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Prime Strong Ideals of S-semigroup

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Abstract

A seminearring is a generalization of the notion nearring in which the elements need not have an additive inverse. A S-semigroup is a generalization of the notion N-group in which the scalars are from seminearring. In this paper, we define various prime strong ideals of a S-semigroup and obtain the relationship among them. We also establish the connections between completely equiprime, equiprime and completely prime strong ideals of a S-semigroup. The obtained results are illustrated with the suitable examples. We also prove that every equiprime strong ideal of a S-semigroup is 3-prime strong ideal. In addition, we show that $(P:Q)_S$ is strong ideal of a seminearring S and prove related results.

Key words and phrases: S-semigroup, Seminearring, Prime ideal.

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1. Introduction

In 1970, Holcombe's [5] work extended the definitions of different types of prime rings to nearrings. In his work, Holcombe classified prime nearrings within the class of all nearrings as 0-prime (or prime), 1-prime and 2-prime nearrings. Subsequently, Groenewald [4] introduced 3-prime nearrings. The notion of equiprime, a generalization of prime rings to nearrings, was introduced by Booth, Groenewald and Veldsman [2]. Later, Veldsman [19] studied results related on equiprime nearrings. Then Kedukodi, Kuncham and Bhavanari [9] defined τ-prime ($\tau = 3, c, e$) fuzzy ideals of nearrings and characterized these structures. In addition, authors introduced the concept of fuzzy cosets based on generalized fuzzy ideals and proved isomorphism theorems.

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Booth and Groenewald [3] established the concept of equiprime N-group. Then Taşdemir and Atagün [15] generalized the concept to ideals of N-group, called equiprime N-ideals. They have also obtained relationship among the τ −prime ($\tau = 3, c, e$) ideals of the N-group. Juglal [6] provided equivalent forms of different prime ideals of N-group and characterized them. The notions of semiprime and strongly prime were extended from nearrings to N-groups. Equiprime ideals, nearrings, N-groups, and related findings were investigated by Kabelo Mogae [12]. Examples of nearrings that become equiprime under certain conditions have been described earlier in the literature. Different notions of primeness defined in nearrings were generalized to the N-group by Juglal, Groenewald and Lee [7]. Juglal also explored characterizations of prime modules. Groenewald and Juglal [8] introduced the idea of a strongly prime nearring module, and characterizations were provided. Further authors studied general classes of N-groups.

Taşdemir, Atagün and Altndiş [16] characterized 3-prime, c-prime, and semiprime N-groups, defined the notion of IFP (Insertion of Factors Property) ideal of N-group, and demonstrated some results in this regard. Taşdemir [17] expanded on this work by introducing the concept of completely equiprime ideals of N-groups and obtained the relation between completely equiprime, equiprime, and completely prime ideals. Furthermore, they established a relationship between IFP ideals and c-e-prime ideals of N-groups.

Koppula, Kedukodi and Kuncham[11] explicitly defined the ideal of seminearring and proved some results. Then various prime strong ideals of seminearring and their corresponding prime radicals were defined and the related results were characterized by Koppula, Kedukodi and Kuncham [10]. Prakash, Koppula, Kedukodi and Kuncham [14] studied S-semigroups by introducing the concept of a strong ideal. Further, quotient structure was defined and proved the classical isomorphism theorems in S-semigroups. For the results and isomorphism theorems on nearrings and N-groups, we refer Pilz [13], Bhavanari and Kuncham [1].

The set of natural numbers under the operations usual addition and multiplication reveals the structure of a seminearring. This exploration is essential, as these unique algebraic structures are not considered in other studies. Furthermore, the definition of an ideal is explicitly established in nearrings and modules over nearrings, highlighting the significance of this field of study. Different prime ideals of a module over a nearring are well established in the literature. In the present work, we make an attempt to effectively generalize these concepts to modules over seminearrings.

We discuss some properties of distinct prime strong ideals of S-semigroup, a module over a seminearring, and their interrelation. Further, we also show that the Noetherian quotient becomes a strong ideal of seminearring. In addition, we obtain results on these strong ideals and illustrate them with suitable examples.

2. Preliminaries

In this section, we provide basic definitions and results which are useful to obtain the results of the present paper.

Throughout this paper S denotes a right seminearring.

