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Remarks on the fixed point theory for quasi-metric spaces

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Motivated by a recent and interesting article by S. Park [Results in Nonlinear Analysis 6 (2023) No. 4, 116–127], we recall several different notions of quasi-metric completeness that appear in the literature and revise how they influence on the fixed point theory in quasi-metric spaces. In particular, we point out that there are several classical fixed point theorems that cannot be directly transferred to the quasi-metric setting without extra conditions, when Park's approach is considered. We also recall some emblematic examples that can help to clarify some aspects of the fixed point theory for these spaces.

Key words and phrases: Quasi-metric space, complete, fixed point theorem

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1. Introduction

According to the current terminology, by a quasi-metric space we mean a pair (X, d) where X is a (nonempty) set and d is a function from $X \times X$ to $[0, \infty)$ that fulfills the following axioms for all $x, y, z \in X$:

(qm1) $d(x, y) = d(y, x) = 0$ if and only if $x = y$;
(qm2) $d(x, y) \leq d(x, z) + d(z, y)$.

In this case, d is said to be a quasi-metric on X.

If the quasi-metric d verifies the next condition stronger than (qm1): $d(y, x) = 0$ if and only if $x = y$, we say that d is a T_1 quasi-metric on X and that (X, d) is a T_1 quasi-metric space.

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The reader is referred to books [11, 16] for a depth study of the theory of quasi-metric spaces and other related structures.

If d is a quasi-metric on a set X, the function $d^{-1}: X\times X\to[0,\infty)$ defined as $d^{-1}(x,y)=d(y,x)$ for all $x,y\in X$, is also a quasi-metric on X while the function $d^{s}: X\times X\to[0,\infty)$ defined as $d^{s}(x,y)=\max\{d(x,y),d^{-1}(x,y)\}$ for all $x,y\in X$, is a metric on X.

Let us recall that each quasi-metric d on a set X induces a T_0 topology τ_d on X that has as a base the family of open balls $\{B_d(x,\varepsilon): x\in X,\varepsilon>0\}$ where $B_d(x,\varepsilon)=\{y\in X: d(x,y)<\varepsilon\}$ for all $x\in X$ and all $\varepsilon>0$ (in particular, τ_d is T_1 if and only if d is a T_1 quasi-metric).

Then, we say that a sequence $(x_n)_{n\in\mathbb{N}}$ in a quasi-metric space (X, d) is τ_d-convergent if it converges in the topological space (X,τ_d). Therefore, $(x_n)_{n\in\mathbb{N}}$ is τ_d-convergent to $x\in X$ if and only if $d(x,x_n)\to 0$.

The lack of symmetry yields the existence of several different notions of Cauchy sequence and completeness for quasi-metric spaces. By using the classical terminology (see, e.g., [11, 12, 30]), next we collect the more representative ones.

Let (X, d) be a quasi-metric space. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be:

Left K-Cauchy if for each $\varepsilon>0$ there is an $n_\varepsilon\in\mathbb{N}$ such that $d(x_n,x_m)<\varepsilon$ whenever $n_\varepsilon\le n\le m$.

Right K-Cauchy if for each $\varepsilon>0$ there is an $n_\varepsilon\in\mathbb{N}$ such that $d(x_m,x_n)<\varepsilon$ whenever $n_\varepsilon\le n\le m$, equivalently, if $(x_n)_{n\in\mathbb{N}}$ is left K-Cauchy in (X,d^{-1}).

Cauchy if it is left K-Cauchy and right K-Cauchy, equivalently, if it is Cauchy in the metric space (X, d^s).

Then, (X, q) is said to be:

Smyth complete if every left K-Cauchy sequence is τ_{d^s}-convergent.

Bicomplete if the metric space (X, d^s) is complete.

Left K-sequentially complete if every left K-Cauchy sequence is τ_d-convergent.

Right K-sequentially complete if every right K-Cauchy sequence is τ_d-convergent.

d-sequentially complete if every Cauchy sequence in (X, d^s) is τ_d-convergent.

d^{-1}-sequentially complete if every Cauchy sequence in (X, d^s) is $\tau_{d^{-1}}$-convergent.

Remark 1.1: *The authors of [4, 18] refer to left K-Cauchy sequences as right-Cauchy sequences, to rigth K-Cauchy sequences as left-Cauchy sequences, to Smyth complete quasi-metric spaces as left-complete quasi-metric spaces and to bicomplete quasi-metric spaces as complete quasi-metric spaces.*

The following implications are obvious:

Smyth complete ⇒ bicomplete ⇒ d-sequentially complete and d^{-1}-sequentially complete,
left K-sequentially complete ⇒ d-sequentially complete, and
right K-sequentially complete ⇒ d-sequentially complete.

