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Blow-up solutions of a system of nonlinear the Klein-Gordon-Fock type wave equations

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Abstract

We consider the initial boundary value problem for a system of strongly damped wave equations with homogeneous Dirichlet boundary conditions and a nonlinear source term. By applying a modification of the concavity method, we demonstrate that the solutions blow up for $p < 3$ with arbitrary positive initial data. Furthermore, we show that the global solvability of the problem for $p \geq 3$.

Key words and phrases: nonlinear generalized Klein-Gordon type equations, blow up, Dirichlet's boundary conditions.

Mathematics Subject Classification: 35A01, 35B44, 35G61, 35L53

1. Introduction

In this note, we consider the initial boundary value problem for the following system

$$u_{tt} - \alpha\Delta u_t - a_1^2\Delta u + \overline{m}_1^2 u + k\,|u|^{p-1}\,u = v^2 u, \quad x \in \Omega, 0 < t < T, \tag{1}$$

$$v_{tt} - \alpha\Delta v_t - a_2^2\Delta v + \overline{m}_2^2 v + k\,|v|^{p-1}\,v = u^2 v, \quad x \in \Omega, 0 < t < T, \tag{2}$$

under the following initial and boundary conditions

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), v(x,0) = v_0(x), v_t(x,0) = v_1(x), \tag{3}$$

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$$u(x,t) = v(x,t) = 0, \ for \ (x,t) \ in \ \partial\Omega \times [0,T), \tag{4}$$

where $\alpha > 0$, the positive numbers $\overline{m}_1$ and $\overline{m}_2$ represent the masses of the scalar fields u and v, respectively. The functions u_0, v_0, u_1 and v_1 are given, Ω is an open bounded connected domain in $\mathbb{R}^n$ with a Lipschitz boundary. We also assume that

$$\begin{cases} 1 < p \leq \dfrac{n}{n-2}, & for \quad n \geq 3 \\ p > 1, & for \quad n \leq 2. \end{cases}$$

Our goal is to investigate the existence of blow-up solutions for the problem (1)-(4). The research on global nonexistence or blow-up solutions is a longstanding topic, extensively explored by numerous researchers in the context of wave equations. Among them, we refer to [4, 10, 12, 15, 16, 19, 20, 26, 28, 34].

The first result on the global nonexistence of the solutions for the strongly damped nonlinear abstract wave equation

$$Pu_{tt} + Au + \nu A u_t = G(u),$$

in a Hilbert space H is established by Levine in [20]. Here both $P \ and \ A$ are positive self-adjoint operators in a Hilbert space,and $G(u)$ is a nonlinear operator that satisfies the condition

$$(u, G(u)) \geq 2(2\alpha_1 + 1)H(u), \tag{5}$$

where $\alpha_1 > 0$ and $H(u)$ is the Fretchèt anti-derivative of G and $(.,.)$ denotes the Euclidean inner product. The main result of blow up solution is obtained here assuming the initial energy of the system is non-positive.

It is obvious that the vector field $F(u,v) = (v^2 u - k|u|^{p-1} u, u^2 v - k|v|^{p-1} v)$ of the system (1)-(2) does not satisfy this condition (5).

In their work, Bilgin and Kalantarov [3] studied the problem of nonexistence of global solutions of a Cauchy problem for the following nonlinear abstract equation

$$Av_{tt} + Bv + Cv_t = G(v), \tag{6}$$

under the initial condition

$$v(0) = v_0, \quad v_t(0) = v_1. \tag{7}$$

where A and B are densely defined self-adjoint positive definite operators in a Hilbert space, and C is a selfadjoint densely defined non-negative operator such that

$$D(B) \subseteq D(C) \subseteq D(A),$$

and the nonlinear operator $G(v)$ meets the condition

$$(G(u,v),(u,v)) - 2(2\alpha_1 + 1)H(u,v) \geq -D_0, \alpha_1 > 0, D_0 \geq 0.$$

Here the authors showed that there is a class of initial data with arbitrary large initial energy for which the solutions of the Cauchy problem (6) blow up in a finite time. M.O. Korpusov in [15], examined homogeneous Dirichlet problem for the nonlinear system of equations of the Klein-Gordon-Fock type

$$\begin{aligned} &u_{tt} + \mu u_t - a^2 \Delta u + m_1^2 u = v^2 u, \quad u|_{\partial\Omega} = 0, \\ &v_{tt} + \mu v_t - b^2 \Delta u + m_2^2 v = u^2 v, \quad v|_{\partial\Omega} = 0, \\ &u(x,0) = u_0(x), u_t(x,0) = u_1(x), v(x,0) = v_0(x), v_t(x,0) = v_1(x), \end{aligned}$$

where $\mu \geq 0, a, b, m_1$ and m_2 are positive numbers and the datum are given functions u_0, u_1, v_0, and v_1 and Ω is a subset in $\mathbb{R}^3$ with the regular boundary $\partial\Omega$ belongs to $\mathbb{C}^{2,\eta}$ for $\eta \in (0,1]$. Using the Faedo-Galerkin approximation he proved the existence-uniqueness of the local weak solutions and the existence of blow-up solutions by using a modification of Levine's concavity method developed in [17].