Definition 2.1: [18] A non-empty set S with respect to the binary operations + and $\cdot$ is said to be a right seminearring if

1. $(S,+)$ is a semigroup with additive identity 0.
2. $(S,\cdot)$ is a semigroup.
3. For all $x, y, z \in S, (x+y)z = xz + yz$
4. For all $a \in S, 0a = 0$

Definition 2.2: [14] Let $(S,+,\cdot)$ be a seminearring and $(\Gamma,+)$ be a semigroup. Then Γ is said to be a S-semigroup, if there exists a mapping $*: S\times\Gamma \to \Gamma$ defined as $*(w,\kappa) \to w*\kappa$ which satisfies the below conditions. For all $\kappa \in \Gamma$ and $w, v \in S$,

1. $(w+v)*\kappa = w*\kappa + v*\kappa$
2. $(w\cdot v)*\kappa = w*(v*\kappa)$
3. $0*\kappa = 0$,

Definition 2.3: A non empty subset M of a semigroup $(S,+)$ is said to be a subsemigroup, if $q,v \in M$, then $q+v \in M$.

Definition 2.4: An additive subsemigroup M of a seminearring S is a subseminearring if $0 \in M$ and $MM \subseteq M$.

Definition 2.5: A subsemigroup M of S is said to be a normal subsemigroup of S, if $s+M \subseteq M+s \ \forall\ s \in S$, where S is a seminearring.

Definition 2.6: A subsemigroup Δ of a S-semigroup Γ is a S-subsemigroup, if $S\Delta \subseteq \Delta$, where S is a seminearring.

Definition 2.7: A subseminearring M of S is said to be left invariant if $SM = \{sa \mid s \in S, a \in M\} \subseteq M$, where S is a seminearring.

Definition 2.8: [14] A nonempty subset T of a S-semigroup R is said to be a strong ideal of R if the following conditions are satisfied.

1. If $a,b \in T$ then $a+b \in T$.
2. $a+T \subseteq T+a, \forall\ a \in R$.
3. For $a,b \in R$, if $a \equiv_T b$ then $a \in T+b$.
4. $s(T+a) \subseteq T+sa \ \ \forall\ s \in S, a \in R$.

3. Prime strong ideals

In this section, we provide the definitions of different prime strong ideals of S-semigroup and obtain the interrelationship between them. The results are illustrated with the suitable examples.

Definition 3.1: Let S be a seminearring and R be a S-semigroup, A and B are any two subsets of S-semigroup. Then the noetherian quotient $(A:B)_S$ is defined as follows:

$$(A:B)_S = \{s \in S \mid sB \subseteq A\}.$$

We represent $(A:B)_S$ by $(x:B)_S$ if A has only one element x.
More specifically if $x=0$, then the set $(0:A)_S$ is called the **annihilator** of A.
If K is a strong ideal of S-semigroup R, we denote it by $K \lhd_S R$.
In the following, we consider R as a S-semigroup.

Example 3.2: Consider the set $S=\mathbb{Z}_3$ under addition modulo 3 and multiplication as defined in the below table.

$\cdot$	0	1	2
0	0	0	0
1	0	0	1
2	0	0	2

Then $(S,+,\cdot)$ forms a seminearring. Let $R = {}_SS$ $(S\times S \to S)$ be a S-semigroup with natural operations.
Now consider the subsets A and B of R, where $A=\{0,1\}, B=\{2\}$.
Then the noetherian quotient $(A:B)_S = \{0,1\}$.

Proposition 3.3: If K is a strong ideal of R and B is any subset of R, then $(K:B)_S = (K:K+B)_S$.

Proof 3.4: Let $a \in (K:B)_S$. Then $aB \subseteq K$.
This implies $ab \in K$, forall $b \in B$.
Since K is a strong ideal of R, we have $a(K+b) \subseteq K + ab \subseteq K$
$\Rightarrow a(k+b) \in K$, for all $k \in K, b \in B$ $a \in (K:K+B)_S$.
On the other hand, let $a \in (K:K+B)_S$.
$\Rightarrow a(K+b) \subseteq K$, for all $b \in B$. Now, $b \in B$ is arbitrarily fixed.
Since K is a strong ideal of R, we get $a(K+b) \subseteq K + ab \subseteq K + K$.
$\Rightarrow k_1 + ab = k_2 + 0$, for some $k_1, k_2 \in K$
$\Rightarrow ab \equiv_K 0ab \in Ka \in (K:B)_S$
Hence $(K:B)_S = (K:K+B)_S$.

Definition 3.5: [10] Let S be a seminearring and K be a strong ideal of S. Then K is an equiprime strong ideal of S if $a,d,f \in S$ with $asd \equiv_K asf$ for every $s \in S$, then either $a \in K$ or $d \equiv_K f$.

Note 1: We say S is an equiprime seminearring if $\{0\}$ is an equiprime strong ideal of S.

Proposition 3.6: Every equiprime seminearring is zero-symmetric seminearring.

