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New fixed point theorems in extended orthogonal $\mathcal{S}$-metric spaces of type (μ,σ) with applications to fractional integral equations

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In this article, we propose the concept of extended orthogonal $\mathcal{S}$-metric spaces of type (μ,σ) and explore several related properties and theorems based on this concept. We provide examples to validate our findings. Finally, we illustrate the practical relevance of our results through applications to the fractional integral equations and Fredholm integral equation, highlighting the advantages of our findings.

Keywords and Phrases: Extended $\mathcal{S}$-metric Space of type (m,s); Extended orthogonal $\mathcal{S}$-metric Spaces of type (m,s); orthogonal metric space; $\mathcal{S}$-metric space.

Mathematics Subject Classification (2020): 47H10, 54H25.

1. Introduction

Brouwer [1] introduced fixed-point(FP) theory in 1911. Banach [2] later invented the Banach contraction principle in 1922. In 2012, Sedghi et al. [3] established S-metric space(SMS) and proved FP theorem on a complete SMS. Sedghi et al. [4] proved a general FP theorem in SMS. Sedghi et al. [5] presented a coupled coincidence point theorems for multi-valued maps on complete SMS using mixed g-monotone mappings. Kim et al. [6] presented some FP theorems for two maps on complete SMS.

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Mlaiki et al. [7] proved new FP theorems applying the set of simulation functions on an SMS. Adewale et al. [8] introduced the idea of rectangular SMS.

In 2017, Gordji et al. [9] introduced the ideas of orthogonal sets and orthogonal metric spaces (OMS). Gordji et al. [10] proved the existence and uniqueness theorem of FP for mappings on a generalized OMS. Senapati et al. [11] used w-distance to verify the well-known Banach's FP theorem in OMS. Gordji et al. [12] proved the existence and uniqueness theorem of FP for mappings on ε-connected OMS. Gungor et al. [13] presented some FP theorems on OMS via altering distance functions. Yang et al. [14] presented an orthogonal (F, ψ)-contraction for the Hardy-Rogers-type mapping. Sawangsup et al. [15] established the concept of an orthogonal F-contraction mapping in OMS. Sawangsup et al. [16] established the concept of orthogonal Z-contraction mappings in OMS. Gunaseelan et al. [17] established some FP theorems in orthogonal complete F-metric spaces. Arul Joseph et al. [18] proved some fixed point theorems on orthogonal b-metric spaces. Ismat et al. [19] introduced the concept of generalized orthogonal F-Suzuki contraction mapping and proved some FP theorems on orthogonal b-metric spaces. Arul Joseph et al. [20] proposed the concept of orthogonally triangular μ-admissible mapping. Gunaseelan et al. [21] used the concept of orthogonal connected contraction maps type I and II to prove the coupled FP theorem in OMS. Singh et al. [22] demonstrate Boyd-Wong and Matkowski type FP theorems for OMS. Zeinab et al. [23] proved the FP theorem for mappings in SMS by decreasing the completeness of SMS using relations.

In 2017, Kamran et al. [24] introduced the concept of extended $\mathfrak{b}$-metric space (EBMS) and proved various FP theorems. Mlaiki [25] proposed the concept of extended $\mathcal{S}_b$-metric spaces. Alqahtani et al. [26] proved the existence of common fixed points in the frame of an EBMS. Kushal et al. [27] proved FP theorems for some classes of contractive mappings. Aydi et al. [28] generalized some FP theorems with Kannan-type contractions in the setting of new EBMS. Huang et al. [29] established the existence of rational type contraction FP in an EBMS context. Nayab et al. [30] established a Hausdorff metric over the family of non-void closed subsets of an EBMS. Qaralleh et al. [31] introduced the notion of extended $\mathcal{S}$-metric space(ESMS) of type (μ,σ). Later, many authors have proved existence and uniqueness of fractional equations [32–38]. Based on the generality of ESMS of type (μ,σ) and orthogonality condition, we are the first who establish a FP result for extended orthogonal $\mathcal{S}$-metric space(EOSMS) of type (μ,σ) with supporting example. Finally, we give applications on Fredholm integral equation and fractional integral equations.

2. Preliminaries

In this section, we recall the main definitions and outcomes connected to SMS, OSMS and ESMS.

Definition 2.1. [3] *Let* ∇ *be a non-void set. A function* $\varphi : \nabla^3 \to [0,\infty)$ *is said to be an S-metric on* ∇, *if for each* $\aleph, \upsilon, \pi, \mathfrak{a} \in \nabla$,

(i) $\varphi(\aleph,\upsilon,\pi) \geq 0$

(ii) $\varphi(\aleph,\upsilon,\pi) = 0$ *iff* $\aleph = \upsilon = \pi$,

(iii) $\varphi(\aleph,\upsilon,\pi) \leq \varphi(\aleph,\aleph,\mathfrak{a}) + \varphi(\upsilon,\upsilon,\mathfrak{a}) + \varphi(\pi,\pi,\mathfrak{a})$

The pair (∇,φ) *is called an SMS.*

Example 2.1. [3] *Let* $\nabla = \mathbb{R}^n$ *and* $||.||$ *is a norm on* ∇, *then* $\varphi(\aleph,\upsilon,\pi) = \|\upsilon + \pi - 2\aleph\| + \|\upsilon - \pi\|$ *is an SMS on* ∇.

Definition 2.2. [9] *Define a binary relation* $\perp$ $(br_\perp)$ *on a non-void set* ∇. *If* $br_\perp$ *satisfies the following criteria:*

$$\exists \aleph_0 \quad (\forall \upsilon \in \nabla, \upsilon \perp \aleph_0) \quad or \quad (\forall \upsilon \in \nabla, \aleph_0 \perp \upsilon),$$

then pair, $(\nabla,\perp)$ *is known as an orthogonal set(OS) and element* $\aleph_0$ *is called an orthogonal element (OE).*

Example 2.2. [9] *Let* $\nabla = 2\mathbb{Z}$ *and a* $br_\perp$ *on* $2\mathbb{Z}$ *as* $\kappa \perp \phi$ *if* $\kappa.\phi = 0$. *Then* $(2\mathbb{Z}, \perp)$ *is an OS with* 0 *as an OE.*

Definition 2.3. [9] *Let* $(\nabla, \perp)$ *be an OS. A sequence* $\{\aleph_\phi\}_{\phi\in\mathbb{N}}$ *is called an orthogonal sequence* ($\perp$*-sequence) if*

$$(\forall \phi \in \mathbb{N}; \aleph_\phi \perp \aleph_{\phi+1}) \quad or \quad (\forall \phi \in \mathbb{N}; \aleph_{\phi+1} \perp \aleph_\phi).$$

Definition 2.4. [10] *Consider a* $br_\perp$ *on a non-void set* ∇ *with metric* $\mathfrak{d}$ *then the triplet* $(\nabla, \perp, \mathfrak{d})$ *is called an OMS. The set* ∇ *is said to be orthogonal complete if every Cauchy* $\perp$*-sequence converges in* ∇.

Definition 2.5. [10] *Let* $(\nabla, \perp, \mathfrak{d})$ *be an OMS and* $\mathfrak{h}: \nabla \to \nabla$. *If for each* $\perp$*-sequence* $\{\aleph_\phi\}_{\phi\in\mathbb{N}} \to \aleph$ *implies* $\mathfrak{h}(\aleph_\phi) \to \mathfrak{h}(\aleph)$ as $\phi \to \infty$, *then* $\mathfrak{h}$ *is called orthogonal continuous(*$\perp$*-continuous) at* $\aleph$.

Definition 2.6. [10] *Consider a* $br_\perp$ *on a non-void set* ∇ *and* $(\nabla, \perp)$ *be an OS. A mapping* $\mathfrak{h}: \nabla \to \nabla$ *is called orthogonal preserving(*$\perp$*-preserving) if* $\mathfrak{h}(\aleph) \perp \mathfrak{h}(\upsilon)$ *whenever* $\aleph \perp \upsilon$.

Definition 2.7. [31] *Let* ∇ *be a non-void set. Suppose that* $\mu,\sigma : \nabla^3 \to [1,\infty)$ *and* $\varphi_{\mu,\sigma} : \nabla^3 \to [0,\infty)$ *are given mappings. For* $\forall \aleph,\upsilon,\pi,\mathfrak{a} \in \nabla$, *let the following conditions hold:*

(i) $\varphi_{\mu\sigma}(\aleph,\upsilon,\pi) \geq 0,$

(ii) $\varphi_{\mu\sigma}(\aleph,\upsilon,\pi) = 0 \quad iff \quad \aleph = \upsilon = \pi,$

(iii) $\varphi_{\mu\sigma}(\aleph,\upsilon,\pi) \leq \mu(\aleph,\upsilon,\pi)\varphi_{\mu\sigma}(\aleph,\aleph,\mathfrak{a}) + \sigma(\aleph,\upsilon,\pi)\varphi_{\mu\sigma}(\upsilon,\upsilon,\mathfrak{a}) + \varphi_{\mu\sigma}(\pi,\pi,\mathfrak{a}).$

Then the pair $(\nabla,\varphi_{\mu\sigma})$ *is called ESMS of type* (μ,σ).

Example 2.3. [31] *Let* $\nabla = \{0,1,2\}$ *and define*

$$\varphi_{\mu\sigma}(\aleph,\upsilon,\pi) = \begin{cases} 0, & \aleph = \upsilon = \pi \\ 1, & \aleph \neq \upsilon \neq \pi \\ \frac{3}{2}, & \aleph \neq \upsilon, \upsilon = \pi \end{cases}$$

Define $\mu,\sigma : \nabla^3 \to [1,\infty)$ as

$$\mu(\aleph,\upsilon,\pi) = 1 + \aleph + \upsilon + \pi$$

and

$$\sigma(\aleph,\upsilon,\pi) = 1 + \aleph\upsilon\pi$$

Then $(\nabla,\varphi_{\mu\sigma})$ *is an ESMS of type* (μ,σ).

3. Main Results

Definition 3.1. *Let a* $br_\perp$ *defined on a non-void set* ∇ *and a function* $\mu,\sigma : \nabla^3 \to [1,\infty)$. *If a mapping* $\varphi_{\mu,\sigma} : \nabla^3 \to [0,\infty)$ *satisfies the following condition,* $\forall \aleph,\upsilon,\pi,\mathfrak{a} \in \nabla$ *with* $\aleph \perp \upsilon \perp \pi \perp \mathfrak{a}$, *then* $(\nabla,\varphi_{\mu\sigma},\perp)$ *is said to be EOSMS of type* (μ,σ).

(i) $\varphi_{\mu\sigma}(\aleph,\upsilon,\pi) \geq 0,$

(ii) $\varphi_{\mu\sigma}(\aleph,\upsilon,\pi) = 0 \quad iff \quad \aleph = \upsilon = \pi,$

(iii) $\varphi_{\mu\sigma}(\aleph,\upsilon,\pi) \leq \mu(\aleph,\upsilon,\pi)\varphi_{\mu\sigma}(\aleph,\aleph,\mathfrak{a}) + \sigma(\aleph,\upsilon,\pi)\varphi_{\mu\sigma}(\upsilon,\upsilon,\mathfrak{a}) + \varphi_{\mu\sigma}(\pi,\pi,\mathfrak{a}).$

Example 3.1. *Let* $\nabla = \mathbb{R}$, *and define the mappings* $\varphi_{\mu\sigma}(\aleph,\upsilon,\pi) : \nabla^3 \to [0,\infty)$ *and* $\mu,\sigma : \nabla^3 \to [1,\infty)$ *with* $\aleph \perp \upsilon \perp \pi$ *if* $\aleph,\upsilon,\pi \geq 0$ *as follows:*

$$\varphi_{\mu\sigma}(\aleph,\upsilon,\pi) = max\{\aleph,\upsilon,\pi\},$$

$$\mu(\aleph,\upsilon,\pi) = |\aleph - \upsilon| + 1, \quad and$$

$$\sigma(\aleph,\upsilon,\pi) = |\aleph - \pi| + |\upsilon - \pi| + 2.$$

It is obvious that conditions (i) and (ii) are satisfied.

