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On the radial solutions of a p-Laplacian equation involving a nonlinear gradient term and initial datum

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Abstract

The present paper establishes the existence, uniqueness and asymptotic behavior of positive radial solutions to the following ordinary differential equation with a positive initial datum

$$(|v'|^{p-2}v')' + \frac{N-1}{r}|v'|^{p-2}v' + rv' + v = 0 \quad \text{for } r > 0,$$

where $p > 2$ and $N > 1$. We start by providing a result on the existence of radial positive solutions using the shooting method and an associated energy function. Next, we derived crucial findings regarding the behavior of entire solutions near infinity. More precisely, we prove that the solutions behave like the function l/r, where l is a positive constant.

Key words and phrases: Elliptic equation, p-Laplace operator, gradient term, radial solutions, shooting method, positive solutions, asymptotic behavior.

Mathematics Subject Classification (2010): 35A01, 35B08, 35B09, 35B40, 35J60

1. Introduction

In this article, our attention is directed towards a radial self-similar solutions of the next elliptic equation

$$(|v'|^{p-2}v')'(r) + \frac{N-1}{r}|v'|^{p-2}v'(r) + rv'(r) + v(r) = 0 \quad \text{if } r > 0, \tag{1}$$

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where $p > 2$ and $N > 1$.

Given our focus on radial regular solutions, we impose the condition $v'(0) = 0$ and we examine the subsequent initial-value problem

$$(\mathcal{P})\begin{cases} (|v'|^{p-2}v')'(r) + \dfrac{N-1}{r}|v'|^{p-2}v'(r) + rv'(r) + v(r) = 0 \quad \text{for } r > 0, & (E1) \\ v(0) = b, \quad v'(0) = 0, & (E2) \end{cases}$$

where $p > 2$ and $N > 1$ and $b > 0$.

The investigation of semi-linear elliptic equations, expressed in the form

$$\Delta u + f(x, u, \nabla u) = 0 \tag{2}$$

has been the subject of numerous studies. In [1], H. Berestycki and L. Nirenberg established specific characteristics of the solutions to (2) under prescribed conditions on the function f. Additionally, in [10], the authors demonstrated the existence of large positive solutions to (2) given certain supplementary hypotheses. Also, in [5], C. Cowan and A. Razani utilized an explicit solution on the unit ball combined with a linearization argument to obtain positive singular solutions for perturbations of the unit ball of equation (2). For more detailed insights into this particular type of equation, we recommend consulting [9, 16, 6], as well as the references contained therein.

When $p > 2$, several studies have focused on the equation

$$\Delta_p u + f(x,u,\nabla u) = 0, \tag{3}$$

within bounded domains. In [7], C. Cowan and A. Razani demonstrated that if the domain is a sufficiently small C^2-perturbation of the unit ball, a singular positive weak solution of (3) exists. Furthermore, in [23], the authors established the existence of a bounded positive classical solution of (3) with additional specified properties.

In this research work, we draw inspiration from this combination of a source term and a gradient term, and delve into the examination of an equation involving the p-Laplace operator with degenerate nature both at $r = 0$ and at points corresponding to $v' = 0$. We refer the reader to the articles [2–4, 8, 17] and [18] for other equations of this type.

We summarize the results obtained for $(\mathcal{P})$ as follows. We prove, by applying a fixed point theorem, that for each $b > 0$ there is a unique function $v(.,b)$ solution of $(\mathcal{P})$ defined in $[0, +\infty]$ so that $|v_b'|^{p-2}v_b'$ is in $C^1([0, +\infty])$. Certain concepts of the proof are derived from the following documents [11, 12, 19–22] and [15]. Next, we carefully study the qualitative characteristics of the solutions to $(\mathcal{P})$. Initially, we demonstrate that each solution of $(\mathcal{P})$ is strictly positive and strictly decreasing. Next, we study the asymptotic behavior of solutions near to infinity. More precisely, inspired by [13, 14] and [24] we prove that $\lim_{r\to+\infty} v(r) = \lim_{r\to+\infty} v'(r) = 0$. We prove also that the function $r\,v(r)$ tends to a strictly positive limit when r tends to $+\infty$.

The subsequent sections of this paper are organized as follows: In Section 2, we introduce a result pertaining to the existence and uniqueness of positive solutions to $(\mathcal{P})$, employing a fixed point theorem and an associated energy function. Section 3 is dedicated to examining the behavior of global solutions and their derivatives in the vicinity of infinity. The last section 4 highlights the main results obtained and outlines the perspectives of this paper.

2. Existence of positive solutions

In this section, we use a Banach fixed point theorem and an energy function to demonstrate that $(\mathcal{P})$ has a unique positive solution. The fundamental theorem of existence is given below. We will refer to the solution of $(\mathcal{P})$ by v_b to represent its dependence on the shooting parameter b.

Theorem 2.1 *For each b > 0, there is a unique solution v_b of ($\mathcal{P}$) defined on [0, +∞] so that* $(|v_b'|^{p-2}v_b')' \in C^1([0,+\infty])$. *Moreover*

$$(|v_b'|^{p-2}v_b')'(0) = -b/N. \tag{4}$$

Proof: To establish this theorem, we will proceed through four steps. Firstly, we demonstrate the local existence of a solution. Next, in the second step we establish the result giving by 4. In the third step, we extend the solution on [0, +∞]. In the last step, we show that this solution is strictly positive on [0, +∞]

Step 1. The local existence of v_b.
Consider a solution v_b of ($\mathcal{P}$) on [0, r_{max}[. Multiplying ($E1$) by r^{N-1} we obtain:

$$\left[r^{N-1}|v_{b'}|^{p-2}v_{b'} + r^N v_b\right]' = (N-1)r^{N-1}v_b, \tag{5}$$