Proof 3.7: Let S be an equiprime seminearring. Then $K = \{0\}$ is an equiprime strong ideal of S. Consider $0 \equiv_K 0 \Rightarrow 0(ss')0 \equiv_K 0(s')0$.
$\Rightarrow s(0ss'0) \equiv_K s(0s'0)$
$\Rightarrow (s0)s'(s0) \equiv_K (s0)s'(0)$
Then by definition of equiprime strong ideal we get $s0 \in K$ or $s0 \equiv_K 0$.

$$s0 \equiv_K 0 \Rightarrow s0 \in K + 0$$

Since $K = \{0\}$, we get $s0 = 0$.
Hence the seminearring is zero symmetric.
The concept of a υ-prime ideal ($\upsilon = 0,1,2,3,c$) of the N-group was introduced in [7]. Here we extend the same to S-semigroups.

Definition 3.8: Let S be a seminearring, R is any S-semigroup and K is a strong ideal of R such that $SR \not\subseteq K$. Then K is called

1. **0-prime** strong: for ideals X of S and ideals Y of R, if $XY \subseteq K$, then $XR \subseteq K$ or $Y \subseteq K$.
2. **1-prime** strong: for left ideals X of S and ideals Y of R, if $XY \subseteq K$, then $XR \subseteq K$ or $Y \subseteq K$.
3. **2-prime** strong: for S-subsemigroups X of S and ideals Y of R, if $XY \subseteq K$, then $XR \subseteq K$ or $Y \subseteq K$.
4. **3-prime** strong: for $a \in S, r \in R$, if $aSr \subseteq K$, then $aR \subseteq K$ or $r \in K$.
5. Completely prime (**c-prime**) strong: for $a \in S, r \in R$, if $ar \in K$, then $aR \subseteq K$ or $r \in K$.
6. Equiprime (**e-prime**) strong: for $a \in S, r_1, r_2 \in R$, if $asr_1 \equiv_K asr_2$ for all $s \in S$, then $aR \subseteq K$ or $r_1 \equiv_K r_2$.

Proposition 3.9: If K is a strong ideal of S-semigroup R then K is c-prime strong $\Rightarrow$ K is 3-prime strong $\Rightarrow$ K is 2-prime strong $\Rightarrow$ K is 0-prime strong.

Example 3.10: Consider the seminearring $(S,+,\cdot)$ where + and $\cdot$ are defined as per the below tables:

+	q	w	e	t
q	q	w	e	t
w	w	q	t	e
e	e	t	q	w
t	t	e	w	q

·	q	w	e	t
q	q	q	q	q
w	q	w	w	w
e	q	e	e	e
t	q	t	t	t

Let $R = {}_SS$ $(S \times S \to S)$ be a S-semigroup.

Then $K=\{q\}$ is a ν-strong ideal of R, where $\nu=c,3,2,1,0$. Note that K is not an e-prime strong ideal of R because, for the values $a=1, r_1=1, r_2=2$, $asr_1 \equiv_K asr_2$ for all $s\in S$ but $ar_2 \notin K$ and $r_1 \not\equiv_K r_2$.

Now, we provide an Example 3.11 to show the e-prime strong ideal.

Example 3.11: Let $(\mathbb{N}\cup\{0\},+,\cdot)$ be a seminearring. Consider $(\mathbb{Z},\oplus)$ is a semigroup, where $\oplus$ is defined as $a\oplus b = a+b-ab$.

Now $*:\mathbb{N}\times\mathbb{Z}\to\mathbb{Z}$ is defined as $n*z=0$ for all $n\in\mathbb{N}, z\in\mathbb{Z}$. Then $\mathbb{Z}$ forms a S-semigroup.

Here $K=\{0,2\}$ is a strong ideal of S-semigroup.

$[-(2z)]_K = \{2(z+1)\}, [-(2z+1)] = \{2z+3\}, \forall\, z\in\mathbb{N}\cup\{0\}$.

Note that K is an e-prime strong ideal of $\mathbb{Z}$.

Proposition 3.12: If $s0=0, \forall\, s\in S$, then every e-prime strong ideal of a S-semigroup R is a 3-prime strong ideal.

Proof 3.13: Let K be an e-prime strong ideal of a S-semigroup R and $a\in S, r\in R$ such that $aSr\subseteq K$. If $r\in K$, then K is 3-prime strong.

Suppose $r\notin K$. Then $aSr\subseteq K \Rightarrow asr\in K$ for all $s\in S$.

As $s0=0\ \forall\, s\in S$, we have $as0=0\in K$ for all $s\in S$. Fix $s\in S$.