(iii) $\mu(\aleph,\upsilon,\pi)\varphi_{\mu\sigma}(\aleph,\aleph,\mathfrak{a})+\sigma(\aleph,\upsilon,\pi)\varphi_{\mu\sigma}(\upsilon,\upsilon,\mathfrak{a})+\varphi_{\mu\sigma}(\pi,\pi,\mathfrak{a}) = (|\aleph-\upsilon+1|)\max\{\aleph,\aleph,\mathfrak{a}\}+$
$(|\aleph-\pi|+|\upsilon-\pi|+2)\ \max\{\upsilon,\upsilon,\mathfrak{a}\}+\max\{\pi,\pi,\mathfrak{a}\}\ \geq \max\{\aleph,\upsilon,\pi\}\geq \varphi_{\mu\sigma}(\aleph,\upsilon,\pi).$

Hence $(\nabla,\varphi_{\mu\sigma},\perp)$ *is an EOSMS of type* (μ,σ).

Lemma 3.2. *In an EOSMS of type* (μ,σ), *we have*

$$\varphi_{\mu\sigma}(\aleph,\aleph,\upsilon)=\varphi_{\mu\sigma}(\upsilon,\upsilon,\aleph).$$

Proof. For $\aleph,\upsilon\in\nabla$ with $\aleph\perp\upsilon$, by the definition of EOSMS of type (μ,σ),

$$\varphi_{\mu\sigma}(\aleph,\aleph,\upsilon)\leq \mu(\aleph,\aleph,\upsilon)\varphi_{\mu\sigma}(\aleph,\aleph,\aleph)+\sigma(\aleph,\aleph,\upsilon)\varphi_{\mu\sigma}(\aleph,\aleph,\aleph)+\varphi_{\mu\sigma}(\upsilon,\upsilon,\aleph) \tag{1}$$

and

$$\varphi_{\mu\sigma}(\upsilon,\upsilon,\aleph)\leq \mu(\upsilon,\upsilon,\aleph)\varphi_{\mu\sigma}(\upsilon,\upsilon,\upsilon)+\sigma(\upsilon,\upsilon,\aleph)\varphi_{\mu\sigma}(\upsilon,\upsilon,\upsilon)+\varphi_{\mu\sigma}(\aleph,\aleph,\upsilon) \tag{2}$$

Hence, we have

$$\varphi_{\mu\sigma}(\aleph,\aleph,\upsilon)=\varphi_{\mu\sigma}(\upsilon,\upsilon,\aleph).$$

Definition 3.2. *Let* $(\nabla, \varphi_{\mu\sigma}, \perp)$ *be an EOSMS of type* (μ,σ) *and* $\{\aleph_\phi\}$ *be an ⊥-sequence in* ∇. *Then*

(a) *An ⊥-sequence* $\{\aleph_\phi\}$ *in* ∇ *is said to be convergent to* $\aleph\in\nabla$ *if for each* $\varepsilon>0,\ \exists\,\mathcal{N}=\mathcal{N}(\varepsilon)\in\mathbb{N}$ *such that* $\varphi_{\mu\sigma}(\aleph_\phi,\aleph_\phi,\aleph)<\varepsilon\ \ \forall\phi\geq\mathcal{N}$.

(b) *An ⊥-sequence* $\{\aleph_\phi\}$ *in* ∇ *is said to be Cauchy if for each* $\varepsilon>0,\ \exists\,\mathcal{N}=\mathcal{N}(\varepsilon)\in\mathbb{N}$ *such that* $\varphi_{\mu\sigma}(\aleph_\phi,\aleph_\phi,\aleph_\psi)<\varepsilon\ \ \forall\phi,\psi\geq\mathcal{N}$.

(c) *An EOSMS of type* (μ,σ) *is said to be complete if every Cauchy ⊥-sequence is convergent.*

Lemma 3.3. *Let* $(\nabla, \varphi_{\mu\sigma}, \perp)$ *be an EOSMS of type* (μ,σ). *If ⊥-sequence* $\{\aleph_\phi\}\to\aleph$ *in* ∇, *then* ℵ *is unique.*

Proof. Assume ⊥-sequence $\{\aleph_\phi\}\to\aleph$ in ∇.

To show that ℵ is unique, suppose that $\exists\,\upsilon\in\nabla$ and $\upsilon\perp\aleph$ with $\upsilon\neq\aleph$ such that $\{\aleph_\phi\}\to\upsilon$. Since ∇ is an EOSMS of type (μ,σ), then

$$\varphi_{\mu\sigma}(\aleph,\aleph,\upsilon)\leq \mu(\aleph,\aleph,\upsilon)\varphi_{\mu\sigma}(\aleph,\aleph,\aleph_\phi)+\sigma(\aleph,\aleph,\upsilon)\varphi_{\mu\sigma}(\aleph,\aleph,\aleph_\phi)+\varphi_{\mu\sigma}(\aleph_\phi,\aleph_\phi,\upsilon)\to 0.$$

which implies $\varphi_{\mu\sigma}(\aleph,\aleph,\upsilon)=0$. Therefore, $\aleph=\upsilon$, which is a contradiction. Hence, ℵ is unique.

Definition 3.3. *Let* $(\nabla, \varphi_{\mu\sigma}, \perp)$ *be an EOSMS of type* (μ,σ). *A self-mapping* $\mathfrak{h}:\nabla\to\nabla$ *is called contraction if* $\exists\,\mathcal{H}\in(0,1)$ *such that*

$$\varphi_{\mu\sigma}(\mathfrak{h}\aleph,\mathfrak{h}\upsilon,\mathfrak{h}\pi)\leq\mathcal{H}\varphi_{\mu\sigma}(\aleph,\upsilon,\pi),\quad \forall\aleph,\upsilon,\pi\in\nabla\quad with\quad \aleph\perp\upsilon\perp\pi.$$

Definition 3.4. *Let* $(\nabla, \varphi_{\mu\sigma}, \perp)$ *be an EOSMS of type* (μ,σ) *and a mapping* $\mathfrak{h}:(\nabla, \varphi_{\mu\sigma}, \perp)\to(\nabla, \varphi_{\mu\sigma}, \perp)$ *then*

(i) $\mathfrak{h}$ *is said to be ⊥-preserving if* $\mathfrak{h}\aleph\perp\mathfrak{h}\upsilon$ *whenever* $\aleph\perp\upsilon$

(ii) $\mathfrak{h}$ *is said to be ⊥-continuous if ⊥-sequence* $\{\aleph_\phi\}$ *in* ∇ *such that* $\aleph_\phi\to\aleph\Rightarrow\mathfrak{h}\aleph_\phi\to\mathfrak{h}\aleph$ *as* $\phi\to\infty$.

Theorem 3.4. *Let* $(\nabla, \varphi_{\mu\sigma}, \perp)$ *be a complete, EOSMS of type* (μ,σ) *such that* $\exists\,\aleph_0\in\nabla$ *and* $\aleph_0\perp\mathfrak{h}\aleph\,\forall\aleph\in\nabla$. *Let* $\mathfrak{h}:\nabla\to\nabla$ *be ⊥-preserving,* $(\varphi_{\mu\sigma},\perp)$*-continuous mappings and the following criteria is satisfied:*

$$\varphi_{\mu\sigma}(\mathfrak{h}\aleph,\mathfrak{h}\upsilon,\mathfrak{h}\pi)\leq\mathcal{H}\varphi_{\mu\sigma}(\aleph,\upsilon,\pi),\quad \forall\aleph,\upsilon,\pi\in\nabla\quad with\quad \aleph\perp\upsilon\perp\pi. \tag{3}$$

where $\mathcal{H}\in(0,1)$. *Suppose that* $\aleph_\phi=\mathfrak{h}^\phi\aleph_0$, *and for* $\psi>\mathfrak{i}$, *we have*

$$\lim_{\psi,\mathrm{i}\to\infty}\frac{\mu(\aleph_{\mathrm{i}+1},\aleph_{\mathrm{i}+1},\aleph_{\psi})+\sigma(\aleph_{\mathrm{i}+1},\aleph_{\mathrm{i}+1},\aleph_{\psi})}{\mu(\aleph_{\mathrm{i}},\aleph_{\mathrm{i}},\aleph_{\psi})+\sigma(\aleph_{\mathrm{i}},\aleph_{\mathrm{i}},\aleph_{\psi})}<\frac{1}{\mathcal{H}}. \tag{4}$$

Additionally, assume that for every $\aleph\in\nabla$,

$$\lim_{\phi\to\infty}[\mu(\aleph_{\phi},\aleph_{\phi},\aleph)+\sigma(\aleph_{\phi},\aleph_{\phi},\aleph)]<\infty. \tag{5}$$

Then, $\mathfrak{h}$ *has a unique fixed point (UFP)* $\omega\in\nabla$.

Proof. Now, define the $\perp$-sequence $\{\aleph_{\phi}\}$ as follows:

$$\aleph_1=\mathfrak{h}\aleph_0,\aleph_2=\mathfrak{h}\aleph_1,\ldots,\aleph_{\phi}=\mathfrak{h}^{\phi}\aleph_0\ \ \forall\phi\in\mathbb{N}.$$

By orthogonal definition, $\exists\ \aleph_0\in\nabla$, such that

$$(\forall\aleph\in\nabla,\aleph_0\perp\aleph)\quad or\quad(\forall\aleph\in\nabla,\aleph\perp\aleph_0).$$

Since $\mathfrak{h}$ is $\perp$-preserving,

$$\aleph_0\perp\mathfrak{h}\aleph_0\quad or\quad \mathfrak{h}\aleph_0\perp\aleph_0.$$

Consider the $\perp$-sequence $\aleph_{\phi}=\mathfrak{h}^{\phi}\aleph_0$. By equation (3), we have

$$\varphi_{\mu\sigma}(\aleph_{\phi},\aleph_{\phi},\aleph_{\phi+1})=\varphi_{\mu\sigma}(\mathfrak{h}^{\phi}\aleph_0,\mathfrak{h}^{\phi}\aleph_0,\mathfrak{h}^{\phi}\aleph_1)\ \le\mathcal{H}^{\phi}\varphi_{\mu\sigma}(\aleph_0,\aleph_0,\aleph_1),\quad\forall\phi\geqslant 0$$

For all natural numbers $\phi<\psi$, we have

$$\begin{aligned}\varphi_{\mu\sigma}(\aleph_{\phi},\aleph_{\phi},\aleph_{\psi})&\le\mu(\aleph_{\phi},\aleph_{\phi},\aleph_{\psi})\varphi_{\mu\sigma}(\aleph_{\phi},\aleph_{\phi},\aleph_{\phi+1})+\sigma(\aleph_{\phi},\aleph_{\phi},\aleph_{\psi})\varphi_{\mu\sigma}(\aleph_{\phi},\aleph_{\phi},\aleph_{\phi+1})\ +\varphi_{\mu\sigma}(\aleph_{\phi+1},\aleph_{\phi+1},\aleph_{\psi})\\&\le\mu(\aleph_{\phi},\aleph_{\phi},\aleph_{\psi})\varphi_{\mu\sigma}(\aleph_{\phi},\aleph_{\phi},\aleph_{\phi+1})+\sigma(\aleph_{\phi},\aleph_{\phi},\aleph_{\psi})\varphi_{\mu\sigma}(\aleph_{\phi},\aleph_{\phi},\aleph_{\phi+1})\\&+\mu(\aleph_{\phi+1},\aleph_{\phi+1},\aleph_{\psi})\varphi_{\mu\sigma}(\aleph_{\phi+1},\aleph_{\phi+1},\aleph_{\phi+2})\ +\sigma(\aleph_{\phi+1},\aleph_{\phi+1},\aleph_{\psi})\varphi_{\mu\sigma}(\aleph_{\phi+1},\aleph_{\phi+1},\aleph_{\phi+2})\\&+\varphi_{\mu\sigma}(\aleph_{\phi+2},\aleph_{\phi+2},\aleph_{\psi})\le\left[\mu(\aleph_{\phi},\aleph_{\phi},\aleph_{\psi})+\sigma(\aleph_{\phi},\aleph_{\phi},\aleph_{\psi})\right]\varphi_{\mu\sigma}(\aleph_{\phi},\aleph_{\phi},\aleph_{\phi+1})\\&+\left[\mu(\aleph_{\phi+1},\aleph_{\phi+1},\aleph_{\psi})+\sigma(\aleph_{\phi+1},\aleph_{\phi+1},\aleph_{\psi})\right]\varphi_{\mu\sigma}(\aleph_{\phi+1},\aleph_{\phi+1},\aleph_{\phi+2})\ +\ldots\\&+\left[\mu(\aleph_{\psi-2},\aleph_{\psi-2},\aleph_{\psi})+\sigma(\aleph_{\psi-2},\aleph_{\psi-2},\aleph_{\psi})\right]\varphi_{\mu\sigma}(\aleph_{\psi-2},\aleph_{\psi-2},\aleph_{\psi-1})\ +\varphi_{\mu\sigma}(\aleph_{\psi-1},\aleph_{\psi-1},\aleph_{\psi}).\end{aligned}$$

