Results in Nonlinear Analysis 8 (2025) No. No. 1, 88–105
https://doi.org/10.31838/rna/2025.08.01.009
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

Fixed Point Theorems of Suzuki-type Contractions in $\mathcal{S}$-metric Spaces with Ternary Relation and Applications

M. V. R. Kameswari[1], A. Bharathi[2], Z. D. Mitrović[3], S. Aljohani[4], A. Aloqaily[4], N. Mlaiki[4]

[1]Department of Mathematics, GITAM School of Science, GITAM (Deemed to be University), Visakhapatnam, 531045, Andhra Prades, India; [2]Department of Basic Sciences and Humanities, RAGHU College of Engineering(A), Dakamarri, Bheemunipatnam Mandal, Visakhapatnam, 53116, India; [3]University of Banja Luka, Faculty of Electrical Engineering, Patre 5, Banja Luka, 78000, Bosnia and Herzegovina; [4]Department of Mathematics and Sciences, Prince Sultan University, Riyadh, 1158, Saudi Arabia.

Abstract

In this paper we define a new class of Suzuki-type contractions and prove some results on fixed points in $\mathcal{S}$-metric spaces with ternary relation. As an application of our results, we prove the existence of solutions for some classes of nonlinear matrix equations and provide a convergence analysis. Also, our results generalize recent results from the literature.

Key words and phrases: $\mathcal{S}$-metric spaces; fixed points; numerical methods; Suzuki-type contractions; ternary relations; nonlinear matrix equation.

Mathematics Subject Classification (2010): 47H10, 54H25

1. Introduction

In 1975, Dass and Gupta [7] introduced contractions of rational type and using these contractions showed the existence of fixed points in complete metric spaces. After that, in 1977, Jaggi [12] introduced another kind of contractions of rational type and obtained some results about fixed points. Sedghi et al., in paper [26], defined $\mathcal{S}$-metric space and studied its properties. Using $\mathcal{S}$-metric spaces,

Email addresses: kmukkavi@gitam.edu (Mukkavilli Vani Rama Kameswari); balla@gitam.in (Alla Bharathi); zoran.mitrovic@etf.unibl.org (Z. D. Mitrović); sjohani@psu.edu.sa (Sarah Aljohani); maloqaily@psu.edu.sa (Ahmad Aloqaily); nmlaiki@psu.edu.sa (Nabil Mlaiki)

Received August 11, 2024; Accepted December 19, 2024; Online January 28, 2025

several authors obtain results about fixed points, see [5, 6, 8, 9, 11, 13, 17, 18, 22, 30]. Khojasteh et al. in the paper [14], results on fixed points for $\mathcal{Z}$-contraction maps are obtained using simulation functions. Many authors used $\mathcal{Z}$-contractions related to simulation functions and obtained results about fixed points in some classes of generalized metric spaces, see [2–6, 10, 15, 19, 25]. Kumam et al., [15] initiated a new idea of Suzuki-type $\mathcal{Z}$-contraction which generalizes Suzuki contractions [29]. In 2019, Mlaiki et al. [17] define a $\mathcal{Z}_S$-contraction using the simulation function and prove the existence of fixed points of such a mapping in complete $\mathcal{S}$-metric spaces. Further, Babu et al., [5, 6] use $\mathcal{S}$-metric space and almost generalized $\mathcal{Z}_S$ contractions of rational type and obtain some results about fixed points. On the other hand, in the papers [24, 28] the authors introduce a new method in the theory of fixed points of metric spaces with binary relations. In this direction, Alam and Imdad [1] get some results about the coincidence points. In 2018, Sawangsup and Sintunavarat [25] define the $\mathcal{Z}_{\mathcal{R}_S}$ contraction and obtain some fixed point results. The notion of $\mathcal{Z}_{\mathcal{R}}$ Suzuki-type contraction, introduced by Hasanuzzaman and Imdad [10], is a generalization of Z-contraction, Suzuki-type $\mathcal{Z}$-contraction and $\mathcal{Z}_{\mathcal{R}}$ contraction. Recently, Wangwe [31] and Kumar and Singh [16] using ternary relations in G-metric spaces obtained results about fixed points for multivalued mappings.