Then $asr\in K+as0 \Rightarrow asr\equiv_K as0$.

Since $s\in S$ is arbitrary, we have $asr\equiv_K as0$ for all $s\in S$.

Because K is an equiprime, we have either $aR\subseteq K$ or $r\equiv_K 0$.

$r\equiv_K 0 \Rightarrow r\in K$, a contradiction.

$\Rightarrow aR\subseteq K$. Hence K is 3-prime strong.

In general, the converse statement of the Proposition 3.12 need not hold.

Example 3.14: Let $S=\{0,1,2,3\}$. Then $(S,+,\cdot)$ is a seminearring with respect to + and $\cdot$ as defined in the following table:

+	0	1	2	3
0	0	1	2	3
1	1	1	1	1
2	2	1	2	1
3	3	1	1	3

·	0	1	2	3
0	0	0	0	0
1	1	1	1	1
2	1	1	2	3
3	1	1	3	2

We have $(S,+)$ is a semigroup.

Now, $*:S\times S\to S$ is defined as follows.

*	0	1	2	3
0	0	0	0	0
1	1	1	1	1
2	1	1	1	1
3	1	1	1	1

Then S is a S-semigroup and $K=\{0\}$ is a strong ideal of S. Note that here $\{0\}$ is a 3-prime strong ideal but not an e-prime strong ideal because, for the values $a=1, r_1=0, r_2=1$, $asr_1\equiv_K asr_2$ for all $s\in S$ implies $ar_2\notin K$ and $r_1\not\equiv_K r_2$.

Remark 3.15: Let S be a seminearring and R be a S-semigroup, K is a strong ideal of R. Then for $s,z\in S, r\in R, p_1\in K$; if $zr=p_1+sr$, then we assume $z\in(K:R)+s$.

If $K=\{0\}$, then $zr=sr$ implies $z\in(0:r)+s\ \forall\, r\in R$.

The above condition holds in case of nearrings obviously as follows.

$$zr=p_1+sr \Rightarrow zr-sr\in K \Rightarrow (z-s)r\in K \Rightarrow z-s\in(K:R) \Rightarrow z\in(K:R)+s.$$

Proposition 3.16: If K and Q are subsemigroups of S-semigroup R, then the following conditions are satisfied. [(i)]

(i) If Q is a S-subsemigroup of R, then $(K:Q)_S$ is a right invariant subsemigroup of S.
(ii) If both K and Q are S-subsemigroups of R, then $(K:Q)_S$ is an invariant subsemigroup of S.
(iii) If K is a strong ideal of R and Q is a S-subsemigroup of R, then $(K:Q)_S$ is a strong ideal of S.

Proof 3.17:

(i) As $0 \in S, 0q = 0 \in K, \forall q \in Q\ 0 \in (K:Q)_S$.
Now, take $x, y \in (K:Q)_S$. Then $xq \in K, yq \in K, \forall q \in Q$.
Let $q \in Q$ be fixed. Then $(x+y)q = xq + yq \in K$.
Thus $(K:Q)_S$ is a subsemigroup of S.
Also given Q is a S-subsemigroup of R implies $SQ \subseteq Q$.
Let $y \in (K:Q)_S \cdot S$. Then $y = as$, for some $a \in (K:Q)_S$ and $s \in S$.
$\Rightarrow aQ \subseteq K$. Let $q \in Q$. Then $yq = (as)q = a(sq) \subseteq aQ \subseteq K$
$yq \in K$ for all $q \in Qy \in (K:Q)_S$.
Hence $(K:Q)_S \cdot S \subseteq (K:Q)_S$. Thus $(K:Q)_S$ is right invariant.

(ii) Suppose K and Q are S-subsemigroups of R. Let $a \in (K:Q)_S$. Then $aQ \subseteq K$.
Now, take $y \in S \cdot (K:Q)_S$. Then $y = sa$, for some $a \in (K:Q)_S$ and $s \in S$.
Let $q \in Q$. Then $yq = (sa)q = s(aq) \subseteq K$.
$yq \in K$ for all $q \in Qy \in (K:Q)_S$.
Hence $S \cdot (K:Q)_S \subseteq (K:Q)_S$. Thus $(K:Q)_S$ is left invariant and together with (1) we get $(K:Q)_S$ is invariant.

