi_2^*,\Gamma\varphi_1^*,\mu)}{2}\end{matrix}\right\}\end{aligned} \tag{12}$$

Letting $n \to \infty$,
then $\mathfrak{L}\zeta(\varphi_1^*,\varphi_2^*,\mu)) \to \zeta(\varphi_1^*,\varphi_2^*,\mu)$.
Hence from equation (12),

$$\tau + F(\zeta(\varphi_1^*,\varphi_2^*,\mu)) \leq F(\zeta(\varphi_1^*,\varphi_2^*,\mu))$$

This leads to a contradiction, thus establishing the uniqueness of the fixed point.

Example 2.3: *Let* $\Omega = [0,1)$ *and* $\zeta : \Omega\times\Omega\times\Omega \to [0,1)$ *be given* $\zeta(\varphi,\pi,\mu) = min\{|\varphi-\pi|,|\pi-\mu|,|\mu-\varphi|\}$ *then* (Ω,ζ) *is a 2-m.s. Let* Γ *defined by* $\Gamma\varphi = \frac{\varphi}{2}$ *for all* $\varphi\in\Omega$*. Then for all* $\varphi,\pi,\mu\in[0,1)$ *where* $\varphi<\pi<\mu, \zeta(\varphi,\pi,\mu) = min\{|\varphi-\pi|,|\pi-\mu|,|\mu-\varphi|\} = |\varphi-\pi|$.
Then

$$|\varphi-\pi| \leq |\pi-\mu| \text{ and } |\varphi-\pi| \leq |\mu-\varphi|$$

implies, $|\varphi-\pi| \leq |\pi-2\mu|$ and $|\varphi-\pi| \leq |\mu-2\varphi|$ since $\varphi<\pi<\mu<2\mu$

$$\frac{1}{2}|\varphi-\pi| \leq \frac{1}{2}|\pi-2\mu| \text{ and } \frac{1}{2}|\varphi-\pi| \leq \frac{1}{2}|\mu-2\varphi|$$

Now,

$$\zeta(\Gamma\varphi,\Gamma\pi,\mu) = \zeta\left(\frac{\varphi}{2},\frac{\pi}{2},\mu\right) = min\left\{\left|\frac{\varphi}{2}-\frac{\pi}{2}\right|,\left|\frac{\pi}{2},\mu\right|,\left|\mu-\frac{\varphi}{2}\right|\right\} = min\left\{\frac{1}{2}|\varphi-\pi|,\frac{1}{2}|\pi-2\mu|,\frac{1}{2}|\mu-2\varphi|\right\}$$

Therefore,

$$F(\zeta(\Gamma\varphi,\Gamma\pi,\mu)) = F\left(\frac{1}{2}|\varphi-\pi|\right) = \frac{1}{2}|\varphi-\pi|.$$

Then there exists $\tau = \frac{1}{2}|\varphi - \pi| > 0$ such that

$$\tau + F(\zeta(\Gamma\varphi, \Gamma\pi, \mu)) \leq F|\varphi - \pi| = F(\zeta(\varphi, \pi, \mu)).$$

Theorem 2.4: *Let Γ be a self-mapping of a 2-m.s (Ω, ζ) and $F: \mathbb{R}^+ \to \mathbb{R}$ be an increasing function. If there exists a value $\tau > 0$ such that for all $\varphi, \pi, \mu \in \Omega$, the following condition is satisfied:*

$$\zeta(\Gamma\varphi, \Gamma\pi, \mu) \geq 0 \Rightarrow \tau + F(\zeta(\Gamma\varphi, \Gamma\pi, \mu)) \leq F(\Im(\varphi, \pi, \mu))$$