Integrating (5) over (0, r) and considering ($E2$), we obtain

$$v_b(r) = b - \int_0^r H(G[v_b](s))ds, \tag{6}$$

where $H(s) = |s|^{\frac{2-p}{p-1}} s,\ s \in \mathbb{R}$ and G is a non-linear function given by

$$G[\Phi](s) = s\Phi(s) + s^{1-N}\int_0^s (1-N)\sigma^{N-1}\Phi(\sigma)d\sigma. \tag{7}$$

Let $R > 0$, then let $C([0, R])$ be the Banach space of continuous functions on [0, R] with the uniform norm $||.||_0$. Thus $G[\Phi]$ is appropriately expressed as an operator from $C([0, R])$ into itself, where R > 0 and $C([0, R])$ denotes the space of continuous functions on [0, R] possessing the standard norm $||.||_0$. We consider b >0, K > 0 and we set the complete metric space $E_{b,K,R}$ giving by

$$E_{b,K,R} = \left\{\psi \in C([0,R]) : ||\psi - b||_0 \le K\right\}. \tag{8}$$

In addition, we introduce the operator Γ on $E_{b,K,R}$ by

$$\Gamma[\Phi](r) = b - \int_0^r H(G[\Phi](s))ds. \tag{9}$$

First, we show that Γ applies $E_{b,K,R}$ in itself for K sufficiently small and R > 0. It is clear that $\Gamma[\Phi] \in C([0,R])$. Using the definition of $E_{b,K,R}$, we determine that $\Phi \in [b-K, b+K]$ and consequently, a simple calculus gives

$$G[\Phi](s) \ge \left[\frac{1}{N}(b-K)\right]s. \tag{10}$$

As

$$\lim_{K\to 0}\left[\frac{1}{N}(b-K)\right] = \frac{b}{N} > 0. \tag{11}$$

Then there is $K_0 \in [0,b]$ so that for all $K \in (0, K_0)$, $G[\Phi]$ has a constant sign. Thus there is $\eta > 0$ so that for all $s \in [0,R]$,

$$G[\Phi](s) \ge \eta s. \tag{12}$$

Since $(H(r)/r)' \le 0$ if $r \in (0,+\infty)$, we obtain

$$|\Gamma[\Phi](r) - b| \le \int_0^r \frac{H(G[\Phi](\sigma))}{G[\Phi](\sigma)}|G[\Phi](\sigma)|d\sigma \le \int_0^r \frac{H(\eta\sigma)}{\eta\sigma}|G[\Phi](\sigma)|d\sigma, \tag{13}$$

for $0 < K < K_0$ and $r \in [0, R]$. However,

$$G[\Phi](s) \le |G[\Phi](s)| \le Cs, \quad C = [2 - 1/N](b + K), \tag{14}$$

hence, for each $r \in [0, R]$ we get

$$|\Gamma[\Phi](r) - b| \le \frac{p-1}{p} C\eta^{\frac{2-p}{p-1}} r^{\frac{p}{p-1}}, \tag{15}$$

with the constant η fixed, R is chosen to be sufficiently small so that

$$\frac{p-1}{p} C\eta^{\frac{2-p}{p-1}} R^{\frac{p}{p-1}} \le m. \tag{16}$$

Hence for each $\Phi \in E_{b,K,R}$ we have

$$|\Gamma[\Phi](r) - b| \le K. \tag{17}$$

This means that $\Gamma[\Phi] \in E_{b,K,R}$.
We will now show that Γ is a contraction on an interval $[0, r_b]$.
Let φ and $\Phi \in E_{b,K,R}$ we have

$$|\Gamma[\varphi](r) - \Gamma[\Phi](r)| \le \int_0^r |H(G[\varphi](s)) - H(G[\Phi](s))| \, ds. \tag{18}$$

Next, we let

$$\psi(s) = \min\left(|G[\varphi](s)|, |G[\Phi](s)|\right)$$

As a consequence of relation (12), we get

$$G[\psi](s) \ge \eta s,$$

for $0 \le s \le r < r_b$. This gives

$$|H(G[\varphi](s)) - H(G[\Phi](s))| \le \frac{H(\psi(s))}{\psi(s)} |G[\varphi](s) - G[\Phi](s)|.$$

Hence

$$|H(G[\varphi](s)) - H(G[\Phi](s))| \le \frac{H(\eta s)}{\eta s} |G[\varphi](s) - G[\Phi](s)|, \tag{19}$$

in addition, from (7) we obtain

$$|G[\varphi](s) - G[\Phi](s)| \le \lambda \| \varphi - \Phi \|_0 \, s; \quad \lambda = [2 - 1/N]. \tag{20}$$

By combining (18), (19) and (20) we obtain

$$|\Gamma[\varphi](r) - \Gamma[\Phi](r)| \le \frac{p-1}{p} \lambda \eta^{\frac{2-p}{p-1}} r^{\frac{p}{p-1}} \| \varphi - \Phi \|_0, \tag{21}$$

for each $r \in [0, r_b]$. Finally, we choose r_b small enough so that

$$\frac{p-1}{p} \lambda \eta^{\frac{2-p}{p-1}} r^{\frac{p}{p-1}} < 1,$$

we deduce that Γ is a contraction. Thus, the fixed point principle involves the existence of a unique fixed point of Γ in $E_{b,K,rb}$, this point is the solution of (6) and therefore the solution of $(\mathcal{P})$.