Recently Y. Ye and L. Li, using the potential well method studied the existence of global solutions and the existence of blow-up solutions for a class of strongly damped wave equations in the system

$$u_{tt} - \Delta u + \mu_1 u_t - \omega_1 \Delta u_t = g_1(u,v), (x,t) \in \Omega \times \mathbb{R}^+,$$
$$v_{tt} - \Delta v + \mu_2 v_t - \omega_2 \Delta v_t = g_2(u,v), (x,t) \in \Omega \times \mathbb{R}^+,$$

under the initial boundary value conditions

$$u(x,0) = u_0(x), u'(x,0) = u_1(x), v(x,0) = v_0(x), v'(x,0) = v_1(x), x \ in \ \Omega$$
$$u(x,t) = v(x,t) = 0, \ for \ (x,t) \ in \ \partial\Omega \times \mathbb{R}^+.$$

Although the system worked by Y. Ye and L. Li reminiscents our system is different from theirs because it includes a nonlinear damping term $|u|^{p-1} u$ on the left-hand side while their damping terms are u_t and $-\Delta u_t$.

2. Preliminaries

Definition 2.1: [35] *Let $\Omega \subset \mathbb{R}^n$ be an open connected domain and $q > 0$ be a real number. $L^q(\Omega)$ denotes the class of all measurable functions h defined on Ω with*

$$\int_\Omega |h(x)|^q \, dx < \infty$$

The functional $\|.\|_q$ defined by

$$\|h\|_q = \left(\int_\Omega |h(x)|^q \, dx\right)^{\frac{1}{q}}$$

is a norm on $L^q(\Omega)$ provided $1 \leq q < \infty$.

Definition 2.2: [35] *Let Y be a Banach Space and $1 \leq q \leq \infty$ the space $L^q(0,T;Y)$ denotes the Banach space of vectors of Y valued measurable functions $f:]0,T[\to Y$ such that $\|f(t)\|_Y \in L^q(0,T)$ with*

$$\|f\|_{L^q(0,T;Y)} = \begin{cases} \left(\int_0^T \|f(t)\|_Y^q \, dt\right)^{\frac{1}{q}} & for \ 1 \leq q < \infty, \\ esssup_{[0,T]} \|f(t)\|_Y & for \ q = \infty. \end{cases}$$

Lemma 2.3: [35] *(Hölder's Inequality)*

Let $1 < \alpha < \infty$ and β denote the conjugate exponent defined by

$$\beta = \frac{\alpha}{\alpha - 1} \text{ that is } \frac{1}{\alpha} + \frac{1}{\beta} = 1,$$

which also satisfies $1 < \beta < \infty$. If $g \in L^\alpha(\Omega)$ and $h \in L^\beta(\Omega)$, then $gh \in L^1(\Omega)$, and

$$\int_\Omega |g(x)h(x)| \, dx \leq \|g\|_\alpha \|h\|_\beta .$$

Lemma 2.4: [36] *(Young's Inequality)*

If $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $a,b > 0, a,b \in \mathbb{R}$, *then* $ab \le \frac{a^p}{p} + \frac{b^q}{q}$.

Now we borrow the Korpusov theorem [15]:

Theorem 2.5: *Assume that a functional $\Psi(t)$ satisfies the following conditions:*

$$\Psi\Psi'' - \alpha_1 \Psi'^2 + \gamma_1 \Psi'\Psi + \beta_1 \Psi \ge 0, \quad \alpha_1 > 1, \quad \beta_1 \ge 0, \quad \gamma_1 \ge 0, \tag{8}$$

where $\Psi(t) \in C^2([0,T])$ $,\Psi(t) \ge 0$ $,\Psi(0) > 0$.