(iii) Let K be a strong ideal of R and Q be a S-subsemigroup of R. Then by (1) we have $(K:Q)_S$ is a right invariant subsemigroup of S. Let $a \in (K:Q)_S$ and $s \in S$. For every $q \in Q, aq \in K$. We have to show that $(K:Q)_S$ is a strong ideal of S.
Let $a, b \in (K:Q)_S$. This implies $aQ \subseteq K, bQ \subseteq K$.
Then $aq \in K, bq \in K$ for all $q \in Q$.
$\Rightarrow aq + bq \in K$ for all $q \in Q$ (since K is a subsemigroup).
$\Rightarrow (a+b)q \in K$ for all $q \in Q\ a + b \in (K:Q)_S$.
Let $z \in s + (K:Q)_S$. Then $z = s + a$ for some $a \in (K:Q)_S, s \in S$.
Since $a \in (K:Q)_S, aQ \subseteq K$. Let $q \in Q$. Then,
$zq = (s+a)q = sq + aq = sq + p_1$ for some $p_1 \in K$.
$zq = p_2 + sq$ (since K is a strong ideal).
Then $zq = p_2 + sqz \in (K:Q)_S + s$. (By Remark 3.15)
Let $s_1 \equiv_{(K:Q)_S} s_2$. Then $a_1 + s_1 = a_2 + s_2$ for some $a_1, a_2 \in (K:Q)_S$.
As $a_1, a_2 \in (K:Q)_S\ a_1 q \in K, a_2 q \in K$ for all $q \in Q$.
Let $q \in Q$, then $(a_1 + s_1)q = (a_2 + s_2)q$
$\Rightarrow a_1 q + s_1 q = a_2 q + s_2 q$ (since $aQ \subseteq K$)
$s_1 q \equiv_K s_2 q\ s_1 q = p + s_2 q$ (since K is a strong ideal)
$\Rightarrow s_1 \in (K:Q)_S + s_2$. (By Remark 3.15)
Since right invariant, right strong ideal part follows.
Let $s, s' \in S$ be such that $z \in s((K:Q)_S + s')$.
Then $z = s(a + s')$ for some $a \in (K:Q)_S$.
Now, take $q \in Q$. Then $zq = s(a+s')q = s(aq + s'q)$
$= s(p_1 + s'q) \subseteq K + ss'q$ (left strong ideal of K).
$zq = p + (ss')qz \in (K:Q)_S + ss'$. (By Remark 3.15)
Hence $(K:Q)_S$ is a strong ideal of the seminearring S.

Corollary 3.18: If R is a S-semigroup, then $(0:r)_S$ is a left strong ideal of S for all $r \in R$.

Lemma 3.19: Suppose L is a left strong ideal of S, $P=(L:S)_S$ is a strong ideal of S that contains all of the right invariant subsets A of S such that $A \subseteq L$ (Specifically, P contains all of the strong ideals of S that are contained in L). Furthermore, if L is left invariant in S, then P is an invariant strong ideal of S.

Lemma 3.20: If L is a left strong ideal of S, then L is left invariant in S if and only if $s0 \in L$ for all $s \in S$.

Proposition 3.21: [11] If L is an e-prime strong ideal of a seminearring S, then
(a) L is left invariant in S.
(b) If $a,b \in S$ and $aSb \subseteq L$, then $a \in L$ or $b \in L$.

Proposition 3.22: If L is an e-prime strong ideal of S and $K=(L:S)_S$, then
(a) $K \subseteq L$ and K is the largest two sided strong ideal of S which is contained in L.
(b) K is an e-prime strong ideal of S.

Proof 3.23: (a) By Lemma 3.19, K is the largest strong ideal of S which contains all strong ideals A of S such that $A \subseteq L$. Let $x \in K$. Then $xS \subseteq L$ (since $x \in K=(L:S)_S$).
Thus $xSx = x(Sx) \subseteq xS \subseteq L$.
Since L is e-prime strong, $x \in L$ by Proposition 3.21(b)
Hence $K \subseteq L$.
(b) Let $a,x,y \in S$ with $a \notin K, x \not\equiv_K y$.
By definition of K, $as \notin L$ and $xs' \not\equiv_L ys'$ for some $s,s' \in S$.
Since L is an e-prime strong ideal, $(as)t(xs') \not\equiv_L (as)t(ys')$ for some $t \in S$.
Since $K \subseteq L$, $(as)t(xs') \not\equiv_K (as)t(ys')$
Since K is a strong ideal of S, we get $a(st)x \not\equiv_K a(st)y$.
Thus K is an e-prime strong ideal of S.

Definition 3.24: Let R be a S-semigroup. Then R is said to be an **equiprime S-semigroup** if the below conditions are met:

1. $SR \neq 0$.
2. $a \in S, a \notin (0:R)_S$ and $r,r' \in R$ with $asr = asr'$ for all $s \in S \Rightarrow r = r'$.
3. $S0_R = 0_R$.