Step 2. $(|v_b'|^{p-2} v_b')' \in C^1([0, r_{max}])$.
From the form of $(E1)$, it is sufficient to verify that v_b is of class C^1 at point $r = 0$, for that we integrate $(E1)$ over $(0, r)$, we obtain

$$|v_b'|^{p-2} v_b'(r) = -rv_b(r) + (N-1)r^{1-N}\int_0^r s^{N-1}v_b(s)ds,$$

then applying the L'Hôpital's rule we obtain

$$\lim_{r\to 0} \frac{|v_b'|^{p-2} v_b'(r)}{r} = -b / N. \tag{22}$$

Hence, according to $(E1)$, we have

$$\lim_{r\to 0}\Big(|v_b'|^{p-2} v_b'\Big)'(r) = -b / N.$$

Step 3. Extend the solution v_b on $[0, +\infty]$.
In this step, we will prove that the solution v_b is global, i.e. it can be extended on $[0, +\infty]$. For this, we introduce the energy function

$$E_b(r) = \frac{(p-1)|v_b'(r)|^p}{p} + \frac{v_b^2(r)}{2} \qquad \forall r \geq 0. \tag{23}$$

It's obvious from $(E1)$ that

$$E_b'(r) = -rv_b'^2(r)\Big[\frac{N-1}{r^2}|v_{b'}(r)|^{p-2} + 1\Big] \qquad \forall r \geq 0. \tag{24}$$

Then E_b is decreasing and as a result it is bounded on $[0, +\infty]$. Consequently v_b and v_b' are also bounded and then $v_b(r)$ exists for each $r \geq 0$.

Step 4. v_b is strictly positive on $[0, +\infty]$.
Proceeding by contradiction, we suppose that $v_b(r_0) = 0$ (where $r_0 > 0$ is the first zero of v_b). Thus $v_b(r) > 0$ on $[0, r_0]$ and $v_b'(r_0) \leq 0$. We integrate (5) over $(0, r_0)$, we get

$$r_0^{N-1}|v_b'|^{p-2} v_b'(r_0) = (N-1)\int_0^{r_0} s^{N-1}v_b(s)ds. \tag{25}$$

According to our assumptions, the right term of the relation (25) is strictly positive. Which is a contradiction. The proof of this theorem is completed. □

3. Asymptotic behavior near infinity

In the present section, we examine the behavior of solutions to $(\mathcal{P})$ near to infinity, taking into consideration the fact that they are strictly positive. First, we present some characteristics of the solutions to $(\mathcal{P})$.

Theorem 3.1. *Let v_b denote a solution of $(\mathcal{P})$. Then*

$$\lim_{r\to+\infty} v_b(r) = \lim_{r\to+\infty} v_{b'}(r) = 0.$$

Proof. Recall that the energy function E_b is positive and decreasing for all $r \geq 0$. Then there is $L \geq 0$ so that $\lim_{r\to+\infty} E_b(r) = L$. We assume by contradiction that $L > 0$, then there is $r_1 > 0$ so that for all $r > r_1$ we have

$$E_b(r) \geq \frac{L}{2}. \tag{26}$$

Next, we define the function

$$I(r) = E_b(r) + \frac{N-1}{2}\frac{|v_{b'}|^{p-2} v_{b'}(r)v_b(r)}{r} + \frac{N-1}{4}v_b^2(r) + \int_0^r sv_b'^2(s)ds. \tag{27}$$

Hence

$$I'(r) = -\frac{(N-1)}{2r}\left[|v_{b'}|^{p} + N\frac{|v_{b'}|^{p-2}\, v_{b'}(r)v_b(r)}{r} + v_b^2(r)\right]. \tag{28}$$

Note that v_b and v_b' are bounded. Hence

$$\lim_{r\to+\infty}\frac{|v_{b'}|^{p-2}\, v_{b'}(r)v_b(r)}{r} = 0.$$

In addition, according to (29), we get for all $r \geq r_1$

$$v_b^2(r) + |v_{b'}|^p\,(r) \geq \frac{1}{2}v_b^2(r) + \frac{p-1}{p}|v_{b'}|^p\,(r) = E_b(r) \geq \frac{L}{2}.$$

Hence, as $N > 1$, there is two constants $\mu > 0$ and $r_2 \geq r_1$ so that

$$I'(r) \leq -\frac{\mu}{r} \qquad \text{for} \quad r \geq r_2. \tag{29}$$

Integrating (29) over $[r_2, r]$, we obtain

$$I(r) \leq I(r_2) - \mu \ln(r / r_2) \qquad \text{for} \quad r > r_2,$$

which implies that $\lim_{r\to+\infty} I(r) = -\infty$. Moreover

$$E_b(r) + \frac{N-1}{2}\frac{|v_{b'}|^{p-2}\, v_{b'}(r)v_b(r)}{r} \leq I(r).$$

Hence $\lim_{r\to+\infty} E_b(r) = -\infty$. This is impossible and therefore $\lim_{r\to+\infty} E_b(r) = 0$. Thus from (23), $\lim_{r\to+\infty} v_b(r) = \lim_{r\to+\infty} v_{b'}(r) = 0.$ □

The following theorem gives information on the monotonicity of solutions to $(\mathcal{P})$.