where

$$\Im(\varphi, \pi, \mu) = \rho_1\left[\zeta(\varphi, \pi, \mu) + \zeta(\varphi, \Gamma\varphi, \mu) + \zeta(\pi, \Gamma\pi, \mu) + \frac{1}{2}[\zeta(\varphi, \Gamma\pi, \mu) + \zeta(\pi, \Gamma\varphi, \mu)]\right]$$
$$+\rho_2[\zeta(\varphi, \Gamma\varphi, \mu) + \zeta(\pi, \Gamma\pi, \mu)] + \rho_3[\zeta(\varphi, \Gamma\pi, \mu) + \zeta(\pi, \Gamma\varphi, \mu)]$$

then ρ_1, ρ_2, ρ_3 be non-negative real numbers satisfying $2\rho_1 + \rho_2 + \rho_3 < 1$ and $\rho_1, \rho_2, \rho_3 \geq 0$. Under these conditions, Γ has an unique fixed point of $\varphi^* \in \Omega$, and for any $\varphi \in \Omega$, the sequence $\{\Gamma^n\varphi\}_{n\in\mathbb{N}}$ converges to φ^*.

Proof. Assume $\varphi_0 \in \Omega$ and consider $\{\varphi_n\}_{n\in N} \in \Omega$ by

$$\varphi_1 = \Gamma\varphi_0, \varphi_2 = \Gamma\varphi_1 = \Gamma^2\varphi_0, \ldots, \varphi_n = \Gamma\varphi_{n-1} = \Gamma^n\varphi_0 \quad \textit{for all } n \in N. \tag{13}$$

If there is an element $\varphi \in N \cup 0$ such that $\zeta(\varphi_n, \Gamma\varphi_n, \mu = 0)$ then φ_n is a fixed point of Γ.

Let's assume that

$$0 < \zeta(\varphi_n, \Gamma\varphi_n, \mu) = \zeta(\Gamma\varphi_{n-1, \Gamma\varphi_n}) \textit{ for all } n \in N. \tag{14}$$

Let $\Pi_n = \zeta(\varphi_n, \varphi_{n+1}, \mu)$

$$\tau + F(\Pi_n) = \tau + F(\zeta(\varphi_n, \varphi_{n+1}, \mu)) = \tau + F(\zeta(\Gamma\varphi_{n-1}, \Gamma\varphi_n, \mu)) \leq F(\Im(\varphi, \pi, \mu))$$

where

$$\begin{aligned}\Im(\varphi_{n-1}, \varphi_n, \mu) &= \rho_1[\zeta(\varphi_{n-1}, \varphi_n, \mu) + \zeta(\varphi_{n-1}, \Gamma\varphi_{n-1}, \mu) + \zeta(\varphi_n, \Gamma\varphi_n, \mu)\\ &\quad + \frac{1}{2}[\zeta(\varphi_{n-1}, \Gamma\varphi_n, \mu) + \zeta(\varphi_n, \Gamma\varphi_{n-1}, \mu)]]\\ &\quad +\rho_2[\zeta(\varphi_{n-1}, \Gamma\varphi_{n-1}, \mu) + \zeta(\varphi_n, \Gamma\varphi_n, \mu)]\\ &\quad +\rho_3[\zeta(\varphi_{n-1}, \Gamma\varphi_n, \mu) + \zeta(\varphi_n, \Gamma\varphi_{n-1}, \mu)]\\ &= \rho_1[\zeta(\varphi_{n-1}, \varphi_n, \mu) + \zeta(\varphi_{n-1}, \varphi_n, \mu) + \zeta(\varphi_n, \varphi_{n+1}, \mu)\\ &\quad + \frac{1}{2}[\zeta(\varphi_{n-1}, \varphi_{n+1}, \mu)]] + \rho_2[\zeta(\varphi_{n-1}, \varphi_n, \mu) + \zeta(\varphi_n, \varphi_{n+1}, \mu)]\\ &\quad +\rho_3[\zeta(\varphi_{n-1}, \varphi_{n+1}, \mu)]\\ &= \left[\frac{5}{2}\varrho_1 + \varrho_2 + \varrho_3\right]\Pi_{n-1} + \left[\frac{3}{2}\varrho_1 + \varrho_2 + \varrho_3\right]\Pi_n\end{aligned}$$