Example 3.25: Consider the seminearring $(S,+,\cdot)$ where + and $\cdot$ are defined as per the below tables:

+	1	2	3	4
1	1	2	3	4
2	2	2	3	4
3	3	3	3	3
4	4	4	3	4

$\cdot$	1	2	3	4
1	1	1	1	1
2	1	2	3	4
3	1	2	3	4
4	1	2	3	4

Let $R = {}_S S$ $(S \times S \to S)$ be a S-semigroup. Here R is an equiprime S-semigroup.

Proposition 3.26: If S is a seminearring and $I \triangleleft S$ such that $L \neq S$, then the subsequent statements are equivalent:
(a) L is an e-prime strong ideal of S.
(b) There exist an equiprime S-semigroup R such that $L=(0:R)_S$.

Proof 3.27: $(a) \Rightarrow (b)$

Let L be an e-prime strong ideal of S, where $L \neq S$ and $R = S / L$. Then R is a S-semigroup with natural operations. Let

$$\begin{aligned} x \in (0:R)_S \quad &\Leftrightarrow xR = 0 \\ &\Leftrightarrow x(s/L) = 0 \ \forall \ s/L \in R = S/L \\ &\Leftrightarrow \frac{xs}{L} = \frac{0}{L} \\ &\Leftrightarrow xs \equiv_L 0 \Leftrightarrow xs \in L \\ &\Leftrightarrow x \in (L:S)_S \end{aligned}$$

Since L is an e-prime strong ideal of S, we have $L = (L:S)_S$ is an e-prime strong ideal of S. Now we have to show R is an equiprime S-semigroup.

1. Suppose $SR = 0$, then $S \subseteq (0:R)_S$ by definition of $(0:R)_S$.
   Since $(0:R)_S = (L:S)_S = L$, we get $S \subseteq L$. Also $L \subseteq S$, since L is a strong ideal of S.
   $\Rightarrow S = L$, a contradiction since $L \neq S$.
   Hence $SR \neq 0$.
2. Suppose $a \in S$ and $a \notin (0:R)_S = L$ and $x, y \in S, r, r' \in R$ such that
   $r = x/L \neq y/L = r'$, then $x \not\equiv_L y$.
   Since L is an e-prime strong ideal of S, $asx \not\equiv_L asy$ for some $s \in S$.
   $\Rightarrow asr = as(x/L) \neq as(y/L) = asr'$
3. Since L is an e-prime strong ideal of S, $SL \subseteq L$.
   Hence for every $s \in S, s0_R = s(0/L) = (s.0)/L = L = 0_R$
   $\Rightarrow S \cdot 0_R = 0_R$.
   Thus R is an equiprime S-semigroup.
   $(b) \Rightarrow (a)$
   Let R be an equiprime S-semigroup. We will show that $L = (0:R)_S$ is an e-prime strong ideal of S. Let $a, x, y \in S$ such that $a \notin L$ and $x \not\equiv_L y$.
   Then $xr \neq yr$ by definition of $(0:R)_S$ for some $r \in R$.
   Since R is an equiprime S-semigroup, $as(xr) \neq as(yr)$ for some $s \in S$.
   $asx \not\equiv_L asy$.
   Hence L is an e-prime strong ideal of S.

Definition 3.28: A S-semigroup R is faithful if $(0:R)_S = 0$ (equivalently, if $s \in S$ with $sr = 0$ for every $r \in R$ implies $s = 0$).

Example 3.29: Consider the set $\mathbb{Z}_5$ under addition modulo 5 and multiplication is defined as per the following table.

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	1	1	1
2	0	2	2	2	2
3	0	3	3	3	3
4	0	4	4	4	4

Then $(\mathbb{Z}_5, +, \cdot)$ forms a seminearring. Define $* : \mathbb{Z}_5 \times \mathbb{Z}_5 \to \mathbb{Z}_5$. Then $R = \mathbb{Z}_5$ forms a S-semigroup. Note that R is a faithful S-semigroup.

Corollary 3.30: If S is a seminearring then S is e-prime if and only if there exists a faithful equiprime S-semigroup.

Definition 3.31: A strong ideal K of R is said be completely equiprime (**c-e-prime**) strong ideal of R if $ar_1 \equiv_K ar_2$ then $aR \subseteq K$ or $r_1 \equiv_K r_2$ for $a \in S, r_1, r_2 \in R$.

Example 3.32: Consider the Example 3.11. Here $K = \{0,2\}$ is a strong ideal of the S-semigroup. Note that K is a c-e-prime strong ideal of S-semigroup.