Consequently, since $\left[\mu(\aleph_{\psi-1},\aleph_{\psi-1},\aleph_{\psi})+\sigma(\aleph_{\psi-1},\aleph_{\psi-1},\aleph_{\psi})\right]\ge 1$,

we have

$$\begin{aligned}\varphi_{\mu\sigma}(\aleph_{\phi},\aleph_{\phi},\aleph_{\psi})&\le\left[\mu(\aleph_{\phi},\aleph_{\phi},\aleph_{\psi})+\sigma(\aleph_{\phi},\aleph_{\phi},\aleph_{\psi})\right]\varphi_{\mu\sigma}(\aleph_{\phi},\aleph_{\phi},\aleph_{\phi+1})\\&+\left[\mu(\aleph_{\phi+1},\aleph_{\phi+1},\aleph_{\psi})+\sigma(\aleph_{\phi+1},\aleph_{\phi+1},\aleph_{\psi})\right]\varphi_{\mu\sigma}(\aleph_{\phi+1},\aleph_{\phi+1},\aleph_{\phi+2})+\ldots\\&+\left[\mu(\aleph_{\psi-2},\aleph_{\psi-2},\aleph_{\psi})+\sigma(\aleph_{\psi-2},\aleph_{\psi-2},\aleph_{\psi})\right]\varphi_{\mu\sigma}(\aleph_{\psi-2},\aleph_{\psi-2},\aleph_{\psi-1})\\&+\left[\mu(\aleph_{\psi-1},\aleph_{\psi-1},\aleph_{\psi})+\sigma(\aleph_{\psi-1},\aleph_{\psi-1},\aleph_{\psi})\right]\varphi_{\mu\sigma}(\aleph_{\psi-1},\aleph_{\psi-1},\aleph_{\psi})\\&=\sum_{\mathrm{i}=\phi}^{\psi-1}\left[\mu(\aleph_{\mathrm{i}},\aleph_{\mathrm{i}},\aleph_{\psi})+\sigma(\aleph_{\mathrm{i}},\aleph_{\mathrm{i}},\aleph_{\psi})\right]\varphi_{\mu\sigma}(\aleph_{\mathrm{i}},\aleph_{\mathrm{i}},\aleph_{\mathrm{i}+1})\\&\le\sum_{\mathrm{i}=\phi}^{\psi-1}\left[\mu(\aleph_{\mathrm{i}},\aleph_{\mathrm{i}},\aleph_{\psi})+\sigma(\aleph_{\mathrm{i}},\aleph_{\mathrm{i}},\aleph_{\psi})\right]\mathcal{H}^{\mathrm{i}}\varphi_{\mu\sigma}(\aleph_0,\aleph_0,\aleph_1).\end{aligned}$$

Hence,

$$\mathcal{L}_{\mathfrak{p}}=\sum_{\mathrm{i}=1}^{\mathfrak{p}-1}[\mu(\aleph_{\mathrm{i}},\aleph_{\mathrm{i}},\aleph_{\psi})+\sigma(\aleph_{\mathrm{i}},\aleph_{\mathrm{i}},\aleph_{\psi})]\mathcal{H}^{\mathrm{i}}.$$

$$\Rightarrow\varphi_{\mu\sigma}(\aleph_{\phi},\aleph_{\phi},\aleph_{\psi})\le[\mathcal{L}_{\psi-1}-\mathcal{L}_{\phi}]\varphi_{\mu\sigma}(\aleph_0,\aleph_0,\aleph_1). \tag{6}$$

using ratio test and the condition (4),

we obtain that $\mathrm{Lim}_{p\to\infty}\mathcal{L}_p < \infty$, and ⊥-sequence ($\mathcal{L}_p$) is Cauchy.

Taking limit as ψ, $\phi\to\infty$ in inequality (6),

$$\Rightarrow \lim_{\psi,\phi\to\infty} \varphi_{\mu\sigma}(\aleph_\phi, \aleph_\phi, \aleph_\psi) = 0.$$

Hence, $\{\aleph_\phi\}$ is a Cauchy ⊥-sequence. Since EOSMS (∇, $\varphi_{\mu\sigma}$, ⊥) of type (μ,σ) is complete, $\exists\ \omega \in \nabla$ such that

$$\lim_{\phi\to\infty} \varphi_{\mu\sigma}(\aleph_\phi, \aleph_\phi, \omega) = \lim_{\phi\to\infty} \varphi_{\mu\sigma}(\omega, \omega, \aleph_\phi) = 0.$$

To show that ω is a FP of $\mathfrak{h}$.

By the definition of $\varphi_{\mu\sigma}$,

$$\begin{aligned}\varphi_{\mu\sigma}(\mathfrak{h}\omega,\mathfrak{h}\omega,\omega) = \varphi_{\mu\sigma}(\omega,\omega,\mathfrak{h}\omega) &\leq \mu(\omega,\omega,\mathfrak{h}\omega)\varphi_{\mu\sigma}(\omega,\omega,\aleph_{\phi+1}) + \sigma(\omega,\omega,\mathfrak{h}\omega)\varphi_{\mu\sigma}(\omega,\omega,\aleph_{\phi+1}) \\ &+ \varphi_{\mu\sigma}(\mathfrak{h}\omega,\mathfrak{h}\omega,\aleph_{\phi+1}) \leq [\mu(\omega,\omega,\mathfrak{h}\omega) + \sigma(\omega,\omega,\mathfrak{h}\omega)]\varphi_{\mu\sigma}(\omega,\omega,\aleph_{\phi+1}) + \mathcal{H}\varphi_{\mu\sigma}(\omega,\omega,\aleph_\phi).\end{aligned}$$

Take limit as $\phi\to\infty$ in the above inequality,

$$\varphi_{\mu\sigma}(\mathfrak{h}\omega,\mathfrak{h}\omega,\omega) = 0,$$

that is, $\mathfrak{h}\omega = \omega$. Therefore, ω is a FP of $\mathfrak{h}$; Hence, the uniqueness of ω follows from Lemma (3.3).

Definition 3.5. *Let* $\mathfrak{h}: \nabla \to \nabla$ *on EOSMS* (∇, $\varphi_{\mu\sigma}$, ⊥) *of type* (μ,σ). *For* $\aleph_0 \in \nabla$, *the set*

$$\mathcal{O}(\aleph_0, \mathfrak{h}) = \left\{\aleph_0, \mathfrak{h}\aleph_0, \mathfrak{h}^2\aleph_0, \mathfrak{h}^3\aleph_0, \ldots\right\}$$

is said to be an orbital of $\mathfrak{h}$ *at* $\aleph_0$.

Definition 3.6. *Let* $\mathfrak{h}: \nabla \to \nabla$ *on EOSMS* (∇, $\varphi_{\mu\sigma}$, ⊥) *of type* (μ,σ). *A mapping* $\mathcal{Q}: \nabla \to \mathbb{R}$ *is said to be* $\mathfrak{h}$*-orbitally lower semi-continuous at* $l \in \nabla$ *if*

$$\{\aleph_\phi\} \subset \mathcal{O}(\aleph_0, \mathfrak{h}) \quad and \quad \aleph_\phi \to \ell \quad as \quad \phi \to \infty \Rightarrow \mathcal{Q}(\ell) \leq \liminf_{\phi\to\infty} \mathcal{Q}(\aleph_\phi).$$

Theorem 3.5. *Let* (∇, $\varphi_{\mu\sigma}$, ⊥) *be a complete, EOSMS of type* (μ,σ) *such that* $\exists\ \aleph_0 \in \nabla$ *and* $\aleph_0 \perp \mathfrak{h}\aleph\ \forall \aleph \in \nabla$. *Let* $\mathfrak{h}: \nabla \to \nabla$ *be ⊥-preserving,* ($\varphi_{\mu\sigma}$, ⊥)*-continuous mappings and the following criteria is satisfied:*

$$\varphi_{\mu\sigma}(\mathfrak{h}\aleph, \mathfrak{h}\aleph, \mathfrak{h}^2\aleph) \leq \mathcal{H}\varphi_{\mu\sigma}(\aleph, \aleph, \mathfrak{h}\aleph), \quad \forall \aleph \in \nabla, \tag{7}$$

where $0 < \mathcal{H} < 1$. *Assume that for every* $\aleph_0 \in \nabla$, *and for* $\psi > \mathrm{i}$, *we have*

$$\lim_{\psi,\mathrm{i}\to\infty} \frac{\mu(\aleph_{\mathrm{i}+1}, \aleph_{\mathrm{i}+1}, \aleph_\psi) + \sigma(\aleph_{\mathrm{i}+1}, \aleph_{\mathrm{i}+1}, \aleph_\psi)}{\mu(\aleph_\mathrm{i}, \aleph_\mathrm{i}, \aleph_\psi) + \sigma(\aleph_\mathrm{i}, \aleph_\mathrm{i}, \aleph_\psi)} < \frac{1}{\mathcal{H}}.$$

Then ⊥-sequence $\{\mathfrak{h}^n\aleph_0\}$ *is convergent to some* $\omega \in \nabla$. *Moreover,* ω *is a FP of* $\mathfrak{h}$ *iff* $\mathcal{Q}(\aleph) = \varphi_{\mu\sigma}(\aleph, \aleph, \mathfrak{h}\aleph)$ *is* $\mathfrak{h}$*-orbitally lower semi-continuous at* ω.

Proof. Now, define the ⊥-sequence $\{\aleph_0\}$ as follows:

$$\aleph_1 = \mathfrak{h}\aleph_0, \aleph_2 = \mathfrak{h}\aleph_1 = \mathfrak{h}^2\aleph_0, \ldots, \aleph_\phi = \mathfrak{h}^\phi\aleph_0, \forall \phi \in \mathbb{N}.$$

By orthogonal definition, $\exists \aleph_0 \in \nabla$, such that

$$(\forall \aleph \in \nabla, \aleph_0 \perp \aleph) \quad or \quad (\forall \aleph \in \nabla, \aleph \perp \aleph_0).$$

Since $\mathfrak{h}$ is ⊥-preserving,

$$\aleph_0 \perp \mathfrak{h}\aleph_0 \quad or \quad \mathfrak{h}\aleph_0 \perp \aleph_0.$$

By equation (7), we obtain

$$\varphi_{\mu\sigma}(\aleph_\phi, \aleph_\phi, \aleph_{\phi+1}) \leq \mathcal{H}\varphi_{\mu\sigma}(\aleph_{\phi-1}, \aleph_{\phi-1}, \aleph_\phi) \leq \mathcal{H}^\phi\varphi_{\mu\sigma}(\aleph_0, \aleph_0, \aleph_1), \quad \forall \phi \geqslant 0$$