$$\tau + F(\Pi_n) \leq F\left[\left[\frac{5}{2}\varrho_1 + \varrho_2 + \varrho_3\right]\Pi_{n-1} + \left[\frac{3}{2}\varrho_1 + \varrho_2 + \varrho_3\right]\Pi_n\right] \tag{15}$$

$$F(\Pi_n) \leq F\left[\left[\frac{5}{2}\varrho_1 + \varrho_2 + \varrho_3\right]\Pi_{n-1} + \left[\frac{3}{2}\varrho_1 + \varrho_2 + \varrho_3\right]\Pi_n\right] - \tau$$

$$\left[1-\frac{3}{2}\varrho_1-\varrho_2-\varrho_3\right]\Pi_n \leq \left[\frac{5}{2}\varrho_1+\varrho_2+\varrho_3\right]\Pi_{n-1}$$

$$\Pi_n \leq \left[\frac{\frac{5}{2}\varrho_1+\varrho-2+\varrho_3}{1-\frac{3}{2}\varrho_1-\varrho_2-\varrho_3}\right]\Pi_{n-1}$$

$$\Pi_n \leq \Pi_{n-1}$$

Thus, the sequence $\{\Pi_n\}_{n\in N}$ is strictly decreasing, which means that $\lim_{n\to\infty}\Pi_n = \Pi$ exists. Assume $\Pi > 0$.
Since F is increasing

$$\lim_{f\to\Pi_n{}^+}\Gamma(\varphi) = F(\Pi+0)$$

In inequality (15), taking the limit n tends to ∞,

$$F(\Pi+0) \leq F(\Pi+0)-\tau$$

which is a contradiction. Therefore, it must be that

$$\lim_{n\to+\infty}\Pi_n = 0 \tag{16}$$

To establish the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, Consider the sequences $\{\alpha(n)\}_{n\in\mathbb{N}}$ and $\{\beta(n)\}_{n\in\mathbb{N}}$ where $\alpha(n) > \beta(n) > n$ for every $n\in\mathbb{N}$, and let $\varepsilon > 0$.

$$\zeta(\varphi_{\alpha(n)},\varphi_{\beta(n)},\mu) \geq \varepsilon,\ \zeta(\varphi_{\alpha(n)-1},\varphi_{\beta(n)},\mu) \leq \varepsilon \qquad \textit{for all } n\in\mathbb{N}. \tag{17}$$

By rectangular inequality

$$\varepsilon \leq \zeta(\varphi_{\alpha(n)},\varphi_{\beta(n)},\mu) \leq \zeta(\varphi_{\alpha(n)},\varphi_{\beta(n)},\varphi_{\alpha(n)-1}) + \zeta(\varphi_{\alpha(n)},\varphi_{\alpha(n)-1},\mu) + \zeta(\varphi_{\alpha(n)-1},\varphi_{\beta(n)},\mu) = \zeta(\Gamma\varphi_{\alpha(n)-1},\varphi_{\alpha(n)-1},\mu)+\varepsilon$$

Relation (16) and the preceding inequality imply that

$$\lim_{n\to\infty}\zeta(\varphi_{\alpha(n)},\varphi_{\beta(n)},\mu) = \varepsilon \tag{18}$$

since $\zeta(\varphi_{\alpha(n)},\varphi_{\beta(n)}) > \varepsilon > 0$, by property of F we get

$$\begin{aligned}\tau + F\zeta(\varphi_{\alpha(n)},\varphi_{\beta(n)},\mu) &= \tau + F\zeta(\Gamma\varphi_{\alpha(n)-1},\Gamma\varphi_{\beta(n)-1},\mu)\\ &\leq F(\Im(\varphi_{\alpha(n)-1},\varphi_{\beta(n)-1},\mu))\end{aligned}$$