If

$$\Psi'(0) > \frac{\gamma_1}{\alpha_1 - 1}\Psi(0), \tag{9}$$

$$\left(\Psi'(0) - \frac{\gamma_1}{\alpha_1 - 1}\Psi(0)\right)^2 > \frac{2\beta_1}{2\alpha_1 - 1}\Psi(0), \tag{10}$$

$\Psi(t) \ge 0$, and $\Psi(0) > 0$, then the time $T > 0$ can not be arbitrarily large the inequality

$$T \le T^* \text{ where} T^* \le \Psi^{1-\alpha_1}(0) A^{-1}$$

where T^* is the maximal existence time interval for $\Psi(t)$ and

$$A^2 = (\alpha_1 - 1)^2 \Psi^{-2\alpha_1}(0)\left[(\Psi'(0) - \frac{\gamma_1}{\alpha_1 - 1}\Psi(0))^2 - \frac{2\beta_1}{2\alpha_1 - 1}\Psi(0)\right]. \tag{11}$$

such that $\lim_{t \to T^*} \sup \Psi(t) = +\infty$.

Definition 2.6: *Assume that u_0, v_0 belong to $H_0^1(\Omega)$ and u_1, v_1 belong to $L^2(\Omega)$. The functions $u(x,t)$ and $v(x,t)$ satisfying the conditions,*

1. $u, v \in L^\infty(0,T; H_0^1(\Omega) \times H_0^1(\Omega))$,
2. $u', v' \in L^\infty(0,T; L^2(\Omega) \times L^2(\Omega))$,
3. $u'', v'' \in L^\infty(0,T; H^{-1}(\Omega) \times H^{-1}(\Omega))$,

$$(iv) \quad \int_0^T \langle L(w), g \rangle \, dt = 0, g(x,t) \in L^1(0,T; H_0^1(\Omega) \times H_0^1(\Omega)), \tag{12}$$

is called a weak generalized solution of (1)-(4), where the bracket $\langle .,. \rangle$ denotes the duality between the Hilbert Space $H_0^1(\Omega) \times H_0^1(\Omega)$ and its dual space $H^{-1}(\Omega) \times H^{-1}(\Omega)$ and $L(w) = (L_1(u,v), L_2(v,u))$ are as below

$$\begin{aligned} L_1(u,v) &= u_{tt} - \alpha\Delta u_t - a_1^2 \Delta u + \overline{m}_1^2 u + k|u|^{p-1}u - v^2 u, \\ L_2(u,v) &= v_{tt} - \alpha\Delta v_t - a_2^2 \Delta v + \overline{m}_2^2 v + k|v|^{p-1}v - vu^2. \end{aligned} \tag{13}$$

Now we state the existence-uniqueness result. Local solvability of the problem (1)-(4) can be proved by the Faedo-Galerkin Approximation method. For the local existence of the solution to this type of problems, we refer to [1, 5, 6, 7, 13, 15, 22, 24, 25, 26].

Since the solution $w = (u,v)$ is not a C^2-function in t to establish our blow-up result we will use a finite-dimensional approximation of solutions $w_m \in C^2([0,T]; H_0^1(\Omega) \times H_0^1(\Omega))$ as in the Faedo-Galerkin Method. For this sake, we ponder the ordinary differential equations system

$$\langle L(w_m), g_j \rangle = 0, \textit{ for } j = 1, 2 \ldots, m, \tag{14}$$

where $w_m=(u_m,v_m)$, $g_j=(g_{1j},g_{2j})$, $\{g_j\}$ are functions in the the basis set of the Hilbert space $H_0^1(\Omega)\times H_0^1(\Omega)$, which we select as the ortho-normalized basis in space $L^2(\Omega)\times L^2(\Omega)$, and $w_m=\sum_{j=1}^{m}c_{mj}(t)g_j$ in equations (14), $L(w_m)=(L_1(u_m,v_m),(L_2(v_m,u_m))$ and the differential operators L_1 and L_2 are given in (13). We assume the initial conditions satisfy

$$w_m(x,0)=\sum_{j=1}^{m}c_{mj}(0)g_j\to w_0=(u_0,v_0)\quad as\quad m\to\infty, \tag{15}$$

strongly in $H_0^1(\Omega)\times H_0^1(\Omega)$ and,

$$w_m'(x,0)=\sum_{j=1}^{m}c_{mj}'(0)g_j\to w_1=(u_1,v_1)\quad as\quad m\to\infty, \tag{16}$$

strongly in $L^2(\Omega)\times L^2(\Omega)$.