For all natural numbers $\phi < \psi$, we have

$$\begin{aligned}\varphi_{\mu\sigma}(\aleph_\phi,\aleph_\phi,\aleph_\psi) &\le \mu(\aleph_\phi,\aleph_\phi,\aleph_\psi)\varphi_{\mu\sigma}(\aleph_\phi,\aleph_\phi,\aleph_{\phi+1}) + \sigma(\aleph_\phi,\aleph_\phi,\aleph_\psi)\varphi_{\mu\sigma}(\aleph_\phi,\aleph_\phi,\aleph_{\phi+1}) + \varphi_{\mu\sigma}(\aleph_{\phi+1},\aleph_{\phi+1},\aleph_\psi)\\
&\le \mu(\aleph_\phi,\aleph_\phi,\aleph_\psi)\varphi_{\mu\sigma}(\aleph_\phi,\aleph_\phi,\aleph_{\phi+1}) + \sigma(\aleph_\phi,\aleph_\phi,\aleph_\psi)\varphi_{\mu\sigma}(\aleph_\phi,\aleph_\phi,\aleph_{\phi+1})\\
&+ \mu(\aleph_{\phi+1},\aleph_{\phi+1},\aleph_\psi)\varphi_{\mu\sigma}(\aleph_{\phi+1},\aleph_{\phi+1},\aleph_{\phi+2}) + \sigma(\aleph_{\phi+1},\aleph_{\phi+1},\aleph_\psi)\varphi_{\mu\sigma}(\aleph_{\phi+1},\aleph_{\phi+1},\aleph_{\phi+2})\\
&+ \varphi_{\mu\sigma}(\aleph_{\phi+2},\aleph_{\phi+2},\aleph_\psi) \le [\mu(\aleph_\phi,\aleph_\phi,\aleph_\psi) + \sigma(\aleph_\phi,\aleph_\phi,\aleph_\psi)]\varphi_{\mu\sigma}(\aleph_\phi,\aleph_\phi,\aleph_{\phi+1})\\
&+ [\mu(\aleph_{\phi+1},\aleph_{\phi+1},\aleph_\psi) + \sigma(\aleph_{\phi+1},\aleph_{\phi+1},\aleph_\psi)]\varphi_{\mu\sigma}(\aleph_{\phi+1},\aleph_{\phi+1},\aleph_{\phi+2}) + \ldots + [\mu(\aleph_{\psi-2},\aleph_{\psi-2},\aleph_\psi)\\
&+ \sigma(\aleph_{\psi-2},\aleph_{\psi-2},\aleph_\psi)]\varphi_{\mu\sigma}(\aleph_{\psi-2},\aleph_{\psi-2},\aleph_{\psi-1}) + \varphi_{\mu\sigma}(\aleph_{\psi-1},\aleph_{\psi-1},\aleph_\psi).\end{aligned}$$

Consequently, since $[\mu(\aleph_{\psi-1},\aleph_{\psi-1},\aleph_\psi) + \sigma(\aleph_{\psi-1},\aleph_{\psi-1},\aleph_\psi)] \ge 1$,
we have

$$\begin{aligned}\varphi_{\mu\sigma}(\aleph_\phi,\aleph_\phi,\aleph_\psi) &\le [\mu(\aleph_\phi,\aleph_\phi,\aleph_\psi) + \sigma(\aleph_\phi,\aleph_\phi,\aleph_\psi)]\varphi_{\mu\sigma}(\aleph_\phi,\aleph_\phi,\aleph_{\phi+1})\\
&+ [\mu(\aleph_{\phi+1},\aleph_{\phi+1},\aleph_\psi) + \sigma(\aleph_{\phi+1},\aleph_{\phi+1},\aleph_\psi)]\varphi_{\mu\sigma}(\aleph_{\phi+1},\aleph_{\phi+1},\aleph_{\phi+2})\\
&+ \ldots + [\mu(\aleph_{\psi-2},\aleph_{\psi-2},\aleph_\psi) + \sigma(\aleph_{\psi-2},\aleph_{\psi-2},\aleph_\psi)]\varphi_{\mu\sigma}(\aleph_{\psi-2},\aleph_{\psi-2},\aleph_{\psi-1})\\
&+ [\mu(\aleph_{\psi-1},\aleph_{\psi-1},\aleph_\psi) + \sigma(\aleph_{\psi-1},\aleph_{\psi-1},\aleph_\psi)]\varphi_{\mu\sigma}(\aleph_{\psi-1},\aleph_{\psi-1},\aleph_\psi)\\
&= \sum_{\mathrm{i}=\phi}^{\psi-1}[\mu(\aleph_\mathrm{i},\aleph_\mathrm{i},\aleph_\psi) + \sigma(\aleph_\mathrm{i},\aleph_\mathrm{i},\aleph_\psi)]\varphi_{\mu\sigma}(\aleph_\mathrm{i},\aleph_\mathrm{i},\aleph_{\mathrm{i}+1})\\
&\le \sum_{\mathrm{i}=\phi}^{\psi-1}[\mu(\aleph_\mathrm{i},\aleph_\mathrm{i},\aleph_\psi) + \sigma(\aleph_\mathrm{i},\aleph_\mathrm{i},\aleph_\psi)]\mathcal{H}^\mathrm{i}\varphi_{\mu\sigma}(\aleph_0,\aleph_0,\aleph_1).\end{aligned}$$

Hence,

$$\mathcal{L}_\mathrm{p} = \sum_{\mathrm{i}=1}^{\mathrm{p}-1}[\mu(\aleph_\mathrm{i},\aleph_\mathrm{i},\aleph_\psi) + \sigma(\aleph_\mathrm{i},\aleph_\mathrm{i},\aleph_\psi)]\mathcal{H}^\mathrm{i}.$$

$$\Rightarrow \varphi_{\mu\sigma}(\aleph_\phi,\aleph_\phi,\aleph_\psi) \le [\mathcal{L}_\psi - \mathcal{L}_\phi]\varphi_{\mu\sigma}(\aleph_0,\aleph_0,\aleph_1). \tag{8}$$

using ratio test and condition (7),
we obtain that $\lim_{\mathrm{p}\to\infty}\mathcal{L}_\mathrm{p} < \infty$, and $\perp$-sequence $\{\mathcal{L}_\mathrm{p}\}$ is Cauchy.
Taking limit as ψ, $\phi\to\infty$ in inequality (8),

$$\Rightarrow \lim_{\psi,\phi\to\infty}\varphi_{\mu\sigma}(\aleph_\phi,\aleph_\phi,\aleph_\psi) = 0.$$

Hence, $\{\aleph_0\}$ is a Cauchy $\perp$-sequence and $\{\aleph_0\} \to \omega \in \nabla$.
Now, assume that $\mathcal{Q}$ is $\mathfrak{h}$-orbitally lower semi-continuous at ω.

$$\varphi_{\mu\sigma}(\omega,\omega,\mathfrak{h}\omega) \le \liminf_{\phi\to\infty}\varphi_{\mu\sigma}(\aleph_\phi,\aleph_\phi,\aleph_{\phi+1}) \le \liminf_{\phi\to\infty}\mathcal{H}^\phi\varphi_{\mu\sigma}(\aleph_0,\aleph_0,\aleph_1) = 0,$$

which implies that $\mathfrak{h}\omega = \omega$.
Conversely, assume that $\mathfrak{h}\omega = \omega$ and $\{\aleph_\phi\} \subset \mathcal{O}(\aleph_0,\mathfrak{h})$ with $\{\aleph\phi\} \to \omega$ as $\phi\to\infty$.
Hence, we have

$$\mathcal{Q}(\omega) = \varphi_{\mu\sigma}(\omega,\omega,\mathfrak{h}\omega) = 0 \le \liminf_{\phi\to\infty}\varphi_{\mu\sigma}(\aleph_\phi,\aleph_\phi,\aleph_{\phi+1}) = \liminf_{\phi\to\infty}\mathcal{Q}(\aleph_\phi),$$

Hence, $\mathcal{Q}$ is $\mathfrak{h}$-orbitally lower semi-continuous at ω.

Example 3.6. *Let* $\nabla = \mathbb{R}$. *Define function* $\varphi_{\mu\sigma}(\aleph,\upsilon,\pi): \nabla^3 \to [0,\infty)$ *and* $\mu,\sigma: \nabla^3 \to [1,\infty)$ *with* $\aleph \perp \upsilon \perp \pi$ *if* $\aleph,\upsilon,\pi \ge 0$ *as follows:*

$$\begin{aligned}\varphi_{\mu\sigma}(\aleph,\upsilon,\pi) &= |\aleph - \pi| + |\upsilon - \pi|,\\
\mu(\aleph,\upsilon,\pi) &= \max\{\aleph,\upsilon\} + \pi + 1, \quad \textit{and}\\
\sigma(\aleph,\upsilon,\pi) &= |\aleph + \upsilon - \pi| + 1.\end{aligned}$$

Suppose that $\mathfrak{h}$: $\nabla \to \nabla$ *is defined by* $\mathfrak{h}(\aleph) = \frac{\aleph}{2}$.

$$\begin{aligned}\varphi_{\mu\sigma}(\mathfrak{h}\aleph, \mathfrak{h}\upsilon, \mathfrak{h}\pi) &= \left|\frac{\mathfrak{h}\aleph}{2} - \frac{\mathfrak{h}\pi}{2}\right| + \left|\frac{\mathfrak{h}\upsilon}{2} - \frac{\mathfrak{h}\pi}{2}\right| \\ &= \left|\frac{\aleph}{4} - \frac{\pi}{4}\right| + \left|\frac{\upsilon}{4} - \frac{\pi}{4}\right| \\ &= \frac{1}{4}\left(|\aleph - \pi| + |\upsilon - \pi|\right) \\ &= \frac{1}{4}\varphi_{\mu\sigma}(\aleph, \upsilon, \pi).\end{aligned}$$

Take $\mathcal{H} = \frac{1}{4}$.

Furthermore, for any $\mathfrak{c} \in \nabla$, $\perp$*-sequence* $\{\aleph_\phi\} = \mathfrak{h}^\phi \mathfrak{c}$ is $\{\aleph_\phi\} = \frac{\mathfrak{c}}{2^\phi}$.
For $\psi > \mathrm{i}$, *we have*

$$\lim_{\psi,\mathrm{i}\to\infty} \frac{\mu(\aleph_{\mathrm{i}+1}, \aleph_{\mathrm{i}+1}, \aleph_\psi) + \sigma(\aleph_{\mathrm{i}+1}, \aleph_{\mathrm{i}+1}, \aleph_\psi)}{\mu(\aleph_{\mathrm{i}}, \aleph_{\mathrm{i}}, \aleph_\psi) + \sigma(\aleph_{\mathrm{i}}, \aleph_{\mathrm{i}}, \aleph_\psi)} = \lim_{\psi,\mathrm{i}\to\infty} \frac{1 + max\left\{\frac{\mathfrak{c}}{2^{\mathrm{i}+1}}, \frac{\mathfrak{c}}{2^{\mathrm{i}+1}}\right\} + \frac{\mathfrak{c}}{2^\psi} + 1 + \frac{\mathfrak{c}}{2^{\mathrm{i}+1}} + \frac{\mathfrak{c}}{2^{\mathrm{i}+1}} - \frac{\mathfrak{c}}{2^\psi}}{1 + max\left\{\frac{\mathfrak{c}}{2^{\mathrm{i}}}, \frac{\mathfrak{c}}{2^{\mathrm{i}}}\right\} + \frac{\mathfrak{c}}{2^\psi} + 1 + \frac{\mathfrak{c}}{2^{\mathrm{i}}} + \frac{\mathfrak{c}}{2^{\mathrm{i}}} - \frac{\mathfrak{c}}{2^\psi}}$$

$$\leq \lim_{\psi,\mathrm{i}\to\infty} \frac{\frac{\mathfrak{c}}{2^{\mathrm{i}+1}} + \frac{\mathfrak{c}}{2^\psi} + \frac{2\mathfrak{c}}{2^{\mathrm{i}+1}} - \frac{\mathfrak{c}}{2^\psi} + 2}{\frac{\mathfrak{c}}{2^{\mathrm{i}}} + \frac{\mathfrak{c}}{2^\psi} + \frac{2\mathfrak{c}}{2^{\mathrm{i}}} - \frac{\mathfrak{c}}{2^\psi} + 2} = \lim_{\psi,\mathrm{i}\to\infty} \frac{\frac{3\mathfrak{c}}{2^{\mathrm{i}+1}} + 2}{\frac{3\mathfrak{c}}{2^{\mathrm{i}}} + 2} = 1 < \frac{1}{\mathcal{H}}.$$

Thus, all condition in Theorem (3.4) *are verified. Hence,* $\mathfrak{h}$ *has a UFP equal to 0.*

To expand the prior theorem, we will use some non-linear functions.