Proposition 3.33: If L is a c-e-prime strong ideal of S-semigroup R, then L is a c-prime strong ideal of R.

Proof 3.34: Suppose L is c-e-prime strong and $ar \in L$ for $a \in S, r \in R$.

If $aR \subseteq L$, then L is c-prime strong ideal. Otherwise, we have $ar \in L \Rightarrow ar \equiv_L 0 \Rightarrow ar \equiv_L a0$ (since $a0 = 0$ in a zero symmetric seminearring).

$\Rightarrow r \equiv_L 0 \Rightarrow r \in L$.

Hence L is c-prime strong ideal.

The example that follows demonstrates that Proposition 3.33 need not hold in converse.

Example 3.35: Consider the seminearring $(S,+,\cdot)$ where + and $\cdot$ are defined as per the below tables:

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

·	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	2	2	2
3	0	3	3	3

Let $R = {}_S S$ $(S \times S \to S)$ be a S-semigroup with natural operations.

Then $L = \{0\}$ is a strong ideal of R. Moreover L is a c-prime strong ideal, but not a c-e-prime strong ideal because, for the values $a = 1, r_1 = 1, r_2 = 2$, $ar_1 \equiv_L ar_2$ implies $ar_1 \notin L$ and $r_1 \not\equiv_L r_2$.

Proposition 3.36: If K is a c-e-prime strong ideal of S-semigroup R, then K is an e-prime strong ideal.

Proof 3.37: Let K be c-e-prime strong and $a \in S, r_1, r_2 \in R$ and $asr_1 \equiv_K asr_2$ for every $s \in S$. Now, arbitrarily fix $s \in S$. Let $sr_1 = r_3, sr_2 = r_4$. Then $ar_3 \equiv_K ar_4$.

Then $asr_1 \equiv_K asr_2 \Rightarrow aR \subseteq K$ or $r_3 \equiv_K r_4$ (since K is c-e-prime strong).

If $aR \subseteq K$, then K is e-prime strong ideal.

Suppose $aR \not\subseteq K$.

As $r_3 \equiv_K r_4$ then $sr_1 \equiv_K sr_2$ and again since K is c-e-prime strong,

we get $sR \subseteq K$ or $r_1 \equiv_K r_2$.

As s is arbitrarily fix, we have $sR \subseteq K, \forall\, s \in S$, which is a contradiction for $aR \not\subseteq K$. Then, we get $r_1 \equiv_K r_2$.

Therefore K is e-prime strong ideal of R.

Definition 3.38: Let L be a strong ideal of seminearring S. Then L is a c-e-prime strong ideal of S if $a, x, y \in S, a \notin L$ and $ax \equiv_L ay$, then $x \equiv_L y$.

Example 3.39: Consider the seminearring $(S,+,\cdot)$ where + and $\cdot$ are defined as per the below tables:

+	1	2	3	4
1	1	2	3	4
2	2	2	3	4
3	3	3	3	3
4	4	4	3	4

·	1	2	3	4
1	1	1	1	1
2	1	2	3	4
3	1	2	3	4
4	1	2	3	4

Here $\{0\}$ is a c-e-prime strong ideal of S.

Proposition 3.40: If K is c-e-prime strong ideal of R, then $(K:R)_S$ is c-e-prime strong ideal of S.

Proof 3.41: Let $a \in S \setminus (K:R)_S$ and $x, y \in S$ such that $ax \equiv_{(K:R)_S} ay$.
Then there exist $i_1, i_2 \in (K:R)_S$ such that $i_1 + ax = i_2 + ay$.
$\Rightarrow (i_1 + ax)r = (i_2 + ay)r$, for some $r \in R$
$\Rightarrow i_1 r + axr = i_2 r + ayr$ $(i_1 \in (K:R)_S \Rightarrow i_1 r \in K)$.
$\Rightarrow p_1 + axr = p_2 + ayr$
Then $axr \equiv_K ayr \Rightarrow ar_1 \equiv_K ar_2$ for some $r_1 = xr, r_2 = yr$.
Since K is c-e-prime, we get $r_1 \equiv_K r_2$.
$\Rightarrow xr \equiv_K yr \Rightarrow xr \in p_1 + yr$.
$\Rightarrow x \equiv_{(K:R)_S} y$ (By Remark 3.15).
Hence $(K:R)_S$ is c-e-prime strong ideal of S.

Definition 3.42: A strong ideal L is called a strongly equiprime (**s-e-prime**) strong ideal of S, if for each $a \in S \setminus L$, there exists a finite subset D of S such that $q, y \in S$ and $adq \equiv_L ady$ for all $d \in D$ implies $q \equiv_L y$.