3. Finite time blow-up solution

Our method of proving the existence of blow-up solutions is based on the Korpusov Lemma given by Theorem 2.5. In this section, we denote by

$$\Psi_m(t)=\| u_m\|_2^2+\| v_m\|_2^2+\alpha\int_0^t\left(\|\nabla u_m\|_2^2+\|\nabla v_m\|_2^2\right)ds+\frac{\alpha}{2}\left(\|\nabla u_{0m}\|_2^2+\|\nabla v_{0m}\|_2^2\right),$$
$$I_m(t)=\| u_m'\|_2^2+\| v_m'\|_2^2+\alpha\int_0^t\left(\|\nabla u_m'\|_2^2+\|\nabla v_m'\|_2^2\right)ds$$

and

$$E_m(t)=\| u_m'\|_2^2+\| v_m'\|_2^2+a_1^2\|\nabla u_m\|_2^2+a_2^2\|\nabla v_m\|_2^2+\overline{m}_1^2\| u_m\|_2^2+$$
$$\overline{m}_2^2\| v_m\|_2^2+\frac{2k}{p+1}\left(\| u_m\|_{p+1}^{p+1}+\| v_m\|_{p+1}^{p+1}\right)-\int_\Omega u_m^2v_m^2\,dx.$$

The following theorem express the main theorem of this study:

Theorem 3.1: *For any initial profiles* $(u_0,v_0)\in H_0^1(\Omega)\times H_0^1(\Omega)$ *and for any* $(u_1,v_1)\in L^2(\Omega)\times L^2(\Omega))$ *there exist a weak generalized solution* $w=(u,v)$ *of problem (1)-(4) that meets the following conditions:*

$$\begin{aligned} w\ &=(u,v)\in L^\infty(0,T;H_0^1(\Omega)\times H_0^1(\Omega)),\\ w'\ &=(u',v')\in L^\infty(0,T;L^2(\Omega)\times L^2(\Omega)),\\ w''&=(u'',v'')\in L^\infty(0,T;H^{-1}(\Omega)\times H^{-1}(\Omega)),\end{aligned}$$

for some $T_0>0$ and all $T\in(0,T_0)$. Here, either $T_0=+\infty$ or $T_0<+\infty$. Moreover, **(I)** For any nonzero initial profiles (u_0,v_0) with sufficiently large initial velocities (u_1,v_1) and $1<p<3$, $k<1$ there exists $0<T_0\le\Psi^{-1/2}(0)A^{-1}$ such that

$$\Psi(t)\ge\frac{1}{(\Psi^{-1/2}(0)-At)^2}\quad\text{and}\quad\lim_{t\to T_0}\sup\Psi(t)=+\infty, \tag{17}$$

where

$$A=\left\{\frac{1}{4}\Psi^{-3}(0)\left[\Psi'(0)^2-(4E(0)+3\alpha(\|\nabla u_0\|_2^2+\|\nabla v_0\|_2^2))\Psi(0)\right]\right\}^{1/2}>0. \tag{18}$$

(II) For $n=3, k\geq 1$ and $p\geq 3$ or $n\leq 2$ for any $p>1$ the systems have global solutions.

We will give a proof of this theorem for $n=3$.

Proof. Now we proceed by multiplying equation (14) by $c_{mk}(t)$ and adding from $k=1$ to $k=m$, we arrive at

$$\begin{aligned}&\frac{1}{2}\frac{d^2\Psi_m(t)}{dt^2}-(\| u_m' \|_2^2+\| v_m' \|_2^2)+a_1^2\|\nabla u_m\|_2^2+a_2^2\|\nabla v_m\|_2^2+\overline{m}_1^2\|u_m\|_2^2\\&+\overline{m}_2^2\|v_m\|_2^2+k(\|u_m\|_{p+1}^{p+1}+\|v_m\|_{p+1}^{p+1})=2\int_\Omega u_m^2v_m^2\,dx.\end{aligned}\tag{19}$$

Similarly, multiplying equality (14) by $c_{mk}'(t)$ and adding up over $k=1,2,..,m$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\| u_m' \|_2^2+\| v_m' \|_2^2\right)+\alpha\left(\|\nabla u_m'\|_2^2+\|\nabla v_m'\|_2^2\right)\\&+\frac{d}{dt}\left(\frac{a_1^2}{2}\|\nabla u_m\|_2^2+\frac{a_2^2}{2}\|\nabla v_m\|_2^2+\frac{\overline{m}_1^2}{2}\|u_m\|_2^2+\frac{\overline{m}_2^2}{2}\|v_m\|_2^2\right)\\&+\frac{k}{p+1}\frac{d}{dt}\left(\|u_m\|_{p+1}^{p+1}+\|v_m\|_{p+1}^{p+1}\right)=\frac{1}{2}\frac{d}{dt}\int_\Omega u_m^2v_m^2\,dx.\end{aligned}\tag{20}$$