Theorem 3.7. *Let* $(\nabla, \varphi_{\mu\sigma}, \perp)$ *be a complete, EOSMS of type* (μ,σ) *such that* $\exists\ \aleph_0 \in \nabla$ *and* $\aleph_0 \perp \mathfrak{h}\aleph\ \forall \aleph \in \nabla$. *Let* $\mathfrak{h}$: $\nabla \to \nabla$ *be* $\perp$*-preserving,* $(\varphi_{\mu\sigma}, \perp)$*-continuous mappings and the following criteria is satisfied:*

$$\varphi_{\mu\sigma}(\mathfrak{h}\aleph, \mathfrak{h}\upsilon, \mathfrak{h}\pi) \leq \chi\left[\varphi_{\mu\sigma}(\aleph, \upsilon, \pi)\right], \quad \forall \aleph, \upsilon, \pi \in \nabla \quad and \quad \aleph \perp \upsilon \perp \pi, \tag{9}$$

where $\chi : [0,\infty) \to [0,\infty)$ *is a non-decreasing function such that for each fixed* $\xi > 0$,

$$\lim_{\phi\to\infty} \chi^\phi(\xi) = 0. \tag{10}$$

Also, assume that $\exists\ \delta > 4$ *and* $\mathcal{L} \in \mathbb{N}$ *such that* $\forall\ \mathfrak{e} \in, \nabla$ *we have*

$$\mu(\aleph_{\phi+1}, \aleph_{\phi+1}, \mathfrak{e}) + \sigma(\aleph_{\phi+1}, \aleph_{\phi+1}, \mathfrak{e}) < \frac{\delta}{2}, \forall \phi \geq \mathcal{L}.$$

Then, $\mathfrak{h}$ *has a UFP in* ∇.

Proof. Let $\aleph \in \nabla$ and construct a $\perp$-sequence,

$$\aleph_0 = \aleph, \aleph_1 = \mathfrak{h}\aleph_0, \aleph_2 = \mathfrak{h}\aleph_1 = \mathfrak{h}^2\aleph_0, \ldots, \aleph_\phi = \mathfrak{h}\aleph_{\phi-1} = \mathfrak{h}^{\phi-1}\aleph_0.$$

By orthogonal definition, $\exists \aleph_0 \in \nabla$, such that

$$(\forall \aleph \in \nabla, \aleph_0 \perp \aleph) \quad or \quad (\forall \aleph \in \nabla, \aleph \perp \aleph_0).$$

Since $\mathfrak{h}$ is $\perp$-preserving,

$$\aleph_0 \perp \mathfrak{h}\aleph_0 \quad or \quad \mathfrak{h}\aleph_0 \perp \aleph_0.$$

For $\varepsilon > 0$, then equation (10) indicates that after sufficiently many iterations of χ, For large ϕ, the value $\chi^{\phi}(\varepsilon)$ will be smaller than $\frac{\varepsilon}{\delta^2}$ as $\phi\to\infty$. $\Rightarrow \chi^{\phi}(\varepsilon) < \frac{\varepsilon}{\delta^2}$.

Now, choose $\phi > \mathcal{L}$. Let $\mathcal{I} = \mathfrak{h}^{\phi}$ and $\lambda = \chi^{\phi}$.

By equation (9) and the increasing property of χ,

$$\begin{aligned}\varphi_{\mu\sigma}(\mathcal{I}\aleph,\mathcal{I}\aleph,\mathcal{I}\upsilon) &= \varphi_{\mu\sigma}\left(\mathfrak{h}^{\phi}\aleph,\mathfrak{h}^{\phi}\aleph,\mathfrak{h}^{\phi}\upsilon\right) \le \chi\left(\varphi_{\mu\sigma}\left(\mathfrak{h}^{\phi-1}\aleph,\mathfrak{h}^{\phi-1}\aleph,\mathfrak{h}^{\phi-1}\upsilon\right)\right)\\ &\le \chi^2\left(\varphi_{\mu\sigma}\left(\mathfrak{h}^{\phi-2}\aleph,\mathfrak{h}^{\phi-2}\aleph,\mathfrak{h}^{\phi-2}\upsilon\right)\right) \cdots \le \chi^{\phi}\left(\varphi_{\mu\sigma}(\aleph,\aleph,\upsilon)\right)\\ &= \gamma\left(\varphi_{\mu\sigma}(\aleph,\aleph,\upsilon)\right).\end{aligned}$$

so, $\varphi_{\mu\sigma}(\aleph_{\phi+1},\aleph_{\phi+1},\aleph_{\phi}) \to 0$ as $\phi\to\infty$.

Hence, $\exists\, \phi \in \mathbb{N}$ such that $\varphi_{\mu\sigma}(\aleph_{\kappa+1},\aleph_{\kappa+1},\aleph_{\kappa}) < \frac{\varepsilon}{2\delta}$.

It shows that $\aleph_{\kappa} \in \mathcal{B}(\aleph_{\kappa},\varepsilon)$, so $\mathcal{B}(\aleph_{\kappa},\varepsilon) \ne \phi$. Thus $\forall \pi \in \mathcal{B}(\aleph_{\kappa},\varepsilon)$, we have

$$\varphi_{\mu\sigma}(\mathcal{I}\pi,\mathcal{I}\pi,\mathcal{I}\aleph_{\kappa}) = \varphi_{\mu\sigma}(\mathcal{I}\aleph_{\kappa},\mathcal{I}\aleph_{\kappa},\mathcal{I}\pi) \le \gamma\left(\varphi_{\mu\sigma}(\aleph_{\kappa},\aleph_{\kappa},\pi)\right) \le \gamma(\varepsilon) = \chi^{\phi}(\varepsilon) < \frac{\varepsilon}{\delta^2}.$$

Also, we have

$$\begin{aligned}\varphi_{\mu\sigma}(\mathcal{I}\pi,\mathcal{I}\pi,\aleph_{\kappa+1}) &\le \left[\mu\left(\varphi_{\mu\sigma}(\mathcal{I}\pi,\mathcal{I}\pi,\aleph_{\kappa})\right) + \sigma\left(\varphi_{\mu\sigma}(\mathcal{I}\pi,\mathcal{I}\pi,\aleph_{\kappa})\right)\right]\varphi_{\mu\sigma}(\mathcal{I}\pi,\mathcal{I}\pi,\mathcal{I}\aleph_{\kappa})\\ &+ \varphi_{\mu\sigma}(\aleph_{\kappa+1},\aleph_{\kappa+1},\aleph_{\kappa}) < \frac{\delta}{2}.\frac{\varepsilon}{\delta^2} + \frac{\varepsilon}{2\delta} = \frac{\varepsilon}{\delta}.\end{aligned}$$

Since, $\varphi_{\mu\sigma}(\mathfrak{h}\aleph_{\kappa},\mathfrak{h}\aleph_{\kappa},\mathfrak{h}\aleph_{\kappa}) = \varphi_{\mu\sigma}(\aleph_{\kappa},\aleph_{\kappa},\aleph_{\kappa}) < \frac{\varepsilon}{\delta}$,

$$\begin{aligned}\varphi_{\mu\sigma}(\aleph_{\kappa},\aleph_{\kappa},\mathcal{I}\pi) &\le \mu(\aleph_{\kappa},\aleph_{\kappa},\mathcal{I}\pi)\varphi_{\mu\sigma}(\aleph_{\kappa},\aleph_{\kappa},\aleph_{\kappa+1}) + \sigma(\aleph_{\kappa},\aleph_{\kappa},\mathcal{I}\pi)\varphi_{\mu\sigma}(\aleph_{\kappa},\aleph_{\kappa},\aleph_{\kappa+1})\\ &+ \varphi_{\mu\sigma}(\mathcal{I}\pi,\mathcal{I}\pi,\aleph_{\kappa+1}) \le \left[\mu(\aleph_{\kappa},\aleph_{\kappa},\mathcal{I}\pi) + \sigma(\aleph_{\kappa},\aleph_{\kappa},\mathcal{I}\pi)\right]\varphi_{\mu\sigma}(\aleph_{\kappa},\aleph_{\kappa},\aleph_{\kappa+1})\\ &+ \varphi_{\mu\sigma}(\mathcal{I}\pi,\mathcal{I}\pi,\aleph_{\kappa+1}) \le \left[\mu(\aleph_{\kappa},\aleph_{\kappa},\mathcal{I}\pi) + \sigma(\aleph_{\kappa},\aleph_{\kappa},\mathcal{I}\pi)\right]\varphi_{\mu\sigma}(\aleph_{\kappa},\aleph_{\kappa},\aleph_{\kappa+1})\\ &+ \left[\mu(\mathcal{I}\pi,\mathcal{I}\pi,\aleph_{\kappa+1}) + \sigma(\mathcal{I}\pi,\mathcal{I}\pi,\aleph_{\kappa+1})\right]\varphi_{\mu\sigma}(\mathcal{I}\pi,\mathcal{I}\pi,\aleph_{\kappa+1})\\ &< \frac{\delta}{2}\left(\frac{\varepsilon}{2\delta}\right) + \frac{\delta}{2}\left(\frac{\varepsilon}{\delta}\right) < \varepsilon.\end{aligned}$$

Therefore, $\mathcal{I}$ maps $\mathcal{B}(\aleph_{\kappa},\varepsilon)$ to itself. Since $\aleph_{\kappa} \in \mathcal{B}(\aleph_{\kappa},\varepsilon)$, we have $\mathcal{I}\aleph_{\kappa} \in \mathcal{B}(\aleph_{\kappa},\varepsilon)$.

By continuing the same procedure, we get $\mathcal{I}^{\psi}_{\aleph_{\kappa}} \in \mathcal{B}(\aleph_{\kappa},\varepsilon), \forall \psi \in \mathbb{N}$.

In otherwords, we have $\aleph_{\ell} \in \mathcal{B}(\aleph_{\kappa},\varepsilon), \forall \ell \ge \kappa$.

Consequently, we obtain $\varphi_{\mu\sigma}(\aleph_{\psi},\aleph_{\psi},\aleph_{\ell}) < \varepsilon, \forall \psi,\ell > \kappa$.

Therefore, $\{\aleph_{\kappa}\}$ is a Cauchy $\perp$-sequence. Since ∇ is complete, $\exists\, \mathfrak{e} \in \nabla$ such that $\aleph_{\kappa} \to \mathfrak{e}$ as $\kappa\to\infty$.

Moreover, we have

$$\mathfrak{e} = \lim_{\kappa\to\infty} \aleph_{\kappa+1} = \lim_{\kappa\to\infty} \aleph_{\kappa} = \mathcal{I}(\mathfrak{e}),$$

Hence, $\mathcal{I}$ has a FP $\mathfrak{e}$.

To prove the uniqueness:

Let $\mathfrak{e}$ and $\mathfrak{v}$ be two FP's of $\mathcal{I}$.

$$\varphi_{\mu\sigma}(\mathfrak{e},\mathfrak{e},\mathfrak{v}) = \varphi_{\mu\sigma}(\mathcal{I}\mathfrak{e},\mathcal{I}\mathfrak{e},\mathcal{I}\mathfrak{v}) \le \chi^{\phi}(\varphi_{\mu\sigma}(\mathfrak{e},\mathfrak{e},\mathfrak{v})) = \gamma(\varphi_{\mu\sigma}(\mathfrak{e},\mathfrak{e},\mathfrak{v})) < \varphi_{\mu\sigma}(\mathfrak{e},\mathfrak{e},\mathfrak{v}).$$

Thus, $\varphi_{\mu\sigma}(\mathfrak{e},\mathfrak{e},\mathfrak{v})=0$. Therefore, $\mathfrak{e}=\mathfrak{v}$. Hence $\mathcal{I}$ has a UFP.

On the other hand,

$$\mathfrak{h}^{\phi\kappa+\mathfrak{r}}(\aleph)=\mathcal{I}^{\kappa}(\mathfrak{h}^{\mathfrak{r}}(\aleph))\to\mathfrak{e}, \quad as \quad \kappa\to\infty.$$

Therefore, $\mathfrak{h}^{\psi}\aleph\to\mathfrak{e}$ as $\psi\to\infty$ $\forall\aleph$. This implies that $\mathfrak{e}=\lim_{\psi\to\infty}\mathfrak{h}\aleph_{\psi}=\mathfrak{h}(\mathfrak{e})$; Hence, $\mathfrak{h}$ possesses a UFP equal to 0.