$$\begin{aligned}&\Im(\varphi_{\alpha(n)-1},\varphi_{\beta(n)-1},\mu)\\ &= \rho_1[\zeta(\varphi_{\alpha(n)-1},\varphi_{\beta(n)-1},\mu)+\zeta(\varphi_{\alpha(n)-1},\Gamma\varphi_{\alpha(n)-1},\mu)+\zeta(\varphi_{\beta(n)-1},\Gamma\varphi_{\beta(n)-1},\mu)\\ &\quad+\frac{1}{2}[\zeta(\varphi_{\alpha(n)-1},\Gamma\varphi_{\beta(n)-1},\mu)+\zeta(\varphi_{\beta(n)-1},\Gamma\varphi_{\alpha(n)-1},\mu)]]\\ &\quad+\rho_2[\zeta(\varphi_{\alpha(n)-1},\Gamma\varphi_{\alpha(n)-1},\mu)+\zeta(\varphi_{\beta(n)-1},\Gamma\varphi_{\beta(n)-1},\mu)]\\ &\quad+\rho_3[\zeta(\varphi_{\alpha(n)-1},\Gamma\varphi_{\beta(n)-1},\mu)+\zeta(\varphi_{\beta(n)-1},\Gamma\varphi_{\alpha(n)-1},\mu)]\\ &= \rho_1[2\Pi_{\alpha(n)-1}+\zeta(\varphi_{\alpha(n)},\varphi_{\beta(n)},\mu)+\zeta(\varphi_{\alpha(n)},\varphi_{\beta(n)-1},\varphi_{\beta(n)})\\ &\quad+\zeta(\varphi_{\alpha(n)-1},\varphi_{\beta(n)-1},\varphi_{\alpha(n)})+2\Pi_{\beta(n)-1}+\frac{1}{2}[\Pi_{\alpha(n)-1}+\Pi_{\beta(n)-1}\end{aligned}$$

$$+2\zeta(\varphi_{\alpha(n)},\varphi_{\beta(n)},\mu)+\zeta(\varphi_{\alpha(n)},\varphi_{\alpha(n)-1},\varphi_{\beta(n)})+\zeta(\varphi_{\beta(n)-1},\varphi_{\alpha(n)},\varphi_{\beta(n)})]]$$
$$+\rho_2[\Pi_{\alpha(n)-1}+\Pi_{\beta(n)-1}]+\rho_3[\Pi_{\alpha(n)-1}+2\zeta(\varphi_{\alpha(n)},\varphi_{\beta(n)},\mu)$$
$$+\Pi_{\beta(n)-1}+\zeta(\varphi_{\beta(n)-1},\varphi_{\alpha(n)},\varphi_{\beta(n)})+\zeta(\varphi_{\alpha(n)},\varphi_{\alpha(n)-1},\varphi_{\beta(n)})]$$

$$\tau+F(\zeta(\varphi_{\alpha(n)},\varphi_{\beta(n)},\mu))\le F\Bigg[\left(\frac{5}{2}\rho_1+\rho_2+\rho_3\right)\Pi_{\alpha(n)-1}$$
$$+\left(\frac{5}{2}\rho_1+\rho_2+\rho_3\right)\Pi_{\beta(n)-1}+(2\rho_1+2\rho_3)\zeta(\varphi_{\alpha(n)},\varphi_{\beta(n)},\mu)$$
$$+\left(\frac{3}{2}\rho_1+\rho_3\right)\zeta(\varphi_{\alpha(n)},\varphi_{beta(n)-1},\varphi_{\beta(n)})+(\rho_1+\rho_3)\zeta(\varphi_{\alpha(n)},\varphi_{\alpha(n)-1},\varphi_{\beta(n)})+\rho_1\zeta(f_{\alpha(n)},\varphi_{\alpha(n)-1},\varphi_{\beta(n)-1})\Bigg]$$