The reverse implications do not hold. For instance, the well-known Sorgenfrey quasi-metric line (see, e.g., [12, Example 1]) provides an example of a bicomplete and right K-sequentially complete quasi-metric space that is not left K-sequentially complete and thus not Smyth complete. Thus, the Sorgenfrey quasi-metric line constitutes an example of a complete non T-orbitally complete quasi-metric space in Park's terminology, for T the identity self mapping on the set of real numbers (this contrast with the claim given in [29, p. 118]).

On the other hand, in Examples 2.4 and 2.6 below we recall instances of left K-sequentially and right K-sequentially complete quasi-metric spaces that are not d^{-1}-sequentially complete and in Example 2.5 below we provide an instance of a d^{-1}-sequentially complete quasi-metric space that is not d-sequentially complete.

2. On the fixed point theory for quasi-metric spaces

In the recent article [29], Park showed that we can obtain full quasi-metric extensions of several relevant fixed point theorems on complete metric spaces in a direct way from the corresponding metric theorems. Park carries out his approach in the setting of bicompleteness and orbital Smyth completeness. In this direction, we point out that important fixed point theorems on metric spaces due to Browder [6], Matkowski [25], Krasnoselskii and Stetsenko [24], Khan et al. [22], and Dutta and Choudhury [14], also can be fully extended to bicomplete quasi-metric spaces in such a way that these extensions are derived from the corresponding metric theorems, as proved in [2].

However, it seems appropriate to emphasize that this procedure is not a general one. In fact, there exist classical fixed point theorems for complete metric spaces that cannot be extended verbatim to the framework of bicomplete quasi-metric spaces without assuming extra conditions. In the next three remarks we recall some of such exceptions.

Remark 2.1. *The famous Boyd and Wong fixed point theorem [5] states that if T is a φ-contraction on a complete metric space (X, d) such that the function $\varphi:[0,\infty)\to[0,\infty)$ is right upper semicontinuous, then T has a unique fixed point.*

The following (see [2, Example 2.14]) is an easy example of a φ-contraction T on a bicomplete quasi-metric space (X, d) where the function $\varphi:[0,\infty)\to[0,\infty)$ is right upper semicontinuous, but T has no fixed points.

Let X = {0, 1} and let d be the quasi-metric on X defined as d(0, 0) = d(1, 1) = d(0, 1) = 0, and d(1, 0) = 1. Since d^s is the discrete metric on X it follows that (X, d) is a bicomplete quasi-metric space. Then, the self mapping T of X defined as T0 = 1 and T1 = 0, is a φ-contraction, where $\varphi:[0,\infty)\to[0,\infty)$ is the (right) upper semicontinuous function given by $\varphi(0)=1$ and $\varphi(t)=t/2$ for all t > 0.

Remark 2.2. *In [26], Meir and Keeler proved their famous fixed point theorem that can be stated as follows: Let T be a self mapping of a complete metric space (X, d). If for each $\varepsilon>0$ there exists $\delta>0$ such that for all $x,y\in X$,*

$$\varepsilon \le d(x,y) < \varepsilon+\delta \Rightarrow d(Tx,Ty) < \varepsilon,$$

then T has a unique fixed point.

As it was observed in [33, p. 2], the self mapping T of the bicomplete quasi-metric space (X, d) of Remark 2.1 satisfies the Meir-Keeler contraction above but has no fixed points.

Remark 2.3. *A simplified form of the renowned Suzuki fixed point theorem [34] states that if T is a self mapping of a complete metric space (X, d) such that there is a constant $r\in(0,1)$ satisfying the next condition for all $x,y\in X$,*

$$d(x,Tx)\le 2d(x,y)\Rightarrow d(Tx,Ty)\le rd(x,y),$$

then, T has a unique fixed point.

In [31, Example 3] it was given the example presented below of a self mapping T of a Smyth complete quasi-metric space (X, d), without fixed points, but for which there is a constant $r\in(0,1)$ satisfying the next condition for all $x,y\in X$,

$$d(x,Tx)\le 2d(x,y)\Rightarrow d(Tx,Ty)\le rd(x,y).$$

Let $=\mathbb{N}\cup\{\infty\}$ and let d be the quasi-metric on X given by d(x, x) = 0 for all $x\in X$, $d(n,\infty)=0$ for all $n\in\mathbb{N}$, $d(\infty,n)=1/n$ for all $n\in\mathbb{N}$, and d(n, m) = 1/m for all $n,m\in\mathbb{N}$ with $n\neq m$ Finally, let T be the self mapping of X defined as $T\infty=1$, and Tn = 2n for all $n\in\mathbb{N}$.