Example 3.8. *Let* ∇ = [0,1]. *Define mappings* $\varphi_{\mu\sigma}(\aleph,\upsilon,\pi):\nabla^3\to[0,\infty)$ *and* $\mu,\sigma:\nabla^3\to[1,\infty)$ *with* $\aleph\perp\upsilon\perp\pi$ *iff* $\aleph,\upsilon,\pi\geq 0$ *as follows:*

$$\varphi_{\mu\sigma}(\aleph,\upsilon,\pi)=\begin{cases}0, & \aleph=\upsilon=\pi\\ \left|\max\{\aleph,\upsilon\}-\pi\right|, & otherwise,\end{cases}$$
$$\mu(\aleph,\upsilon,\pi)=\max\{\aleph,\upsilon\}+\pi+1, \quad and$$
$$\sigma(\aleph,\upsilon,\pi)=\min\{\aleph,\upsilon\}+\pi+1.$$

and

$$\chi(\mathfrak{u})=\frac{\mathfrak{u}}{2}$$

Let $\mathfrak{h}$ *be a self-mapping on* ∇ *defined by* $\mathfrak{h}(\aleph)=\frac{\aleph}{2}$.

Note that

$$\varphi_{\mu\sigma}(\mathfrak{h}\aleph,\mathfrak{h}\upsilon,\mathfrak{h}\pi)=\left|\max\left\{\frac{\aleph}{2},\frac{\upsilon}{2}\right\}-\frac{\pi}{2}\right|=\frac{1}{2}\varphi_{\mu\sigma}(\aleph,\upsilon,\pi)=\chi\left(\varphi_{\mu\sigma}(\aleph,\upsilon,\pi)\right).$$

Furthermore, for any $\aleph\in\nabla$, *we have* $\aleph_{\phi}=\frac{\aleph}{2^{\phi}}$.

Also, for any $\mathfrak{c}\in\nabla$, *we obtain*

$$\mu(\aleph_{\phi},\aleph_{\phi},\mathfrak{c})+\sigma(\aleph_{\phi},\aleph_{\phi},\mathfrak{c})=\max\left\{\frac{\aleph}{2^{\phi}},\frac{\aleph}{2^{\phi}}\right\}+\mathfrak{c}+1+\min\left\{\frac{\aleph}{2^{\phi}},\frac{\aleph}{2^{\phi}}\right\}+\mathfrak{c}+1$$
$$=2\mathfrak{c}+2+\frac{\aleph}{2^{\phi-1}}\leq 4+\frac{1}{2^{\phi-1}}.$$

Let us choose $\mathcal{L}$ = 3, *As* ϕ *increases the term* $\left(\frac{1}{2^{\phi-1}}\right)$ *decreases. For sufficiently large* ϕ, *the sum will be less than* $\frac{9}{2}$.

$$\Rightarrow\mu(\aleph_{\phi},\aleph_{\phi},\mathfrak{c})+\sigma(\aleph_{\phi},\aleph_{\phi},\mathfrak{c})<\frac{9}{2},$$

And in this case, we pick δ = 9. *Thus, all the criteria of Theorem* (3.7) *are verified. Hence,* $\mathfrak{h}$ *has a UFP equal to 0.*

4. Application of fixed point theorem to Fredholm integral equation and Fractional Integrals

4.1. Fixed Point Approximation to Fredholm integral equation:

Consider, $\nabla=\mathcal{C}[\mathfrak{a},\mathfrak{c}]$ be the space of continuous real-valued functions on $[\mathfrak{a},\mathfrak{c}]$. The function $\varphi_{\mu\sigma}(\aleph,\upsilon,\pi):\nabla^3\to[0,\infty)$ is defined by

$$\varphi_{\mu\sigma}\left(\aleph(\xi),\upsilon(\xi),\pi(\xi)\right)=\left\|\aleph(\xi)-\pi(\xi)\right\|_{\infty}+\left\|\upsilon(\xi)-\pi(\xi)\right\|_{\infty}$$

where $\|\mathfrak{a}(\xi)\|_\infty = \max_{\xi\in[\mathfrak{a},\mathfrak{c}]}|\mathfrak{a}(\xi)|, \quad \forall \aleph,\upsilon,\pi \in \nabla \quad and \quad \aleph \perp \upsilon \perp \pi.$

Also, define $\mu,\sigma : \nabla^3 \to [1,\infty)$ by

$$\mu\left(\aleph(\xi),\upsilon(\xi),\pi(\xi)\right) = \max_{\xi\in[\mathfrak{a},\mathfrak{c}]}\left(\max_{\xi\in[\mathfrak{a},\mathfrak{c}]}\left\{|\aleph(\xi)|,|\upsilon(\xi)|\right\} + |\pi(\xi)| + 1\right),$$

and

$$\sigma\left(\aleph(\xi),\upsilon(\xi),\pi(\xi)\right) = \min_{\xi\in[\mathfrak{a},\mathfrak{c}]}\left(\min_{\xi\in[\mathfrak{a},\mathfrak{c}]}\left\{|\aleph(\xi)|,|\upsilon(\xi)|\right\} + |\pi(\xi)| + 2\right).$$

$$\forall \aleph,\upsilon,\pi \in \nabla.$$

Define the following binary relation $\perp$ in ∇, $\aleph \perp \upsilon$ if $\aleph(\xi)\upsilon(\xi) \geq \upsilon(\xi)$, for almost every $\xi \in [\mathfrak{a},\mathfrak{c}]$. It is obvious that $(\nabla, \varphi_{\mu\sigma}, \perp)$ be a complete, EOSMS of type (μ,σ).
The Fredholm integral equation is:

$$\aleph(\xi) = \mathfrak{j}(\xi) + \int_{\mathfrak{a}}^{\mathfrak{c}} \mathcal{M}(\xi,\mathfrak{t},\aleph(\mathfrak{t}))d\mathfrak{t}, \quad \xi,\mathfrak{t} \in [\mathfrak{a},\mathfrak{c}], \tag{11}$$

where $\mathfrak{j} : [\mathfrak{a},\mathfrak{c}] \to \mathbb{R}$ and $\mathcal{M} : [\mathfrak{a},\mathfrak{c}]\times[\mathfrak{a},\mathfrak{c}]\times\mathbb{R} \to \mathbb{R}$ are both continuous functions.
Let $\mathfrak{h} : (\nabla,\varphi_{\mu\sigma},\perp) \to (\nabla,\varphi_{\mu\sigma},\perp)$ be given by

$$\mathfrak{h}(\aleph(\xi)) = \mathfrak{j}(\xi) + \int_{\mathfrak{a}}^{\mathfrak{c}} \mathcal{M}(\xi,\mathfrak{t},\aleph(\mathfrak{t}))d\mathfrak{t}, \quad \xi,\mathfrak{t} \in [\mathfrak{a},\mathfrak{c}].$$

Now, we show that equation (11) has a unique solution under the following condition:

$$|\mathcal{M}(\xi,\mathfrak{t},\pi_1) - \mathcal{M}(\xi,\mathfrak{t},\pi_2)| \leq \frac{1}{\mathfrak{c}-\mathfrak{a}}|\pi_1 - \pi_2|, \quad for\ each \quad \xi,\mathfrak{t} \in [\mathfrak{a},\mathfrak{c}].$$

Note that if $\aleph \in \nabla$ is a FP of $\mathfrak{h}$, then $\aleph$ is a solution to the equation (11).

First we claim that for every $\aleph \in \nabla$, $\mathfrak{h}\aleph \in \nabla$.

To see this, for every $\xi \in [0,1], \aleph \in \nabla$,

we have

$$\mathfrak{h}\aleph(\xi) = \mathfrak{j}(\xi) + \int_{\mathfrak{a}}^{\mathfrak{c}} \mathcal{M}(\xi,\mathfrak{t},\aleph(\mathfrak{t}))d\mathfrak{t} \geq 1$$

We conclude that $\mathfrak{h}\aleph(\xi) > 1$ and since $\mathfrak{h}\aleph \in \nabla$

Now verify that the hypothesis in Theorem (3.5) is satisfied. to do this, we show that

(a) $\exists \aleph_0 \in \nabla$ such that $\aleph_0 \perp \mathfrak{h}\aleph \, \forall \aleph \in \nabla$.
(b) $\mathfrak{h}$ is $\perp$ –preserving
(c) $\mathfrak{h}$ is $(\varphi_{\mu\sigma},\perp)$-contraction
(d) $\mathfrak{h}$ is $\perp$-continuous

Proof. (a) Put $\aleph_0 = \mathfrak{a}$ (the constant function $\aleph_0 = \mathfrak{a}$), we have $\mathfrak{a} \perp \mathfrak{h}\aleph \, \forall \aleph \in \nabla$

(b) we recall that $\mathfrak{h}$ is $\perp$-preserving if for every $\aleph,\upsilon \in \nabla, \aleph \perp \upsilon$, we have $\mathfrak{h}\aleph \perp \mathfrak{h}\upsilon$. we have shown above that $\mathfrak{h}\aleph(\xi) > 1$ for every $\xi \in [0,1]$, which implies that $\mathfrak{h}\aleph(\xi)\mathfrak{h}\upsilon(\xi) \geq \mathfrak{h}\upsilon(\xi) \ \forall \xi \in [0,1]$.. So $\mathfrak{h}\aleph \perp \mathfrak{h}\upsilon$.

(c) Let $\aleph,\upsilon \in \nabla, \aleph \perp \upsilon$ and $\xi \in [0,1]$, we have

$$\begin{aligned}\|\mathfrak{h}\aleph - \mathfrak{h}^2\aleph\|_\infty &= \max_{\xi\in[\mathfrak{a},\mathfrak{c}]}|\mathfrak{h}\aleph(\xi) - \mathfrak{h}^2\aleph(\xi)| = \max_{\xi\in[\mathfrak{a},\mathfrak{c}]}\left|\int_{\mathfrak{a}}^{\mathfrak{c}}[\mathcal{M}(\xi,\mathfrak{t},\mathfrak{h}\aleph(\mathfrak{t})) - \mathcal{M}(\xi,\mathfrak{t},\aleph(\mathfrak{t}))]d\mathfrak{t}\right| \\ &\leq \max_{\xi\in[\mathfrak{a},\mathfrak{c}]}\frac{1}{\mathfrak{c}-\mathfrak{a}}|\aleph(\xi) - \mathfrak{h}\aleph(\xi)|\int_{\mathfrak{a}}^{\mathfrak{c}}d\mathfrak{t} \leq \max_{\xi\in[\mathfrak{a},\mathfrak{c}]}|\aleph(\xi) - \mathfrak{h}\aleph(\xi)| = \|\aleph - \mathfrak{h}\aleph\|_\infty,\end{aligned}$$

for any $\aleph \in \nabla$. Consequently, we obtain

$$\varphi_{\mu\sigma}(\mathfrak{h}\aleph,\mathfrak{h}\aleph,\mathfrak{h}^2\aleph) \le \mathcal{H}\varphi_{\mu\sigma}(\aleph,\aleph,\mathfrak{h}\aleph), \quad where \quad 0<\mathcal{H}<1,$$

(d) Let $\{\aleph_{\mathfrak{q}}\}$ be an $\perp$-sequence in ∇ such that $\{\aleph_{\mathfrak{q}}\}$ converges to some $\aleph \in \nabla$. Since $\mathfrak{h}$ is $\perp$-preserving, $\{\mathfrak{h}\aleph_{\mathfrak{q}}\}$ is an $\perp$-sequence, too.

For each $\mathfrak{q} \in \mathbb{N}$, we have

$$\left|\mathfrak{h}\aleph_{\mathfrak{q}} - \mathfrak{h}\aleph\right| \le \mathcal{H}\left|\aleph_{\mathfrak{q}} - \aleph\right|, \quad 0<\mathcal{H}<1$$

As $\mathfrak{q}\to\infty$, it follows that $\mathfrak{h}$ is $\perp$-continuous.