Limit as $n\to\infty$,

$$\tau+F(\varepsilon+0)\le F(\varepsilon+0)$$

This is a contradiction. Hence the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.Since (Ω,ζ) is a complete m.s then the sequence $\{\varphi_n\}$ for $n\in\mathbb{N}$ converges to a certain limit φ^* within Ω. If $\exists\{\alpha(n)\}_{n\in N}$ of natural number such that
$\varphi_{\alpha(n)+1}=\Gamma\varphi_{\alpha(n)}=\Gamma\varphi^*$ then $\lim_{n\to\infty}\varphi_{\alpha(n)+1}=\varphi^*$. Thus $\Gamma\varphi^*=\varphi^*$. Assume that $\Gamma\varphi^*\ne\varphi^*$, we obtained

$$\tau+F(\zeta(\Gamma\varphi_n,\Gamma\varphi^*,\mu))\le F(\Im(\varphi_n,\varphi^*,\mu))$$

where

$$\Im(\varphi_n,\varphi^*,\mu)=\rho_1[\zeta(\varphi_n,\varphi^*,\mu)+\zeta(\varphi_n,\Gamma\varphi_n,\mu)+\zeta(\varphi^*,\Gamma\varphi^*,\mu)$$
$$+\frac{1}{2}\Big[\zeta(\varphi_n,\Gamma\varphi^*,\mu)+\zeta(\varphi^*,\Gamma\varphi_n,\mu)\Big]+\rho_2[\zeta(\varphi_n,\Gamma\varphi_n,\mu)+\zeta(\varphi^*,\Gamma\varphi^*,\mu)]$$
$$+\rho_3[\zeta(\varphi_n,\Gamma\varphi^*,\mu)+\zeta(\varphi^*,\Gamma\varphi_n,\mu)]$$

$$\begin{aligned}\tau+F(\zeta(\Gamma\varphi_n,\Gamma\varphi^*,\mu))\le\ & F[\rho_1[\zeta(\varphi_n,\varphi^*,\mu)+\zeta(\varphi_{n+1},\varphi_n,\mu)+\zeta(\varphi^*,\Gamma\varphi^*,\mu)\\ &+\frac{1}{2}\Big[\zeta(\varphi_n,\Gamma\varphi^*,\mu)+\zeta(\varphi^*,\varphi_{n+1},\mu)\Big]\Big]\\ &+\rho_2[\zeta(\varphi_n,\varphi_{n+1},\mu)+\zeta(\varphi^*,\Gamma\varphi^*,\mu)]\\ &+\rho_3[\zeta(\varphi_n,\Gamma\varphi^*,\mu)+\zeta(\varphi^*,\varphi_{n+1},\mu)]]\end{aligned}$$

taking $n\to\infty$ we get

$$\begin{aligned}\zeta(\varphi^*,\Gamma\varphi^*,\mu)\ &\le\rho_1\zeta(\varphi^*,\Gamma\varphi^*,\mu)+\rho_2\zeta(\varphi^*,\Gamma\varphi^*,\mu)\\ &\le(\rho_1+\rho_2)\zeta(\varphi^*,\Gamma\varphi^*,\mu)\\ &\le\zeta(\varphi^*,\Gamma\varphi^*,\mu)\end{aligned}$$

which is contraction and therefore, $\varphi^*=\Gamma\varphi^*$. Let φ^* and π^* be two distinct fixed points of Γ in Ω. Then $\zeta(\Gamma\varphi^*,\Gamma\pi^*,\mu)=\zeta(\varphi^*,\pi^*,\mu)>0$, we have

$$\tau+F(\zeta(\varphi^*,\pi^*,\mu))=\tau+F(\zeta(\Gamma\varphi^*,\Gamma\pi^*,\mu))\le F(\Im(\varphi^*,\pi^*,\mu))$$