This example shows that Theorem 6.3 and Corollary 6.5 in [29] are not true.

Furthermore, fixed point theorems stated in the realm of left K-sequentially, right K-sequentially, d-sequentially or d^{-1}-sequentially complete quasi-metric spaces, cannot be derived from the corresponding ones to complete metric spaces. Indeed, if (*X, d*) is a quasi-metric space endowed with one of these types of completeness, we get that completeness of the metric space (X, d^s) is not guaranteed. Next we recall some emblematic quasi-metric spaces where this scenario occurs.

Example 2.4. *Let d be the quasi-metric on* $\mathbb{N}$ *given by* $d(n,n) = 0$ *for all* $n \in \mathbb{N}$, *and* $d(n, m) = 1/m$ *for all* $n,m \in \mathbb{N}$ *with* $n \neq m$. *Note that* τ_d *is a compact topology on* $\mathbb{N}$, *so* $(\mathbb{N}, d)$ *is left K- and right K-sequentially complete. However, it is not* d^{-1}*-sequentially complete because* $\tau_{d^{-1}}$ *is the discrete topology on* $\mathbb{N}$ *and* $(x_n)_{n\in\mathbb{N}}$ *is a Cauchy sequence in the metric space* $(\mathbb{N}, d^s)$.

Example 2.5. *Denote by e the quasi-metric* d^{-1} *of the preceding example. Thus,* $e(n, n) = 0$ *for all* $n \in \mathbb{N}$, *and* $e(n, m) = 1/n$ *for all* $n,m \in \mathbb{N}$ *with* $n \neq m$. *Then,* $(\mathbb{N}, e)$ *is not e-sequentially complete but* $(\mathbb{N}, e^{-1})$ *is left K- and right K-sequentially complete.*

The topology τ_d of Example 2.4 is T_1 but not Hausdorff. The quasi-metric space of the following example has the same completeness properties than Example 2.4 but its topology is metrizable.

Example 2.6. *It is well known that the Alexandroff (or the one-point) compactification of* $\mathbb{N}$ *consists of the set* $\mathbb{N} \cup \{\infty\}$ *endowed with the compact and metrizable topology* τ_A, *where each natural number is an isolated point and the neighborhoods of* ∞ *are of the form* $(\mathbb{N} \cup \{\infty\}) \setminus F$, *where F is a finite subset of* $\mathbb{N}$. *We endow* $\mathbb{N} \cup \{\infty\}$ *with the following quasi-metric d such that* $\tau_d = \tau_A$ *and* $(\mathbb{N} \cup \{\infty\}, d)$ *is left K- and right K-sequentially complete but not* d^{-1}*-sequentially complete:*

$$d(\infty,\infty) = 0,\ d(\infty,n) = 1/n \text{ for all } n \in \mathbb{N},\ d(n,m) = |1/n - 1/m| \text{ for all } n,m \in \mathbb{N}, \text{ and } d(n,\infty) = 1 \text{ for all } n \in \mathbb{N},$$

We conclude this note by recalling the usefulness of the aforementioned types of completeness in the fixed point theory of quasi-metric spaces.

Thus, left K-sequential completeness provides a suitable context to obtain reasonable quasi-metric extensions of Downing-Kirk fixed point theorem [13] (see [9]). On the other hand, right K-sequential completeness provides a suitable context to obtain full quasi-metric extensions of Carisit-Kirk's fixed point theorem [7, 23] and Ekeland Variational Principle [15] as shown in [10, 20], while d-sequential completeness allows us to extend Kannan fixed point theorem [19] and generalized forms of Ćirić fixed point theorem [8] among others, [3, 7, 32]. Finally, d^{-1}-sequential completeness provides an appropriate framework to obtain a full quasi-metric extension of Caristi-Kirk's fixed point theorem when w-distances in the sense of Park [28] are involved [21] (in [1] were obtained, via w-distances, versions of Caristi-Kirk's fixed point theorem, equilibrium version of Ekeland Variational Principle and of Nadler's fixed point theorem [27], in the more restrictive context of T_1 Smyth complete quasi-metric spaces).

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