Therefore, all the criteria of Theorem (3.5) are fulfilled, and $\mathfrak{h}$ has a UFP. Hence, unique solution exists for the Fredholm integral equation.

4.2. FP Approximation to Atangana-Baleanu Fractional Integrals

Recently, the fractional calculus and the fractional differential problem have attracted attentions of many researchers, also they have applications in science and dynamic fields [39, 40]. In 2016, the fractional integral in the form of equation was introduced by Atangana and Baleanu. Using the FP theorem, we demonstrate the Atangana–Baleanu fractional integral equation's existence and uniqueness.

Consider, $\nabla = \mathcal{C}[\mathfrak{a},\mathfrak{c}]$ and the function $\varphi_{\mu\sigma}(\aleph,\upsilon,\pi):\nabla^3 \to [0,\infty)$ is defined by

$$\varphi_{\mu\sigma}\left(\aleph(\xi),\upsilon(\xi),\pi(\xi)\right) = \left\|\aleph(\xi)-\pi(\xi)\right\|_\infty + \left\|\upsilon(\xi)-\pi(\xi)\right\|_\infty$$

where $\left\|\mathfrak{a}(\xi)\right\|_\infty = \max_{\xi\in[\mathfrak{a},\mathfrak{c}]}\left|\mathfrak{a}(\xi)\right|, \quad \forall \aleph,\upsilon,\pi \in \nabla \quad and \quad \aleph \perp \upsilon \perp \pi.$

Also, define $\mu,\sigma:\nabla^3 \to [1,\infty)$ by

$$\mu\left(\aleph(\xi),\upsilon(\xi),\pi(\xi)\right) = \max_{\xi\in[\mathfrak{a},\mathfrak{c}]}\left(\max_{\xi\in[\mathfrak{a},\mathfrak{c}]}\left\{\left|\aleph(\xi)\right|,\left|\upsilon(\xi)\right|\right\} + \left|\pi(\xi)\right| + 1\right),$$

and

$$\sigma\left(\aleph(\xi),\upsilon(\xi),\pi(\xi)\right) = \min_{\xi\in[\mathfrak{a},\mathfrak{c}]}\left(\min_{\xi\in[\mathfrak{a},\mathfrak{c}]}\left\{\left|\aleph(\xi)\right|,\left|\upsilon(\xi)\right|\right\} + \left|\pi(\xi)\right| + 2\right), \forall \aleph,\upsilon,\pi \in \nabla.$$

and define the relation $\perp$ in ∇: $\aleph \perp \upsilon$ if $\aleph(\xi)\upsilon(\xi) \ge \upsilon(\xi)$, then $(\nabla, \varphi_{\mu\sigma}, \perp)$ is an EOSMS of type (μ,σ).

The Atangana-Baleanu fractional integral equation is

$$ {}^{\mathcal{AB}}_{\ \ell}\mathcal{I}^{\delta}_{\xi}\aleph(\xi) = \frac{1-\delta}{\eta(\delta)}\aleph(\xi) + \frac{\delta}{\eta(\delta)\Gamma(\delta)}\int_\ell^\xi \aleph(\mathfrak{t})(\xi-\mathfrak{t})^{\delta-1}d\mathfrak{t}, \tag{12}$$

where $\delta \in (0,1]$, $\aleph(\mathfrak{t}) \in \nabla$ and $\xi,\mathfrak{t} \in [0,1]$. Also, η is the normalization function such that $\eta(0) = \eta(1) = 1$.

To prove that unique solution exists for the Atangana-Baleanu fractional integral in equation (12) under the following condition:

$$\frac{1-\delta}{\eta(\delta)} + \frac{(\xi-\ell)^\delta}{\eta(\delta)\Gamma(\delta)} < \mathcal{H}, \quad where \quad \mathcal{H} \in (0,1). \tag{13}$$

Define $\mathfrak{h}_{\mathcal{AB}}:(\nabla,\varphi_{\mu\sigma},\perp) \to (\nabla,\varphi_{\mu\sigma},\perp)$ by

$$\mathfrak{h}_{\mathcal{AB}}\aleph(\xi) = \frac{1-\delta}{\eta(\delta)}\aleph(\xi) + \frac{\delta}{\eta(\delta)\Gamma(\delta)}\int_\ell^\xi \aleph(\mathfrak{t})(\xi-\mathfrak{t})^{\delta-1}d\mathfrak{t}. \tag{14}$$

First we claim that for every $\aleph \in \nabla$, $\mathfrak{h}_{\mathcal{AB}}\aleph \in \nabla$.

To see this, for every $\xi,\mathfrak{t} \in [0,1]$, $\aleph \in \nabla$,
we have

$$\mathfrak{h}_{\mathcal{AB}}\aleph(\xi) = \frac{1-\delta}{\eta(\delta)}\aleph(\xi) + \frac{\delta}{\eta(\delta)\Gamma(\delta)}\int_\ell^\xi \aleph(\mathfrak{t})(\xi-\mathfrak{t})^{\delta-1}d\mathfrak{t} \ge 1.$$

We conclude that $\mathfrak{h}_{\mathcal{AB}}\aleph(\xi) > 1$ and since $\mathfrak{h}_{\mathcal{AB}}\aleph \in \nabla$.

Now verify that the hypothesis in Theorem (3.5) is satisfied. to do this, we show that

(a) $\exists \aleph_0 \in \nabla$ such that $\aleph_0 \perp \mathfrak{h}_{\mathcal{AB}}\aleph \ \forall \aleph \in \nabla$.
(b) $\mathfrak{h}_{\mathcal{AB}}$ is $\perp$-preserving.
(c) $\mathfrak{h}_{\mathcal{AB}}$ is $(\varphi_{\mu\sigma}, \perp)$-contraction.
(d) $\mathfrak{h}_{\mathcal{AB}}$ is $\perp$-continuous.

Proof. (a) Put $\aleph_0 = \mathfrak{a}$ (the constant function), we have $\mathfrak{a} \perp \mathfrak{h}_{\mathcal{AB}}\aleph \ \forall \aleph \in \nabla$.
(b) Since $\mathfrak{h}_{\mathcal{AB}}$ is $\perp$-preserving if for every $\aleph, \upsilon \in \nabla$, $\aleph \perp \upsilon$, we have $\mathfrak{h}_{\mathcal{AB}}\aleph \perp \mathfrak{h}_{\mathcal{AB}}\upsilon$. we have shown above that $\mathfrak{h}_{\mathcal{AB}}\aleph(\xi) > 1$ for every $\xi \in [0,1]$, which implies that $\mathfrak{h}_{\mathcal{AB}}\aleph(\xi)\mathfrak{h}_{\mathcal{AB}}\upsilon(\xi) \geq \mathfrak{h}_{\mathcal{AB}}\upsilon(\xi) \ \forall \xi \in [0,1]$. So $\mathfrak{h}_{\mathcal{AB}}\aleph \perp \mathfrak{h}_{\mathcal{AB}}\upsilon$.
(c) Let $\aleph, \upsilon \in \nabla$, $\aleph \perp \upsilon$ and $\xi, \mathfrak{t} \in [0,1]$, we have

$$\left\|\mathfrak{h}_{\mathcal{AB}}\aleph - \mathfrak{h}^2_{\mathcal{AB}}\aleph\right\|_\infty = \max_{\xi\in[\mathfrak{a},\mathfrak{c}]}\left|\mathfrak{h}_{\mathcal{AB}}\aleph(\xi) - \mathfrak{h}^2_{\mathcal{AB}}\aleph(\xi)\right| = \max_{\xi\in[\mathfrak{a},\mathfrak{c}]}\left|\left(\frac{1-\delta}{\eta(\delta)}\aleph(\xi) + \frac{\delta}{\eta(\delta)\Gamma(\delta)}\int_\ell^\xi \aleph(\mathfrak{t})(\xi-\mathfrak{t})^{\delta-1}d\mathfrak{t}\right)\right.$$
$$\left. - \left(\frac{1-\delta}{\eta(\delta)}\mathfrak{h}_{\mathcal{AB}}\aleph(\xi) + \frac{\delta}{\eta(\delta)\Gamma(\delta)}\int_\ell^\xi \mathfrak{h}_{\mathcal{AB}}\aleph(\mathfrak{t})(\xi-\mathfrak{t})^{\delta-1}d\mathfrak{t}\right)\right|$$

$$\begin{aligned}
\left\|\mathfrak{h}_{\mathcal{AB}}\aleph - \mathfrak{h}^2_{\mathcal{AB}}\aleph\right\|_\infty &= \max_{\xi\in[\mathfrak{a},\mathfrak{c}]}\left|\left(\frac{1-\delta}{\eta(\delta)}\left[\aleph(\xi) - \mathfrak{h}_{\mathcal{AB}}\aleph(\xi)\right] - \frac{\delta}{\eta(\delta)\Gamma(\delta)}\int_\ell^\xi (\xi-\mathfrak{t})^{\delta-1}d\mathfrak{t}\left[\aleph(\mathfrak{t}) - \mathfrak{h}_{\mathcal{AB}}\aleph(\mathfrak{t})\right]\right)\right| \\
&\leq \max_{\xi\in[\mathfrak{a},\mathfrak{c}]}\left|\left(\frac{1-\delta}{\eta(\delta)}\left|\aleph(\xi) - \mathfrak{h}_{\mathcal{AB}}\aleph(\xi)\right| - \frac{\delta}{\eta(\delta)\Gamma(\delta)}\int_\ell^\xi (\xi-\mathfrak{t})^{\delta-1}d\mathfrak{t}\left|\aleph(\mathfrak{t}) - \mathfrak{h}_{\mathcal{AB}}\aleph(\mathfrak{t})\right|\right)\right| \\
&= \max_{\xi\in[\mathfrak{a},\mathfrak{c}]}\left|\left(\frac{1-\delta}{\eta(\delta)}\left|\aleph(\xi) - \mathfrak{h}_{\mathcal{AB}}\aleph(\xi)\right| - \frac{\delta}{\eta(\delta)\Gamma(\delta)}\left[\frac{(\xi-\mathfrak{t})^\delta}{\delta}\right]_\ell^\xi \left|\aleph(\mathfrak{t}) - \mathfrak{h}_{\mathcal{AB}}\aleph(\mathfrak{t})\right|\right)\right| \\
&= \max_{\xi\in[\mathfrak{a},\mathfrak{c}]}\left|\left(\frac{1-\delta}{\eta(\delta)}\left|\aleph(\xi) - \mathfrak{h}_{\mathcal{AB}}\aleph(\xi)\right| + \frac{\delta}{\eta(\delta)\Gamma(\delta)}\frac{(\xi-\ell)^\delta}{\delta}\left|\aleph(\mathfrak{t}) - \mathfrak{h}_{\mathcal{AB}}\aleph(\mathfrak{t})\right|\right)\right| \\
&= \max_{\xi\in[\mathfrak{a},\mathfrak{c}]}\left|\left(\frac{1-\delta}{\eta(\delta)}\left|\aleph(\xi) - \mathfrak{h}_{\mathcal{AB}}\aleph(\xi)\right| + \frac{(\xi-\ell)^\delta}{\eta(\delta)\Gamma(\delta)}\left|\aleph(\mathfrak{t}) - \mathfrak{h}_{\mathcal{AB}}\aleph(\mathfrak{t})\right|\right)\right| \\
&\leq \max_{\xi\in[\mathfrak{a},\mathfrak{c}]}\left|\left(\frac{1-\delta}{\eta(\delta)} + \frac{(\xi-\ell)^\delta}{\eta(\delta)``(\delta)}\right)\left|\aleph(\xi) - \mathfrak{h}_{\mathcal{AB}}\aleph(\xi)\right|\right| \\
&\leq \mathcal{H}\max_{\xi\in[\mathfrak{a},\mathfrak{c}]}\left|\aleph(\xi) - \mathfrak{h}_{\mathcal{AB}}\aleph(\xi)\right| = \mathcal{H}\left\|\aleph - \mathfrak{h}_{\mathcal{AB}}\aleph\right\|_\infty.
\end{aligned}$$

for any $\aleph \in \nabla$. Consequently, we obtain

$$\varphi_{\mu\sigma}\left(\mathfrak{h}_{\mathcal{AB}}\aleph, \mathfrak{h}_{\mathcal{AB}}\aleph, \mathfrak{h}^2_{\mathcal{AB}}\aleph\right) \leq \mathcal{H}\varphi_{\mu\sigma}\left(\aleph, \aleph, \mathfrak{h}_{\mathcal{AB}}\aleph\right), \quad where \quad 0 < \mathcal{H} < 1.$$

(d) Let $\{\aleph_\wp\}$ be an $(\varphi_{\mu\sigma}, \perp)$-sequence in ∇ such that $\{\aleph_\wp\}$ converges to some $\aleph \in \nabla$. Since $\mathfrak{h}_{\mathcal{AB}}$ is $\perp$-preserving, $\left\{\mathfrak{h}_{\mathcal{AB}}\aleph_\wp\right\}$ is an $(\varphi_{\mu\sigma}, \perp)$-sequence.
For each $\wp \in \mathbb{N}$, we have

$$\left|\mathfrak{h}_{\mathcal{AB}}\aleph_\wp - \mathfrak{h}_{\mathcal{AB}}\aleph\right| \leq \mathcal{H}\left|\aleph_\wp - \aleph\right|, \quad 0 < \mathcal{H} < 1.$$

As $\wp \to \infty$, $\mathfrak{h}_{\mathcal{AB}}$ is $\perp$-continuous.