Integrating this equality over $[0,t]$ we get

$$\begin{aligned}&I_m(t)+\alpha\int_0^t\left(\|\nabla u_m'\|_2^2+\|\nabla v_m'\|_2^2\right)ds+a_1^2\|\nabla u_m\|_2^2+a_2^2\|\nabla v_m\|_2^2+\overline{m}_1^2\|u_m\|_2^2\\&+\overline{m}_2^2\|v_m\|_2^2+\frac{2k}{p+1}\left(\|u_m\|_{p+1}^{p+1}+\|v_m\|_{p+1}^{p+1}\right)-E(0)=\int_\Omega u_m^2v_m^2\,dx.\end{aligned}\tag{21}$$

Plugging the left hand side of (21) for $\int_\Omega u_m^2v_m^2\,dx$ into (19) we find the inequality ;

$$\begin{aligned}&\frac{1}{2}\Psi_m''(t)-\left(\| u_m' \|_2^2+\| v_m' \|_2^2\right)+a_1^2\|\nabla u_m\|_2^2+a_2^2\|\nabla v_m\|_2^2\\&+\overline{m}_1^2\|u_m\|_2^2+\overline{m}_2^2\|v_m\|_2^2+k\|u_m\|_{L^{p+1}}^{p+1}+k\|v_m\|_{L^{p+1}}^{p+1}\\&\geq 2I_m(t)+2\alpha\int_0^t\left(\|\nabla u_m'\|_2^2+\|\nabla v_m'\|_2^2\right)ds+2a_1^2\|\nabla u_m\|_2^2+2a_2^2\|\nabla v_m\|_2^2\\&+2\overline{m}_1^2\|u_m\|_2^2+2\overline{m}_2^2\|v_m\|_2^2+\frac{4k}{p+1}\left(\|u_m\|_{p+1}^{p+1}+\|v_m\|_{p+1}^{p+1}\right)-2E_m(0).\end{aligned}\tag{22}$$

Using the definition of I_m again we rewrite above inequality as

$$\begin{aligned}&\frac{1}{2}\Psi_m''(t)-3I_m+2E_m(0)\geq\alpha\int_0^t(\|\nabla u_m'\|_2^2+\|\nabla v_m'\|_2^2)ds+a_1^2\|\nabla u_m\|_2^2+a_2^2\|\nabla v_m\|_2^2\\&+\overline{m}_1^2\|u_m\|_2^2+\overline{m}_2^2\|v_m\|_2^2+\frac{k(3-p)}{p+1}(\|u_m\|_{p+1}^{p+1}+\|v_m\|_{p+1}^{p+1}).\end{aligned}\tag{23}$$

For $1<p<3$ and $k<1$, the right-hand side of this inequality is non-negative and hence we have

$$\Psi_m''(t)-6I_m(t)+4E_m(0)\geq 0.\tag{24}$$

Now we establish the inequality:

$$(\Psi_m')^2\leq 4\Psi_mI_m+2\alpha\Psi_m(\|\nabla u_{0m}\|_2^2+\|\nabla v_{0m}\|_2^2).\tag{25}$$

Differentiating $\Psi_m(t)$ and applying Hölder's inequality we get

$$\begin{aligned}\Psi'_m(t) &= 2\int_\Omega u_m u'_m dx + 2\int_\Omega v_m v'_m dx + \alpha \| \nabla u_m \|_2^2 + \alpha \| \nabla v_m \|_2^2 \\ &\leq 2 \| u_m \|_2 \| u'_m \|_2 + 2 \| v_m \|_2 \| v'_m \|_2 + \alpha \| \nabla u_m \|_2^2 + \alpha \| \nabla v_m \|_2^2 .\end{aligned} \tag{26}$$

and

$$\alpha \| \nabla u_m \|_2^2 = \alpha \int_0^t \frac{d}{ds} \| \nabla u_m \|_2^2 (s) ds + \alpha \| \nabla u_{0m} \|_2^2 = 2\alpha \int_0^t \int_\Omega (\nabla u'_m, \nabla u_m) dx ds + \alpha \| \nabla u_{0m} \|_2^2 . \tag{27}$$