Theorem 3.2. *Let v_b be a solution of $(\mathcal{P})$, then v_b is strictly decreasing.*

Proof. We proceed by means of contradiction. Then let $v_0 > 0$ be the first zero of v_b'. Thus from $(E1)$ we have $(|v_{b'}|^{p-2}\, v_{b'})'(r_0) = -v_b(r_0) < 0.$ Furthermore, we can see from (22) that $v_b'(r) < 0$ for $r \approx 0$. By continuity of v_b' and definition of r_0, there is $\varepsilon > 0$ so that v_b' is strictly negative and strictly increasing on $]r_0 - \varepsilon, r_0[$. This implies that $(|v_{b'}|^{p-2}\, v_{b'})'(r) > 0$ for all $r \in]r_0 - \varepsilon, r_0[$, and consequently, if we tend r to r_0, we obtain $(|v_{b'}|^{p-2}\, v_{b'})'(r_0) \geq 0.$ Which is contradictory. □

Theorem 3.3. *Let v_b denote a solution of $(\mathcal{P})$. Then*

$$\lim_{r\to+\infty} r^k v_b(r) \in [0, +\infty[\qquad \text{for all} \quad 0 < k < 1.$$

The proof of Theorem 3.3 requires the following result, which gives information on the monotonicity of the functions $r^c v_b(r)$ for $c > 0$. The reason why we define for each function v_b verifying $(E1)$, the function T_c by

$$T_c(r) = r^{1-c}(r^c v_b(r))', \qquad c > 0 \quad \text{and} \quad r > 0, \tag{30}$$

that is

$$T_c(r) = c v_b(r) + r v_{b'}(r), \qquad c > 0 \quad \text{and} \quad r > 0. \tag{31}$$

Since $v_b'(r) < 0$ on $(0, +\infty)$, we have from $(E1)$,

$$(p-1)|v_{b'}|^{p-2}\,(r) T_c'(r) = (p-1)\left(c - \frac{N-p}{p-1}\right)|v_{b'}|^{p-2}\, v_{b'}(r) - r^2 v_{b'}(r) - r v_b(r). \tag{32}$$

Hence for all r_0 so that $T_c(r_0) = 0$, we have

$$(p-1)\,|v_{b'}|^{p-2}\,(r_0)T_c'(r_0) = r_0 v_b(r_0)\left[c-1+(p-1)c^{p-1}\left(\frac{N-p}{p-1}-c\right)\frac{|v_b|^{p-2}\,(r_0)}{r_0^p}\right]. \tag{33}$$

The following proposition is devoted to study the sign of $T_c(r)$ if r is sufficiently large.

Proposition 3.4. *Consider a solution v_b of ($\mathcal{P}$), and let $c > 0$. Then $T_c(r)$ keeps a constant sign if r is sufficiently large in both cases*

(*i*) $c \neq 1$,

(*ii*) $c = 1 \neq \dfrac{N-p}{p-1}$.

Proof. (i) Let r_0 sufficiently large so that $T_c(r_0) = 0$. Since v_b is strictly positive and converges to 0 at infinity, then from (33), we have $T_c'(r_0) > 0$ if $c > 1$ or else $T_c'(r_0) < 0$ if $c < 1$. As a consequence, $T_c(r) \neq 0$ if r is sufficiently large.

For (ii), similarly if r_0 is large enough so that $T_1(r_0) = 0$, then

$$|v_{b'}|^{p-2}\,(r_0)T_1'(r_0) = \left(\frac{N-p}{p-1}-1\right)\frac{v_b^{p-1}(r_0)}{r_0^{p-1}}. \tag{34}$$

Hence $T_1'(r_0) \neq 0$ and therefore $T_1(r_0) \neq 0$ when r is large enough. □

The demonstration of Theorem 3.3 is now within our reach.

Proof. Let $0 < k < 1$. According to (*i*) of Proposition 3.4, the function $T_k(r)$ keeps a constant sign if r is sufficiently large. Assume that $T_k(r)$ is positive when r is sufficiently large. Then, according to (31) and the fact that $v_b' < 0$, we have

$$r\,|v_{b'}(r)| \leq k v_b(r), \tag{35}$$

when r is sufficiently large. However, using ($E1$), we obtain

$$(|v_b'|^{p-2}\,v_b')'(r) = \frac{N-1}{r}|v_b'|^{p-1}\,(r) + r\,|v_b'(r)| - v_b(r). \tag{36}$$

Using (35), we find that if r is sufficiently large,

$$(|v_b'|^{p-2}\,v_b')'(r) \leq (N-1)k^{p-1}\frac{v_b^{p-1}(r)}{r^p} + k v_b(r) - v_b(r), \tag{37}$$

Which is equivalent to

$$(|v_b'|^{p-2}\,v_b')'(r) \leq v_b(r)\left[k-1+(N-1)k^{p-1}\frac{v_b^{p-2}(r)}{r^p}\right], \tag{38}$$

if r is sufficiently large. As v_b is strictly positive, $0 < k < 1$ and $\lim_{r\to+\infty} v_b(r) = 0$, we deduce that $(|v_b'|^{p-2}\,v_b')'(r) < 0$ if r is sufficiently large. Moreover, since $v_b'(r) < 0$ for each $r > 0$, then $\lim_{r\to+\infty} v_{b'}(r) \in [-\infty, 0[$. However, this is contradictory since v_b is strictly positive. It follows that $T_k(r)$ is negative if r is sufficiently large and so, by virtue of (30), the function $r^k v_b(r)$ is decreasing when r is sufficiently large. Consequently $\lim_{r\to+\infty} r^k v_b(r) \in [0, +\infty[$. □

Now, after establishing the existence of $\lim_{r\to+\infty} r^k v_b(r)$ for all $0 < k < 1$, the question of convergence of the function $r v_b(r)$ arises. The answer lies in the subsequent theorem.

Theorem 3.5. *Consider a solution v_b of $(\mathcal{P})$. Then*

$$\lim_{r\to+\infty} rv_b(r) = l > 0 \tag{39}$$

Moreover if $\frac{N-p}{p-1} \neq 1$, then

$$\lim_{r\to+\infty} r^2 v_b'(r) = -l < 0. \tag{40}$$

The proof of Theorem 3.5 requires the next preliminary results.