Thus, all the criteria of Theorem (3.5) are satisfied, and $\mathfrak{h}_{\mathcal{AB}}$ has a UFP. As a result, unique solution exists for the Atangana-Baleanu fractional integral equation.

4.3. Riemann–Liouville Fractional Integrals Fixed Point Approximation:

Using the fixed point theorem, we demonstrate the Riemann-Liouville equation's existence and uniqueness.
The general form of Riemann–Liouville fractional integral is

$$ {}^{\mathcal{RL}}_{\ell}\mathcal{I}^{\delta}_{\xi} = \frac{1}{\Gamma(\delta)}\int_{\ell}^{\xi}\aleph(\mathfrak{t})(\xi-\mathfrak{t})^{\delta-1}d\mathfrak{t};\quad \Gamma(\delta)>0, \tag{15}$$

where $\delta\in\mathbb{R}$, $\aleph(\xi)\in\nabla$ and $\xi,\mathfrak{t}\in[0,1]$.

Let $\nabla=\mathcal{C}[\mathfrak{a},\mathfrak{c}]$ and the function $\varphi_{\mu\sigma}(\aleph,\upsilon,\pi):\nabla^3\to[0,\infty)$ is defined by

$$\varphi_{\mu\sigma}\left(\aleph(\xi),\upsilon(\xi),\pi(\xi)\right)=\|\aleph(\xi)-\pi(\xi)\|_{\infty}+\|\upsilon(\xi)-\pi(\xi)\|_{\infty}$$

where $\|\mathfrak{a}(\xi)\|_{\infty}=\max\limits_{\xi\in[\mathfrak{a},\mathfrak{c}]}|\mathfrak{a}(\xi)|$, $\forall\aleph,\upsilon,\pi\in\nabla$ *and* $\aleph\perp\upsilon\perp\pi$.

Also, define $\mu,\sigma:\nabla^3\to[1,\infty)$ by

$$\mu\left(\aleph(\xi),\upsilon(\xi),\pi(\xi)\right)=\max_{\xi\in[\mathfrak{a},\mathfrak{c}]}\left(\max_{\xi\in[\mathfrak{a},\mathfrak{c}]}\left\{|\aleph(\xi)|,|\upsilon(\xi)|\right\}+|\pi(\xi)|+1\right),$$

and

$$\sigma\left(\aleph(\xi),\upsilon(\xi),\pi(\xi)\right)=\min_{\xi\in[\mathfrak{a},\mathfrak{c}]}\left(\min_{\xi\in[\mathfrak{a},\mathfrak{c}]}\left\{|\aleph(\xi)|,|\upsilon(\xi)|\right\}+|\pi(\xi)|+2\right).$$
$$\forall\aleph,\upsilon,\pi\in\nabla.$$

and define the relation $\perp$ in ∇: $\aleph\perp\upsilon$ if $\aleph(\xi)\upsilon(\xi)\geq\upsilon(\xi)$, then $(\nabla,\varphi_{\mu\sigma},\perp)$ is an EOSMS of type (μ,σ).

To show that equation (15) has a unique solution under the following condition:

$$\frac{1}{\Gamma(\delta+1)}\frac{(\xi-\mathfrak{t})^{\delta-1}(\xi-\ell)^{\delta}}{\left|(\xi-\mathfrak{t})^{\delta-1}\right|}<\mathcal{H},$$

where $\mathcal{H}\in(0,1)$ and $\xi\neq\mathfrak{t}$.

Also define an operator $\mathfrak{h}:(\nabla,\varphi_{\mu\sigma},\perp)\to(\nabla,\varphi_{\mu\sigma},\perp)$ by

$$\mathfrak{h}\aleph(\xi)=\frac{1}{\Gamma(\delta)}\int_{\ell}^{\xi}\aleph(\mathfrak{t})(\xi-\mathfrak{t})^{\delta-1}d\mathfrak{t}. \tag{16}$$

First we claim that for every $\aleph\in\nabla$, $\mathfrak{h}\aleph\in\nabla$.

To see this, for every $\xi\in[0,1]$, $\aleph\in\nabla$,
we have

$$\mathfrak{h}\aleph(\xi)=\frac{1}{\Gamma(\delta)}\int_{\ell}^{\xi}\aleph(\mathfrak{t})(\xi-\mathfrak{t})^{\delta-1}d\mathfrak{t}\geq 1.$$

We conclude that $\mathfrak{h}\aleph(\xi)>1$ and since $\mathfrak{h}\aleph\in\nabla$.

Now verify that the hypothesis in Theorem (3.5) is satisfied. to do this, we show that

(a) $\exists\aleph_0\in\nabla$ such that $\aleph_0\perp\mathfrak{h}\nabla\ \forall\aleph\in\nabla$.
(b) $\mathfrak{h}$ is $\perp$-preserving.
(c) $\mathfrak{h}$ is $(\varphi_{\mu\sigma},\perp)$-contraction.
(d) $\mathfrak{h}$ is $\perp$-continuous.

Proof. (a) Put $\aleph_0=\mathfrak{a}$ (the constant function), we have $\mathfrak{a}\perp\mathfrak{h}\aleph\ \forall\aleph\in\nabla$.
(b) Since $\mathfrak{h}$ is orthogonal preserving if for every $\aleph,\upsilon\in\nabla$, $\aleph\perp\upsilon$, we have $\mathfrak{h}\aleph\perp\mathfrak{h}\upsilon$. we have shown above that $\mathfrak{h}\aleph(\xi)>1$ for every $\xi\in[0,1]$, which implies that $\mathfrak{h}\aleph(\xi)\mathfrak{h}\upsilon(\xi)\geq\mathfrak{h}\upsilon(\xi)\ \forall\xi\in[0,1]$. So $\mathfrak{h}\aleph\perp\mathfrak{h}\upsilon$.

(c) Let $\aleph, \upsilon \in \nabla$, $\aleph \perp \upsilon$ and $\xi \in [0,1]$, we have

$$\begin{aligned}
\left\|\mathfrak{h}\aleph - \mathfrak{h}^2\aleph\right\|_\infty &= \max_{\xi\in[\mathfrak{a},\mathfrak{c}]}\left|\mathfrak{h}\aleph(\xi) - \mathfrak{h}^2\aleph(\xi)\right| = \max_{\xi\in[\mathfrak{a},\mathfrak{c}]}\left|\frac{1}{\Gamma(\delta)}\int_\ell^\xi \aleph(t)(\xi-t)^{\delta-1}dt - \frac{1}{\Gamma(\delta)}\int_\ell^\xi \mathfrak{h}\aleph(t)(\xi-t)^{\delta-1}dt\right| \\
&\le \max_{\xi\in[\mathfrak{a},\mathfrak{c}]}\left|\left(\frac{1}{\Gamma(\delta)}\int_\ell^\xi(\xi-t)^{\delta-1}dt\right)\right|\left|\aleph(t)-\mathfrak{h}\aleph(t)\right| = \max_{\xi\in[\mathfrak{a},\mathfrak{c}]}\frac{1}{\Gamma(\delta)}\left(\int_\ell^\xi\left|(\xi-t)^{\delta-1}\right|dt\right)\left|\aleph(t)-\mathfrak{h}\aleph(t)\right| \\
&= \max_{\xi\in[\mathfrak{a},\mathfrak{c}]}\frac{1}{\Gamma(\delta)}\frac{(\xi-t)^{\delta-1}}{\left|(\xi-t)^{\delta-1}\right|}\left(\int_\ell^\xi\left|(\xi-t)^{\delta-1}\right|dt\right)\left|\aleph(t)-\mathfrak{h}\aleph(t)\right| \\
&= \max_{\xi\in[\mathfrak{a},\mathfrak{c}]}\frac{1}{\Gamma^2(\delta)}\frac{(\xi-t)^{\delta-1}}{\left|(\xi-t)^{\delta-1}\right|}\left(\frac{(\xi-t)^\delta}{\delta}\Bigg|_\ell^\xi\right)\left|\aleph(t)-\mathfrak{h}\aleph(t)\right| \\
&= -\max_{\xi\in[\mathfrak{a},\mathfrak{c}]}\frac{1}{\Gamma(\delta)}\frac{(\xi-t)^{\delta-1}}{\left|(\xi-t)^{\delta-1}\right|}\left(\frac{(\xi-\ell)^\delta}{\delta}\right)\left|\aleph(t)-\mathfrak{h}\aleph(t)\right| \\
&= \max_{\xi\in[\mathfrak{a},\mathfrak{c}]}\frac{1}{\Gamma(\delta+1)}\frac{(\xi-t)^{\delta-1}(\xi-\ell)^\delta}{\left|(\xi-t)^{\delta-1}\right|}\left|\aleph(t)-\mathfrak{h}\aleph(t)\right| \le \mathcal{H}\max_{\xi\in[\mathfrak{a},\mathfrak{c}]}\left|\aleph(t)-\mathfrak{h}\aleph(t)\right| = \mathcal{H}\left\|\aleph-\mathfrak{h}\aleph\right\|_\infty.
\end{aligned}$$

for any $\aleph \in \nabla$. Consequently, we obtain

$$\varphi_{\mu\sigma}(\mathfrak{h}\aleph, \mathfrak{h}\aleph, \mathfrak{h}^2\aleph) \le \mathcal{H}\varphi_{\mu\sigma}(\aleph, \aleph, \mathfrak{h}\aleph), \quad where \quad 0 < \mathcal{H} < 1.$$

(d) Let $\{\aleph_\wp\}$ be an $(\varphi_{\mu\sigma}, \perp)$-sequence in ∇ such that $\{\aleph_\wp\}$ converges to some $\aleph \in \nabla$. Since $\mathfrak{h}$ is $\perp$-preserving, $\{\mathfrak{h}\aleph_\wp\}$ is an $(\varphi_{\mu\sigma}, \perp)$-sequence.

For each $\wp \in \mathbb{N}$, we have

$$\left|\mathfrak{h}\aleph_\wp - \mathfrak{h}\aleph\right| \le \mathcal{H}\left|\aleph_\wp - \aleph\right|, \quad 0 < \mathcal{H} < 1.$$

As $\wp \to \infty$, $\mathfrak{h}$ is $\perp$-continuous.

Therefore, all the criteria of Theorem (3.5) are fulfilled, and $\mathfrak{h}$ has a UFP. Hence, unique solution exists for the Riemann–Liouville fractional integral equation.

5. Conclusions

In this article, we proposed the concept of an EOSMS of type (μ,σ). Then, we proved the generalized fixed point theorems for EOSMS of type (μ,σ). Some examples are given in this new space. Finally, we presented applications to check the existence and uniqueness of the solutions to the fractional and fredholm integral equations. In future study, researchers can contribute to a deeper understanding of EOSMS of type (μ,σ) and foster advancements with implications across diverse fields.

Competing Interests

The authors declare that they have no competing interests.

Authors contributions

Every author contributed equally to each part of the paper.

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