Applying Hölder's inequality to the right-hand side of (27) we obtain that

$$\alpha \| \nabla u_m \|_2^2 \leq 2\alpha \left(\int_0^t \| \nabla u'_m \|_2^2 \, ds \right)^{\frac{1}{2}} \left(\int_0^t \| \nabla u_m \|_2^2 \, ds \right)^{\frac{1}{2}} + \alpha \| \nabla u_{0m} \|_2^2, \tag{28}$$

and similarly,

$$\alpha \| \nabla v_m \|_2^2 \leq 2\alpha \left(\int_0^t \| \nabla v'_m \|_2^2 \, ds \right)^{\frac{1}{2}} \left(\int_0^t \| \nabla v_m \|_2^2 \, ds \right)^{\frac{1}{2}} + \alpha \| \nabla v_{0m} \|_2^2 . \tag{29}$$

Using (26), (28) and (29), we prove (25). Now, multiplying the (24) $\Psi(t)$ using (25) gives

$$\Psi''_m \Psi_m - \frac{3}{2} (\Psi'_m)^2 + \left(4E_m(0) + 3\alpha \left(\| \nabla u_{0m} \|_2^2 + \| \nabla v_{0m} \|_2^2 \right) \right) \Psi_m \geq 0. \tag{30}$$

Now, the axioms of the Theorem 2.5 are satisfied for

$$\begin{gathered}\alpha_1 = \frac{3}{2}, \quad \beta_1 = 4E_m(0) + 3\alpha \left(\| \nabla u_{0m} \|_2^2 + \| \nabla v_{0m} \|_2^2 \right), \gamma_1 = 0 \; and \; \Psi'_m(0) > 0, \\ \Psi'_m(0)^2 > \left(4E_m(0) + 3\alpha \left(\| \nabla u_{0m} \|_2^2 + \| \nabla v_{0m} \|_2^2 \right) \right) \Psi_m(0), E_m(0) > 0.\end{gathered}$$

Therefore the blow-up time is $T_0 \leq \Psi_m^{-1/2}(0) A_m^{-1}$, where

$$A_m^2 = \frac{1}{4} \Psi_m^{-3}(0) \left[\Psi'_m(0)^2 - (4E_m(0) + 3\alpha \left(\| \nabla u_0 \|_2^2 + \| \nabla v_0 \|_2^2 \right) \Psi_m(0) \right],$$

and

$$\Psi_m(t) \geq \frac{1}{(\Psi_m^{-1/2}(0) - A_m t)^2} . \tag{31}$$

Now we proceed by taking the limit as $m \to \infty$ for a subsequence $\{\Psi_m(t)\}$:
Essentially, considering the limit properties (15)-(16),

$$\Psi_m(0) \to \Psi(0) = \int_\Omega (| u_0 |^2 + | v_0 |^2) \, dx + \frac{\alpha}{2} (\| \nabla u_0 \|_2^2 + \| \nabla v_0 \|_2^2), \tag{32}$$

$$\Psi'_m(0) \to \Psi'(0) = 2\int_\Omega (u_0 u_1 + v_0 v_1) \, dx + \alpha (\| \nabla u_0 \|_2^2 + \| \nabla v_0 \|_2^2), \tag{33}$$

$$\begin{aligned}E_m(0) \to E(0) = &\| u_1 \|_2^2 + \| v_1 \|_2^2 + a_1^2 \| \nabla u_0 \|_2^2 + a_2^2 \| \nabla v_0 \|_2^2 + \\ &\overline{m}_1^2 \| u_0 \|_2^2 + \overline{m}_2^2 \| v_0 \|_2^2 + \frac{2k}{p+1} (\| u_0 \|_{p+1}^{p+1} + \| v_0 \|_{p+1}^{p+1}) - \int_\Omega u_0^2 v_0^2 \, dx,\end{aligned} \tag{34}$$

$$A_m^2 \to A^2 = \frac{1}{4}\Psi^{-3}(0)\Big[\Psi'(0)^2 - (4E(0) + 3\alpha(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2))\Psi(0)\Big]. \tag{35}$$

The following lemma can easily be proven by using the technique given in [8].

Lemma 3.2: *The sequence*

$$\Psi_m(t) = \|u_m\|_2^2 + \|v_m\|_2^2 + \alpha\int_0^t (\|\nabla u_m\|_2^2 + \|\nabla v_m\|_2^2)ds + \frac{\alpha}{2}(\|\nabla u_{0m}\|_2^2 + \|\nabla v_{0m}\|_2^2). \tag{36}$$

has a subsequence that we still denote by the same notation converges uniformly to

$$\Psi(t) = \|u\|_2^2 + \|v\|_2^2 + \alpha\int_0^t (\|\nabla u\|_2^2 + \|\nabla v\|_2^2)ds + \frac{\alpha}{2}(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2), \tag{37}$$

where $t \in [0,T]$.