Lemma 3.6. *Let v_b be a solution of $(\mathcal{P})$. Assume in addition that there is $\eta > 0$, $\gamma \geq 0$ and $r_0 > 0$ so that*

$$v_b(r) \leq \eta(1+r)^{-\gamma} \quad \text{for each} \quad r \geq r_0. \tag{41}$$

Then there is a constant m depending on η, γ and r_0 so that

$$|v_{b'}(r)| \leq m(1+r)^{-\gamma-1} \quad \text{for all} \quad r \geq r_0. \tag{42}$$

Proof. Let us define the function ζ as follows

$$\zeta(r) = \exp\left[\frac{1}{p-1}\int_{r_0}^{r} s\,|v_{b'}(s)|^{2-p}\,ds\right]. \tag{43}$$

Remember that $v_b' < 0$, hence $\zeta(r)$ is properly expressed for all $r \geq r_0$ and it is strictly increasing and infinitely differentiable. By multiplying $(E1)$ by

$$(p-1)r^{\frac{N-p}{(p-1)}}\zeta'(r) = |v_{b'}|^{2-p}\, r^{\frac{N-1}{p-1}}\zeta(r), \tag{44}$$

we obtain, for all $r \geq r_0$

$$\begin{aligned}&(p-1)r^{(N-1)/(p-1)}\zeta(r)v_{b''} + (N-1)r^{(N-p)/(p-1)}\zeta(r)v_{b'} + (p-1)r^{(N-1)/(p-1)}\zeta'(r)v_{b'}\\ &= -(p-1)r^{(N-p)/(p-1)}\zeta'(r)v_b.\end{aligned}$$

This equation can be written as follows

$$W'(r) = -(p-1)r^{(N-p)/(p-1)}\zeta'(r)v_b, \tag{45}$$

with

$$W(r) = (p-1)v_{b'}(r)r^{\frac{N-1}{p-1}}\zeta(r), \tag{46}$$

for all $r \geq r_0$. Integrating (45) over (r_0, r) for all $r \geq r_0$, we obtain

$$v_{b'}(r) = \frac{r^{(1-N)/(p-1)}}{\zeta(r)}v_{b'}(r_0)r_0^{(N-1)/(p-1)} - \frac{r^{(1-N)/(p-1)}}{\zeta(r)}\int_{r_0}^{r} s^{(N-p)/(p-1)}\zeta'(r)v_b(s)ds.$$

As $v_b'(r) < 0$ and $\zeta'(r) > 0$, then for all $r \geq r_0$,

$$|v_{b'}(r)| \leq \frac{r^{(1-N)/(p-1)}}{\zeta(r)}r_0^{(N-1)/(p-1)}\,|v_{b'}(r_0)| + \frac{r^{(1-N)/(p-1)}}{\zeta(r)}J, \tag{47}$$

with

$$J = \int_{r_0}^{r} s^{(N-p)/(p-1)}\zeta'(r)v_b(s)ds.$$

As $v_b{}'$ is continuous on [0, +∞] and $\lim_{r\to+\infty} v_{b'}(r)=0$, then there is a constant η_0 depending on r_0 exists, so that

$$|v_{b'}(r)|^{2-p} \ge \eta_0. \tag{48}$$

Hence

$$\zeta(r) \ge \eta_2 \exp(\eta_1 r^2) \qquad \text{for all} \quad r \ge r_0, \tag{49}$$

where $\eta_1 = \dfrac{1}{2(p-1)}\eta_0$ and $\eta_2 = \exp[-\eta_1 r_0^2]$. By majoring the first term of the right-hand side of the inequality (47), we obtain

$$\frac{r^{(1-N)/(p-1)} r_0^{(N-1)/(p-1)}}{\zeta(r)} |v_{b'}(r_0)| \le \frac{1}{\eta_2} |v_{b'}(r_0)| \exp[-\eta_1 r^2]. \tag{50}$$

Then, to obtain an estimation of the second term of the right-hand side of the inequality (47), we use the assumption (41). So for all $r \ge 2r_0$ we have

$$J \le C\int_{r_0}^{r/2} s^{(N-p)/(p-1)}\zeta'(r)(1+s)^{-\sigma}\,ds + C\int_{r/2}^{r} s^{(N-p)/(p-1)}\zeta'(r)(1+s)^{-\sigma}\,ds. \tag{51}$$

One can easily verify that

$$\int_{r_0}^{r/2} s^{\frac{N-p}{p-1}}\zeta'(r)(1+s)^{-\sigma}\,ds \le (1+r_0)^{-\sigma} \max_{r_0, r/2}(s^{\frac{N-p}{p-1}})\zeta(r/2), \tag{52}$$

also

$$\int_{r/2}^{r} s^{\frac{N-p}{p-1}}\zeta'(r)(1+s)^{-\sigma}\,ds \le (1+r/2)^{-\sigma} \max_{r/2, r}(s^{\frac{N-p}{p-1}})\zeta(r). \tag{53}$$

Therefore, we note that

$$\frac{\zeta(r/2)}{\zeta(r)} = \exp\left[-\frac{1}{p-1}\int_{r/2}^{r} s\,|v_{b'}(r_0)|^{2-p}\,ds\right], \tag{54}$$

from (48) we deduce that

$$\frac{\zeta(r/2)}{\zeta(r)} \le \exp[-\eta_3 r^2], \tag{55}$$

where $\eta_3 = \dfrac{3}{8(p-1)}\eta_0$. Combining (51–55), we obtain

$$r^{\frac{1-N}{p-1}} \frac{J}{\zeta(r)} \le \delta(1+r)^{-\sigma-1} + \delta r^{\frac{1-N}{p-1}} \max_{(r_0, r/2)}(s^{\frac{N-p}{p-1}}) \exp\left[-\eta_3 r^2\right], \tag{56}$$

with $\delta > 0$ is a constant that depends on N, p, r_0 and σ. Finally, by combining (50) and (56), we deduce the estimation (42). This completes the demonstration. □

Lemma 3.7. *Let v_b be a solution of $(\mathcal{P})$ with $v(0) = b$. Then*

$$0 < v_b(r) \le \frac{C}{r} \qquad \textit{for } r \textit{ large enough}, \tag{57}$$

where C is a strictly positive constant.