where,

$$\begin{aligned}\Im(\varphi^*,\pi^*,\mu) &= \rho_1[\zeta(\varphi^*,\pi^*,\mu)+\zeta(\varphi^*,\Gamma\varphi^*,\mu)+\zeta(\pi^*,\Gamma\pi^*,\mu)\\ &\quad +\frac{1}{2}\Big[\zeta(\varphi^*,\Gamma\pi^*,\mu)+\zeta(\pi^*,\Gamma\varphi^*,\mu)\Big]\Big]+\rho_2[\zeta(f^*,\Gamma\varphi^*,\mu)+\zeta(\pi^*,\Gamma\pi^*,\mu)]\\ &\quad +\rho_3[\zeta(\varphi^*,\Gamma\pi^*,\mu)+\zeta(\pi^*,\Gamma\varphi^*,\mu)]\\ &= \rho_1[2\zeta(\varphi^*,\pi^*,\mu)]+\rho_3[2\zeta(\varphi^*,\pi^*,\mu)]\\ \Im(\varphi^*,\pi^*,\mu) &= (2\rho_1+2\rho_2)\zeta(\varphi^*,\pi^*,\mu)\tau+F(\zeta(\varphi^*,\pi^*,\mu))\\ &\le F((2\rho_1+2\rho_2)\zeta(\varphi^*,\pi^*,\mu))\le\zeta(\varphi^*,\pi^*,\mu)\end{aligned}$$

Indeed, this leads to a contradiction, thus proving that the fixed point is unique.

3. Applications

Fixed point theory is a notable approach for addressing integral equations. It is frequently utilized in complex-valued b-metric spaces to identify unique common solutions for systems of integral equations. Throughout the paper, we denote the following:

- We write Ω as a 2-m.s.
- Θ :is defined as the set of functions ϑ where each $\vartheta:\mathbb{R}_+\to\mathbb{R}_+$ is Lebesgue integrable , summable on each a compact subset of $\mathbb{R}_+$, satisfying the following conditions:
  1. $\int_0^{\varepsilon}\vartheta(j)\eth j>0$ for each ε and
  2. $\int_0^{a+b}\vartheta(j)\eth j\le\int_0^{a}\vartheta(j)\eth j+\int_0^{b}\vartheta(j)\eth j$

Theorem 3.1: *Let* (Ω,ζ) *be a complete 2-m.s and* $\Gamma:\Omega\to\Omega$ *be a self mappings satisfying the relation*

$$\int_0^{\tau+F(\zeta(\Gamma\varphi,\Gamma\pi,\mu))}\vartheta(j)\eth j\le\int_0^{\mathfrak{L}F(\zeta(\varphi,\pi,\mu))}\vartheta(j)\eth j$$

where $\vartheta\in\Theta$ and

$$\mathfrak{L}(\varphi,\pi,\mu)=max\left\{\zeta(\varphi,\pi,\mu),\zeta(\varphi,\Gamma\varphi,\mu),\zeta(\pi,\Gamma\pi,\mu),\frac{\zeta(\varphi,\Gamma\pi,\mu)+\zeta(\pi,\Gamma\varphi,\mu)}{2}\right\}$$

for all $\varphi,\pi,\mu\in\Omega$, then Γ have unique fixed point in Ω.

Proof. Let φ_n be a sequence in Ω such that $\varphi_{n+1}=\Gamma\varphi_n$ for $n\in\mathbb{N}\cup 0$ where $\varphi_0\in\Omega$ is an initial approximation.

If $\varphi_{n+1}=\varphi_n\Rightarrow\Gamma\varphi_n=\varphi_n$ then φ_n is a fixed point of Γ and this complete the proof.