Now we take the limit as $m \to \infty$ in (31) and we obtain the following inequality

$$\Psi(t) \geq \frac{1}{(\Psi^{-1/2}(0) - At)^2}.$$

Based on this we conclude that for (u_0, v_0) *in* $H_0^1(\Omega) \times H_0^1(\Omega)$ and all $(u_1, v_1) \in L^2(\Omega) \times L^2(\Omega)$ satisfying conditions,

$$\Psi'(0) > [(4E(0) + 3\alpha(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2))\Psi(0))]^{\frac{1}{2}} > 0, \quad E(0) > 0, \tag{38}$$

where $\Psi(0), \Psi'(0)$ and E(0) are given in (32),(33),(34). Then there exists a finite $0 < T_0$ satisfying $AT_0 \leq \Psi^{\frac{-1}{2}}(0)$. As a result , the (17) holds.

Verification of the compatibility conditions (38): First, for sufficiently large (u_0, v_0) in $H_0^1(\Omega) \times H_0^1(\Omega)$ and with $\alpha > 0$ small enough and $1 < p < 3, k < 1$ and suitable coefficients $a_1, a_2, \overline{m}_1, \overline{m}_2$ the following inequality is true since we can control the first term on the left-hand side with the first term on the right-hand side and the others term on the left-hand side by integral of $u_0^2 v_0^2$:

$$\begin{aligned} &\frac{(2 - 4\sqrt{\alpha})(\|u_0\|_2^2 + \|v_0\|_2^2)(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2)}{4\Psi(0)} \\ &+ (a_1^2\|\nabla u_0\|_2^2 + a_2^2\|\nabla v_0\|_2^2 + \overline{m}_1^2\|u_0\|_2^2 + \overline{m}_2^2\|v_0\|_2^2) + \frac{2k}{p+1}(\|u_0\|_{p+1}^{p+1} + \|v_0\|_{p+1}^{p+1}) \\ &+ \frac{3\alpha}{4}(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2) < \frac{\alpha^2(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2)^2}{4\Psi(0)} + \int_\Omega u_0^2 v_0^2\, dx. \end{aligned} \tag{39}$$

For $\lambda > 0$, substituting $(u_1, v_1) = (\lambda u_0, \lambda v_0)$ into (33) and (34) we get

$$\begin{aligned} \Psi'(0) &= 2\int_\Omega ((u_0 u_1 + v_0 v_1)\, dx + \alpha(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2) \\ &= 2\lambda\int_\Omega (|u_0|^2 + |v_0|^2)\, dx + \alpha(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2) > 0, \end{aligned} \tag{40}$$

and for λ sufficiently large

$$\begin{aligned} E(0) = \lambda^2(\|u_0\|_2^2 + \|v_0\|_2^2) &+ a_1^2\|\nabla u_0\|_2^2 + a_2^2\|\nabla v_0\|_2^2 + \overline{m}_1^2\|u_0\|_2^2 \\ &+ \overline{m}_2^2\|v_0\|_2^2 + \frac{2k}{p+1}(\|u_0\|_{p+1}^{p+1} + \|v_0\|_{p+1}^{p+1}) - \int_\Omega u_0^2 v_0^2\, dx > 0. \end{aligned}$$

Thanks to the inequality (40), we see that inequality in (38) is equivalent to the inequality

$$\Psi'(0)^2 > (4E(0) + 3\alpha(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2))\Psi(0) > 0. \tag{41}$$

To prove (41), we proceed by sequentially replacing both sides of (41) with the following:

$$\begin{aligned}\Psi'(0)^2 &= 4\lambda^2(\|u_0\|_2^2 + \|v_0\|_2^2)^2 + \alpha^2(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2)^2 \\ &+4\alpha\lambda(\|u_0\|_2^2 + \|v_0\|_2^2)(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2),\end{aligned} \tag{42}$$

and

$$\begin{aligned}&(4E(0) + 3\alpha(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2))\Psi(0)) \\ &= 2\alpha\lambda^2(\|u_0\|_2^2 + \|v_0\|_2^2)(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2) + 4\lambda^2(\|u_0\|_2^2 + \|v_0\|_2^2)^2 \\ &+ 3\alpha(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2)\Psi(0) + 4\Psi(0)(a_1^2\|\nabla u_0\|_2^2 + a_2^2\|\nabla v_0\|_2^2 \\ &+ \overline{m}_1^2\|u_0\|_2^2 + \overline{m}_2^2\|v_0\|_2^2 + \frac{2k}{p+1}\Big(\|u_0\|_{p+1}^{p+1} + \|v_0\|_{p+1}^{p+1}) - \int_\Omega u_0^2 v_0^2\, dx\Big),\end{aligned}$$