Proof. By multiplying ($E1$) by v_b/r, it is easy to obtain

$$\frac{v_b^2(r)}{r} = \frac{|v_{b'}|^p(r)}{r} - v_b v_{b'} - \frac{N}{r^2} v_b |v_{b'}|^{p-2} v_{b'} - \left[\frac{v_b |v_{b'}|^{p-2} v_{b'}}{r}\right]'. \tag{58}$$

Using the energy function given by (23), we obtain

$$\frac{E_b(r)}{r} = \frac{3p-2}{2p}\frac{|v_{b'}|^p(r)}{r} - \frac{1}{2} v_b v_{b'} - \frac{N}{2r^2} v_b |v_{b'}|^{p-2} v_{b'} - \frac{1}{2}\left[\frac{v_b |v_{b'}|^{p-2} v_{b'}}{r}\right]'. \tag{59}$$

However, according to Theorem T3.2, $v_b' < 0$, then

$$\begin{aligned}\frac{E_b(r)}{r} &= \frac{3p-2}{2p}\frac{|v_{b'}|^p(r)}{r} - \frac{1}{2} v_b v_{b'} + \frac{N}{2r^2} v_b |v_{b'}|^{p-1} - \frac{1}{2}\left[\frac{v_b |v_{b'}|^{p-2} v_{b'}}{r}\right]' \\ &= \frac{3p-2}{2p}\frac{|v_{b'}|^p(r)}{r} - \frac{1}{4}(v_b^2)' + \frac{N}{2r^2} v_b |v_{b'}|^{p-1} - \frac{1}{2}\left[\frac{v_b |v_{b'}|^{p-2} v_{b'}}{r}\right]'.\end{aligned}$$

We integrate the last equality on (r, R), then we get

$$\begin{aligned}\int_r^R \frac{E_b(s)}{s} ds = {} & \frac{3p-2}{2p}\int_r^R \frac{|v_{b'}(s)|^p}{s} ds - \frac{1}{4} v_b^2(R) + \frac{1}{4} v_b^2(r) \\ & + \frac{N}{2}\int_r^R \frac{v_b(s)|v_{b'}|^{p-1}(s)}{s^2} ds - \frac{1}{2}\frac{v_b(R)|v_{b'}|^{p-2} v_{b'}(R)}{R} \\ & + \frac{1}{2}\frac{v_b(r)|v_{b'}|^{p-2} v_{b'}(r)}{r}.\end{aligned}$$

Using $v_b > 0$ and $v_b' < 0$ again, we obtain

$$\begin{aligned}\int_r^R \frac{E_b(s)}{s} ds \le {} & \frac{3p-2}{2p}\int_r^R \frac{|v_{b'}(s)|^p}{s} ds + \frac{1}{4} v_b^2(r) + \\ & \frac{N}{2}\int_r^R \frac{v_b(s)|v_{b'}|^{p-1}(s)}{s^2} ds + \frac{1}{2}\frac{v_b(R)|v_{b'}(R)|^{p-1}}{R}.\end{aligned}$$

Since v_b is bounded, more precisely $v_b(r) \le b$, we can assume that there is $\sigma \ge 0$ and $C > 0$ so that

$$v_b(r) \le C r^{-\sigma} \tag{60}$$

for r large enough. Thus, the Lemma 3.6 implies that

$$|v_{b'}(r)| \le C r^{-\sigma-1} \tag{61}$$

when r is large enough. It follows that $\frac{|v_{b'}|^p(r)}{r}$ and $\frac{v_b |v_{b'}|^{p-1}}{r^2}$ are in $L^1(]r_0, +\infty[)$ for all $r_0 > 0$. We tend $R \to +\infty$ and use the fact that $\lim_{r\to+\infty} v_b(r) = \lim_{r\to+\infty} v_{b'}(r) = 0$, we obtain

$$\int_r^\infty \frac{E_b(s)}{s} ds \le \frac{3p-2}{2p}\int_r^\infty \frac{|v_{b'}(s)|^p}{s} ds + \frac{1}{4} v_b^2(r) + \frac{N}{2}\int_r^\infty \frac{v_b(s)|v_{b'}|^{p-1}(s)}{s^2} ds. \tag{62}$$

We set

$$\Lambda(r) = \int_r^\infty \frac{E_b(s)}{s} ds. \tag{63}$$

From (23) we have $v_b^2(r) \le 2E_b(r)$, consequently (63) implies that

$$\begin{aligned} \Lambda(r) + \frac{r}{2}\Lambda'(r) &= \Lambda(r) - \frac{E_b(r)}{2} \\ &\le \Lambda(r) - \frac{v_b^2(r)}{4}. \end{aligned} \tag{64}$$

Hence from (62) we obtain

$$\Lambda(r) + \frac{r}{2}\Lambda'(r) \le \frac{3p-2}{2p}\int_r^{\infty} \frac{|v_{b'}(s)|^p}{s} ds + \frac{N}{2}\int_r^{\infty} \frac{v_b(s)|v_{b'}|^{p-1}(s)}{s^2} ds. \tag{65}$$

Consequently, according to (60), (61) and (65), we obtain

$$[r^2\Lambda(r)]' \le Cr^{1-p(\sigma+1)}, \quad \text{when } r \text{ is large enough.} \tag{66}$$