Assume $\varphi_n\ne\varphi_{n+1}$ First to prove $\zeta(\varphi_{n+1},\varphi_n,\mu)=0$

$$\int_0^{\tau+F(\zeta(\varphi_{n+1},\varphi_n,\mu))}\vartheta(j)\eth j=\int_0^{\tau+F(\zeta(\Gamma\varphi_n,\Gamma\varphi_{n-1},\mu))}\vartheta(j)\eth j\le\int_0^{F(\mathfrak{L}(\zeta(\varphi_n,\varphi_{n-1},\mu)))}\vartheta(j)\eth j$$

where

$$\begin{aligned}\mathfrak{L}(\zeta(\varphi_n,\varphi_{n-1},\mu)) &= max\{\zeta(\varphi_n,\varphi_{n-1},\mu),\zeta(\varphi_n,\Gamma\varphi_n,\mu),\zeta(\varphi_{n-1},\Gamma\varphi_{n-1},\mu),\\ &\quad \frac{\zeta(\varphi_n,\Gamma\varphi_{n-1},\mu)+\zeta(\varphi_{n-1},\Gamma\varphi_n,\mu)}{2}\}\\ &= max\left\{\zeta(\varphi_n,\varphi_{n-1},\mu),\zeta(\varphi_n,\varphi_{n+1},\mu)\right\}\end{aligned}$$

If $\mathfrak{L}(\zeta(\varphi_n,\varphi_{n-1},\mu))=\zeta(\varphi_n,\varphi_{n+1},\mu)$

$$\begin{aligned}
&\int_0^{\tau+F(\zeta(\varphi_n,\varphi_{n+1},\mu))}\vartheta(j)\eth j\le\int_0^{F(\zeta(\varphi_n,\varphi_{n+1},\mu))}\vartheta(j)\eth j\\
&\tau+F(\zeta(\varphi_n,\varphi_{n+1},\mu))\le F(\zeta(\varphi_n,\varphi_{n+1},\mu))\\
&F(\zeta(\varphi_n,\varphi_{n+1},\mu))\le F(\zeta(\varphi_n,\varphi_{n+1},\mu))-\tau\\
&\zeta(\varphi_n,\varphi_{n+1},\mu)\le\zeta(\varphi_n,\varphi_{n+1},\mu))-\tau
\end{aligned}$$

which is contradiction, since $\tau>0$.

Therefore $\mathfrak{L}(\zeta(\varphi_n,\varphi_{n-1},\mu))=\zeta(\varphi_n,\varphi_{n-1},\mu)$

$$\begin{aligned}
&F(\zeta(\varphi_n,\varphi_{n+1},\mu))\le F(\zeta(\varphi_n,\varphi_{n-1},\mu))\\
&\zeta(\varphi_n,\varphi_{n+1},\mu)\le\zeta(\varphi_n,\varphi_{n-1},\mu)
\end{aligned}$$

Therefore, $\zeta(\varphi_n,\varphi_{n+1})$ is a monotonically decreasing sequence of real numbers that is bounded below, and consequently, it converges. since,

$$\tau+F(\zeta(\varphi_n,\varphi_{n+1},\mu))=\tau+\zeta(\Gamma\varphi_n,\Gamma\varphi_{n-1},\mu)\le F(\mathfrak{L}\zeta(\varphi_n,\varphi_{n-1},\mu))$$

we have,

$$\begin{aligned}
\int_0^{F(\zeta(\varphi_n,\varphi_{n+1},\mu))}\vartheta(j)\eth j&\le\int_0^{F(\zeta(\varphi_n,\varphi_{n-1},\mu))-\tau}\vartheta(j)\eth j\\
&\le\int_0^{F(\zeta(\varphi_{n-1},\varphi_{n-2},\mu))-2\tau}\vartheta(j)\eth j\\
&\vdots\\
&\le\int_0^{F(\zeta(\varphi_1,\varphi_0,\mu))-n\tau}\vartheta(j)\eth j\\
F(\zeta(\varphi_n,\varphi_{n+1},\mu))&\le F(\zeta(\varphi_1,\varphi_0,\mu))-n\tau
\end{aligned}$$