and we obtain the inequality

$$\begin{aligned}&4\lambda^2(\|u_0\|_2^2 + \|v_0\|_2^2)^2 + 2\alpha\lambda^2(\|u_0\|_2^2 + \|v_0\|_2^2)(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2) \\ &+ 3\alpha(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2)\Psi(0) + 4\Psi(0)[a_1^2\|\nabla u_0\|_2^2 + a_2^2\|\nabla v_0\|_2^2 + \overline{m}_1^2\|u_0\|_2^2 \\ &+ \overline{m}_2^2\|v_0\|_2^2 + \frac{2k}{p+1}(\|u_0\|_{p+1}^{p+1} + \|v_0\|_{p+1}^{p+1}) - \int_\Omega u_0^2 v_0^2\, dx] < 4\lambda^2(\|u_0\|_2^2 + \|v_0\|_2^2)^2 \\ &+ 4\alpha\lambda(\|u_0\|_2^2 + \|v_0\|_2^2)(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2) + \alpha^2(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2)^2.\end{aligned} \tag{43}$$

Now we let $\lambda^2 = \frac{1}{\alpha}$ *for sufficiently small* $\alpha > 0$ and obtain,

$$\begin{aligned}&\frac{(2-4\sqrt{\alpha})(\|u_0\|_2^2 + \|v_0\|_2^2)(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2)}{4\Psi(0)} + (a_1^2\|\nabla u_0\|_2^2 + a_2^2\|\nabla v_0\|_2^2 + \overline{m}_1^2\|u_0\|_2^2 + \overline{m}_2^2\|v_0\|_2^2) \\ &+ \frac{2k}{p+1}(\|u_0\|_{p+1}^{p+1} + \|v_0\|_{p+1}^{p+1}) + \frac{3\alpha}{4}(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2) < \frac{\alpha^2(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2)^2}{4\Psi(0)} + \int_\Omega u_0^2 v_0^2\, dx.\end{aligned}$$

which is true by assumption (39).

4. Global Existence

We consider the equality (20), we have

$$E_m(t) + 2\alpha\int_0^t(\|\nabla u_m'\|_2^2 + \|\nabla v_m'\|_2^2)ds = E_m(0). \tag{44}$$

Using Young inequality we have

$$\int_\Omega u_m^2 v_m^2 dx \le \int_\Omega \frac{u_m^{2\alpha}}{\alpha}dx + \int_\Omega \frac{v_m^{2\beta}}{\beta}dx = \frac{2}{p+1}\int_\Omega |u_m|^{p+1}\, dx + \frac{p-1}{p+1}\int_\Omega |v_m|^{\frac{2(p+1)}{p-1}}\, dx.$$

Now, by applying Young inequality to the second integral on the right side, for $\gamma = \frac{p-1}{2}$ *and* $\delta = \frac{p-1}{p-3}$, we get

$$\frac{p-1}{p+1}\int_\Omega |v_m|^{\frac{2(p+1)}{p-1}}\, dx \le \frac{2}{p+1}\int_\Omega |v_m|^{p+1}\, dx + C(\delta, p, \Omega).$$

If $k \geq 1$ and $p \geq 3$ then we get

$$\int_\Omega u_m^2 v_m^2 dx \leq \frac{2k}{p+1}\int_\Omega |u_m|^{p+1}\, dx + \frac{2k}{p+1}\int_\Omega |v_m|^{p+1}\, dx + C(p,k,\Omega).$$

Hence by (44) we have

$$\| u_m' \|_2^2 + \| v_m' \|_2^2 + a_1^2 \| \nabla u_m \|_2^2 + a_2^2 \| \nabla v_m \|_2^2 + \overline{m}_1^2 \| u_m \|_2^2 + \overline{m}_2^2 \| v_m \|_2^2 - C(p,k,\Omega) \leq E_m(t) \leq E_m(0),$$

which implies

$$\| u_m' \|_2^2 + \| v_m' \|_2^2 + a_1^2 \| \nabla u_m \|_2^2 + a_2^2 \| \nabla v_m \|_2^2 + \overline{m}_1^2 \| u_m \|_2^2 + \overline{m}_2^2 \| v_m \|_2^2 \leq E_m(0) + C(p,k,\Omega) \leq C.$$

Hence the solution does not have a finite time blow up.

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