A simple integration of (66) yields

$$\Lambda(r) \le Cr^{-2}, \quad \text{if } r \text{ is large enough.} \tag{67}$$

Now, taking into consideration the fact that the energy function E_b is non-increasing, we obtain

$$\Lambda(r) \ge \int_r^{2r} \frac{E_b(s)}{s} ds \ge \frac{E_b(2r)}{2} \ge \frac{v_b^2(2r)}{4}. \tag{68}$$

Hence, according to the estimation (67) we have

$$v_b(r) \le Cr^{-1}, \quad \text{when } r \text{ is large enough.} \tag{69}$$

The demonstration is achieved. □

Lemma 3.8. *Consider a solution v_b of $(\mathcal{P})$. Assume that $\lim_{r\to+\infty} rv_b(r) = 0$. Then for each real $m > 0$,*

$$\lim_{r\to+\infty} r^m v_b(r) = \lim_{r\to+\infty} r^m v_b'(r) = 0. \tag{70}$$

Proof. We set

$$I_v(r) = r\Big[v_b(r) + \frac{|v_b'|^{p-2}v_b'(r)}{r}\Big]. \tag{71}$$

Then, we have

$$I_{v'}(r) = -(N-1)\frac{|v_b'|^{p-2}v_b'(r)}{r}. \tag{72}$$

According to the Lemmas 3.6 et 3.7, the function $r \to r^{-1}|v_b'|^{p-1}$ is integrable on $(r_0, +\infty)$ for all $r_0 > 0$. Hence $I_v' \in L^1(r_0, +\infty)$. As $\lim_{r\to+\infty} rv_b(r) = 0$ and $\lim_{r\to+\infty} v_b'(r) = 0$, then $\lim_{r\to+\infty} I_v(r) = 0$. As a result

$$I_v(r) = -\int_r^{+\infty} I_{v'}(t)dt.$$

Hence, from the expressions of I_v and $I_v{}'$ we have

$$v_b(r) = \frac{-|v_b'|^{p-2}v_{b'}(r)}{r} + \frac{N-1}{r}\int_r^{+\infty} \frac{|v_{b'}(s)|^{p-2}v_{b'}(s)}{s} ds. \tag{73}$$

Since $v_b{}' < 0$, then

$$v_b(r) \le \frac{1}{r}|v_b'|^{p-1} + \frac{N-1}{r}\int_r^{+\infty} \frac{|v_{b'}(s)|^{p-1}}{s} ds \tag{74}$$

Since $\lim_{r\to+\infty} rv_b(r)=0$, then, according to Lemma 3.6 and (74), we have if r is large enough

$$v_b \le Cr^{1-2p},$$

for some $C > 0$. Let us define the sequence

$$m_k=(p-1)m_{k-1}+p \quad \text{and} \quad m_0=1.$$

Then $\lim_{r\to+\infty} m_k = +\infty$, Consequently, the theorem is obtained by induction, starting with $m_0 = 1$. This completes the proof. □

Lemma 3.9. *Let v_b be a solution of $(\mathcal{P})$. We define the function J_v by*

$$J_v(r)=r^N\left[v_b(r)+\frac{|v_{b'}(r)|^{p-2}\,v_{b'}(r)}{r}\right]. \tag{75}$$

Then, for each $r > 0$, we have $J_v(r) > 0$ and $J_v'(r) > 0$.

Proof. It is easy to verify that

$$J_{v'}(r)=(N-1)r^{N-1}v_b(r).$$

Since $v_b(r)$ is strictly positive, then for all $r > 0$, J_v is strictly increasing. Moreover, since $\left(|v_b'|^{p-2}\,v_b'\right)'(0)=-b/N$ is finite, then $J_v(0) = 0$. Therefore $J_v(r) > 0$ for all $r > 0$. □

Now, we can proceed to prove Theorem 3.5.

Proof. Recalling Lemmas 3.6 and 3.7, we have the function $r\to r^{-1}|v_b'|^{p-1}$ is integrable on $(r_0,+\infty)$ for all $r_0 > 0$. Thus $I_v'\in L^1(r_0,+\infty)$. Therefore

$$\lim_{r\to+\infty} I_v(r)=\int_{r_0}^{\infty} I_v'(r)dr+I_v(r_0) \tag{77}$$

exists and is finite. Moreover, (57) and (42) implies that

$$|v_{b'}|^{p-1}\le Cr^{2-2p}, \tag{78}$$

if r is large enough. Then it follows from (71) and (78) that $\lim_{r\to+\infty} rv_b(r)=l$ exists and is finite. Suppose now that $l = 0$, then by virtue of Lemma 3.8, we have

$$\lim_{r\to+\infty} r^m v_b(r)=\lim_{r\to+\infty} r^m v_{b'}(r)=0,$$

for all $m > 0$ and then $\lim_{r\to+\infty} J_v(r)=0$. But this is a contradiction as J_v is strictly positive and strictly increasing for all $r > 0$ (according to Lemma 3.9). Therefore $\lim_{r\to+\infty} rv_b(r)=l>0$.