Taking limit as $n\to\infty$ we obtain from the above that

$$\lim_{n\to\infty}F(\zeta(\varphi_n,\varphi_{n+1},\mu))=-\infty\Rightarrow\lim_{n\to\infty}\zeta(\varphi_n,\varphi_{n+1},\mu)=0.$$

since (Ω,ζ) be complete 2-m.s. we have $n,m\in\mathbb{N},n>m$.

$$\begin{aligned}
\lim_{n,m\to\infty}\int_0^{F(\zeta(\varphi_n,\varphi_m,\mu))}\vartheta(j)\eth j&=\lim_{n,m\to\infty}\int_0^{F(\zeta(\Gamma\varphi_{n-1},\Gamma\varphi_{m-1},\mu))}\vartheta(j)\eth j\\
&\le\lim_{n,m\to\infty}\int_0^{\zeta(\varphi_{n-1},\varphi_{m-1},\mu)-\tau}\vartheta(j)\eth j\\
&\vdots\\
&\le\lim_{n,m\to\infty}\int_0^{\zeta(\varphi_{n-m-1},\varphi_0,\mu)-(m+1)\tau}\vartheta(j)\eth j
\end{aligned}$$

$$\lim_{n,m\to\infty}F(\zeta(\varphi_n,\varphi_m,\mu))\le\lim_{n,m\to\infty}\zeta(\varphi_{n-m-1},\varphi_0,\mu)-(m+1)\tau=-\infty\zeta(\varphi_n,\varphi_m,\mu)=0$$

Thus, φ_n is a Cauchy sequence. Given that (Ω,ζ) is a complete m.s, there exists an $\varphi\in\Omega$ such that

$$\begin{aligned}
&\lim_{n\to}\zeta(\varphi_n,\varphi,\mu)=0\\
&\lim_{n\to}\int_0^{F(\zeta(\varphi_n,\Gamma\varphi,\mu))}\vartheta(j)\eth j=\lim_{n\to\infty}\int_0^{F(\zeta(\Gamma\varphi_{n-1},\Gamma\varphi,\mu))}\vartheta(j)\eth j\le\lim_{n\to\infty}\int_0^{F(\zeta(\varphi_{n-1},\varphi,\mu))-\tau}\vartheta(j)\eth j\le\lim_{n\to\infty}\int_0^{F(\zeta(\varphi_{n-1},\varphi,\mu))}\vartheta(j)\eth j\\
&\lim_{n\to\infty}F(\zeta(\varphi_n,\Gamma\varphi,\mu))\le\lim_{n\to\infty}F(\zeta(\varphi_{n-1},\varphi,\mu))\\
&\lim_{n\to}\infty\zeta(\varphi_n,\Gamma\varphi,\mu)\le\lim_{n\to\infty}\zeta(\varphi_{n-1},\varphi,\mu)=0\\
&\lim_{n\to\infty}\zeta(\varphi_n,\Gamma\varphi,\mu)=0(i.e).,\Gamma\varphi=\lim_{n\to\infty}\varphi_n=\varphi
\end{aligned}$$

Thus φ is a fixed point of $\{\Gamma_n\}_{n-1}^{\infty}$

Suppose $\varphi \neq \delta$ is another fixed point.

Then

$$\int_0^{F(\zeta(\varphi,\delta,\mu))} \vartheta(j)\eth j \leq \int_0^{\tau+F(\zeta(\varphi,\delta,\mu))} \vartheta(j)\eth j = \int_0^{\tau+F(\zeta(\Gamma\varphi,\Gamma\delta,\mu))} \vartheta(j)\eth j \leq \int_0^{F(\zeta(\varphi,\delta,\mu))} \vartheta(j)\eth j$$

a contradiction.

Therefore, $\varphi = \delta$.

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