Example 3.43: Consider the seminearring from the Example 3.35. Here $L = \{0\}$ is a s-e-prime strong ideal of S. Note that L is not a c-e-prime strong ideal of S.

Proposition 3.44: If L is a c-e-prime strong ideal of S, then L is a s-e-prime strong ideal of S.

Proof 3.45: Let $a \in S \setminus L$ and $D = \{a\}$. Then D is finite; for any $q, y \in S, adq \equiv_L ady$ for every $d \in D \Rightarrow dq \equiv_L dy$ for every $d \in D$, since L is a c-e-prime strong ideal.
Then, $aq \equiv_L ay$ (since $D = \{a\}$).
Since L is c-e-prime strong ideal, we have $q \equiv_I y$.
Thus I is a s-e-prime strong ideal of S.

Corollary 3.46: If L is a c-e-prime strong ideal of S, then L is an e-prime strong ideal of S.

Remark 3.47: Let S be a seminearring and $(L:S)_S$ be an ideal of S.
Then for $x, y, s \in S$, we assume $xs \in L + ys \Leftrightarrow x \in (L:S)_S + y$.
This holds good in nearrings obviously as follows.
$xs \in L + ys \Leftrightarrow xs - ys \in L \Leftrightarrow (x-y)s \in L \Leftrightarrow x - y \in (L:S)_S \Leftrightarrow x \in (L:S)_S + y$.

Proposition 3.48: If L is a completely equiprime strong ideal of S, then $K = (L:S)_S$ is a completely equiprime strong ideal of S which is contained in L and $K \lhd S$.

Proof 3.49: $K \subseteq L$ and $K \lhd S$ follows from Proposition 3.22 and Corollary 3.46.
Given $K = (L:S)_S$. Let $a, x, y \in S, a \notin K$ and $ax \equiv_K ay$.
$\Rightarrow axs \equiv_L ays$, since $K = (L:S)_S$ and by Remark 3.47.
Since L is c-e-prime strong ideal of S, $xs \equiv_L ys$ for all $s \in S$.
Thus $x \equiv_K y$ and K is a c-e-prime strong ideal of S.

The idea of semi-primeness was generalized from rings to N-groups by Groenewald et al. [7]. We extend this concept to S-semigroups as follows:

Definition 3.50 Semiprime strong ideals on S-semigroup

Let K be a strong ideal of S-semigroup R such that $SR \not\subseteq K$. Then K is called

1. 0-semiprime, if $D^2M \subseteq K$ then $DM \subseteq K$ for ideals D of S.
2. 1-semiprime, if $D^2M \subseteq K$ then $DM \subseteq K$ for left ideals D of S.
3. 2-semiprime, if $D^2M \subseteq K$ then $DM \subseteq K$ for S-subsemigroups D of S.
4. 3-semiprime, if $bSbr \subseteq K$ then $br \in K$ for all $b \in S$ and $r \in R$.
5. completely semiprime (c-semiprime), if $b^2r \in K$ then $br \in K$ for all $b \in S$ and $r \in R$.

Example 3.51: Consider the seminearring $(S,+,\cdot)$ where + and $\cdot$ are defined as per the below tables:

$+$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	0	5	6	7	4
2	2	3	0	1	6	7	4	5
3	3	0	1	2	7	4	5	6
4	4	7	6	5	0	3	2	1
5	5	4	7	6	1	0	3	2
6	6	5	4	7	2	1	0	3
7	7	6	5	4	3	2	1	0

$\cdot$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	0	2	0	0	0	0
3	0	3	2	1	4	5	6	7
4	0	4	2	6	4	0	6	2
5	0	5	0	5	0	5	0	5
6	0	6	2	4	4	0	6	2
7	0	7	0	7	0	5	0	5

Let $R=\{0,1,2,3\}$ with respect to $+'$ defined as:

$+'$	0	1	2	3
0	0	1	2	3
1	1	0	2	3
2	2	2	2	2
3	3	3	2	3

Then $(R,+')$ is a semigroup.

Now, $*:S\times R\to R$ is defined as $n*\alpha=0$ for all $n\in S,\alpha\in R$. Then R forms a S-semigroup with respect to $*$. Here $\{0\},\{0,1\}$ are strong ideals of R.

Note that $T=\{0,1\}$ is υ-semiprime ($\upsilon=0,1,2,3,c$).

Proposition 3.52: If K is a strong ideal of S-semigroup R, then K is c-semiprime $\Rightarrow$ K is 3-semiprime $\Rightarrow$ K is 2-semiprime $\Rightarrow$ K is 1-semiprime $\Rightarrow$ K is 0-semiprime.

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