Now, to show that $r^2v_b'(r)$ converges to $-l$ at infinity, we make the following logarithmic transformation:

$$U(t)=rv_b(r) \quad \text{with} \quad t=\ln(r). \tag{79}$$

So U satisfies the following equation

$$w'(t)+\Gamma w(t)+e^{2(p-1)t}U'(t)=0, \tag{80}$$

where

$$w(t)=|y(t)|^{p-2}\,y(t), \tag{81}$$

$$y(t) = U'(t) - U(t) = r^2 v_b'(r) \tag{82}$$

and

$$\Gamma = N - 2p + 1. \tag{83}$$

From Proposition 3.4 and (30), we know that $r\ v_b$ is strictly monotone for large r if $\frac{N-p}{p-1} \neq 1$. According to (82) we have

$$U'(t) = y(t) + U(t) = r(rv_b)'. \tag{84}$$

Therefore U is strictly monotone for large t. Since $\lim_{t\to+\infty} U(t) = l > 0$, then necessarily $\lim_{t\to+\infty} U'(t) = 0$. Afterwards, by (82), $\lim_{t\to+\infty} y(t) = -\lim_{t\to+\infty} U(t) = -l < 0$, that is $\lim_{r\to+\infty} r^2 v_b'(r) = -l < 0$. □

Theorem 3.6. *Consider a solution* v_b *of* $(\mathcal{P})$.

(i) *If* $\frac{N-p}{p-1} > 1$, *then* $r\ v_b$ *is strictly increasing and* $r^2 v_b'$ *is strictly decreasing, for large* r.

(ii) *If* $\frac{N-p}{p-1} < 1$, *then* $r\ v_b$ *is strictly decreasing and* $r^2 v_b'$ *is strictly increasing, for large* r.

Proof. First, we use the logarithmic change (79) and we show that $w(t)$ is strictly monotone if t is large. Suppose that there is a large t_0 so that $w'(t_0) = 0$, then from (80), we have

$$w''(t_0) = (1-2p)e^{2(p-1)t} U'(t_0). \tag{85}$$

Since U is strictly monotone for large t, then $w''(t_0) \neq 0$. Hence $w'(t) \neq 0$ for large t, that is w is strictly monotone for large t. Moreover, since

$$\lim_{t\to+\infty} w(t) = \lim_{t\to+\infty} |\, y(t)\,|^{p-2}\ y(t) = -l^{p-1} < 0, \tag{86}$$

then necessarily $\lim_{t\to+\infty} w't = 0$ and thus according to (80), we get

$$\begin{aligned} \lim_{t\to+\infty} e^{2(p-1)t} U'(t) &= -(N-2p+1) \lim_{t\to+\infty} w(t) \\ &= (N-2p+1) l^{p-1}. \end{aligned} \tag{87}$$

That is from (84) and (31), we have

$$\lim_{t\to+\infty} r^{2p-1}(rv_b)' = \lim_{t\to+\infty} r^{2p-1} T_1(r) = (N-2p+1) l^{p-1}. \tag{88}$$

According to our hypotheses, we have $\Gamma = N - 2p + 1 \neq 0$. Then the monotonicity of $r\ v_b$ depends on the sign of Γ.

Now, to see the monotonicity of $r^2 v_b'$, we use again the logarithmic change (79), we get

$$y'(t) = \frac{1}{p-1} |\, w\,|^{(2-p)/(p-1)}\ w'(t) = r(r^2 v_b')' = r^2 T_1'(r). \tag{89}$$

Since $w(t)$ is strictly monotone for large t, then y is also strictly monotone for large t. Therefore $r^2 v_b'$ is strictly monotone for large r, that is $T_1'(r) \neq 0$ for large r. On the other hand, we have from (88), $\lim_{r\to+\infty} T_1(r) = 0$. Therefore, If $\frac{N-p}{p-1} > 1$, we have $T_1(r)$ is strictly positive, strictly monotone if r is large and converges to 0 at infinity. Then necessarily $T_1'(r) < 0$ if r is large, hence $(r^2 v_b')' < 0$ for large r. If $\frac{N-p}{p-1} < 1$, in the same way, $T_1(r)$ is strictly negative, strictly monotone if r is large and converges to 0 at infinity. Then necessarily $T_1'(r) < 0$ if r is large and so $(r^2 v_b')' > 0$ if r is large. □

Theorem 3.11. *Assume that* $\frac{N-p}{p-1} \neq 1$. *Let* v_b *be a solution of* $(\mathcal{P})$. *Then*

$$\lim_{r\to+\infty} r^3 v_{b''}(r) = 2l > 0. \tag{90}$$

Proof. Since $v_b{}' < 0$, then from (1) and expression (31), we write

$$(p-1)\,|\,v_b'\,|^{p-2}\, v_b''(r) = -\frac{N-1}{r}\,|\,v_b'\,|^{p-2}\, v_b'(r) - T_1(r). \tag{91}$$

Hence

$$v_{b''}(r) = \frac{-v_{b'}(r)}{(p-1)r}\left[N-1+\frac{rT_1(r)}{|\,v_{b'}\,|^{p-2}\, v_{b'}(r)}\right]. \tag{92}$$

Using again $v_b{}' < 0$, then we obtain

$$r^3 v_b''(r) = \frac{r^2\,|\,v_b'(r)\,|}{(p-1)}\left[N-1-\frac{rT_1(r)}{|\,v_b'(r)\,|^{p-1}}\right]. \tag{93}$$

Since $\lim_{r\to+\infty} r^2\,|\,v_b'(r)\,| = l > 0$, then by (88),

$$\lim_{r\to+\infty} \frac{rT_1(r)}{|\,v_b'(r)\,|^{p-1}} = N-2p+1 \neq 0. \tag{94}$$

Hence from (93), we obtain $\lim_{r\to+\infty} r^3 v_b''(r) = 2l > 0.$ This completes the proof. □

4. Conclusion and perspectives

In this work, we proved the existence of entire solutions of $(\mathcal{P})$ through the utilization of Banach's fixed point Theorem and the energy method. Next, we studied the asymptotic behavior near infinity. More precisely, we demonstrated that any solution of $(\mathcal{P})$ is strictly positive, strictly decreasing and behaves like $1/r$ near infinity. The study was based on the fact that the dimension $N > 1$. The unidimensional case remains an open question to be examined in future research.

7. References

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