In section 2, we give some preliminaries related to $\mathcal{S}$ metric spaces. In section 3, we present some basic definitions on ternary relations. In section 4, we define Suzuki type $\mathcal{Z}_{\mathcal{R}_S}$ contraction under a ternary relation $\mathcal{R}$ and obtain fixed points for such contractions in $\mathcal{S}$ metric space. Furthermore, an example is provided to validate our results which shows the authenticity of Suzuki-type $\mathcal{Z}_{\mathcal{R}_S}$ contraction over those previously mentioned contractions [5, 6]. In section 5, we get fixed point results for $\theta_s - \eta_s$ Suzuki-type $\mathcal{Z}_{\mathcal{R}_S}$ contractions Finally, in section 6, we apply our results to the solutions of some classes of nonlinear matrix equations and provide a convergence analysis of the solutions.

2. Preliminaries

Here we give some definitions and results that we will use.

Definition 2.1. [14] *The function* $\varsigma : [0,+\infty)\times[0,+\infty) \to (-\infty,+\infty)$ *is a simulation function if satisfies the following:*

(i) $\varsigma(0,0) = 0$,

(ii) $\varsigma(\mathfrak{r},\mathfrak{v}) < \mathfrak{v} - \mathfrak{r}$, *for all* $\mathfrak{v},\mathfrak{r} > 0$,

(iii) *if the sequences* $\{\mathfrak{r}_p\}$ *and* $\{\mathfrak{v}_p\}$ *in* $(0,+\infty)$ *are such that*

$$\lim_{p\to+\infty} \mathfrak{r}_p = \lim_{p\to+\infty} \mathfrak{v}_p = t \in (0,+\infty) \text{ then } \limsup_{p\to+\infty} \varsigma(\mathfrak{r}_p,\mathfrak{v}_p) < 0.$$

We denote the family of all simulation functions by $\mathcal{Z}$.

Definition 2.2. [26] *The* $\mathcal{S}$ *metric is a function* $\mathcal{S} : \bar{E}\times\bar{E}\times\bar{E} \to [0,+\infty)$, *where* $\bar{E} \neq \emptyset$, *which has fulfills the following conditions:*

(i) $\mathcal{S}(\omega,\xi,z) = 0$ *if* $\omega = \xi = z$,

(ii) $\mathcal{S}(\omega,\xi,z) \le \mathcal{S}(\omega,\omega,\ell) + \mathcal{S}(\xi,\xi,\ell) + \mathcal{S}(z,z,\ell)$,
for all $\omega,\xi,z,\ell \in \bar{E}$.

The pair $(\bar{E}, \mathcal{S})$ *then called an* $\mathcal{S}$ *metric space.*

From now on, $\bar{E} = (\bar{E},\mathcal{S})$ stands for $\mathcal{S}$ metric space.

Definition 2.3. [26]

(i) A sequence $\{\omega_n\} \subseteq \bar{E}$ *is convergent to a point* $\omega \in \bar{E}$ *if* $\mathcal{S}(\omega_n,\omega_n,\omega) \to 0$ *as* $n \to +\infty$, *i.e., for a given* $\varepsilon > 0$ *there exists* $n_0 \in \mathbb{N}$, $\mathcal{S}(\omega_n,\omega_n,\omega) < \varepsilon$, *for all* $n \ge n_0$, *it is denoted by* $\lim_{m,n\to+\infty} \omega_n = \omega$.

(ii) A sequence $\{\omega_n\} \subset \bar{E}$ *is a Cauchy sequence in* $\bar{E}$ *if*

$$\lim_{n,m\to+\infty} \mathcal{S}(\omega_n,\omega_n,\omega_m)$$

exists and it is finite.

(iii) If each Cauchy sequence in $\bar{E}$ is convergent to a point in $\bar{E}$, then $\bar{E}$ is complete.

Lemma 2.1 *[5, 27] Let $\{\omega_n\}$ be a sequence in $\bar{E}$ such that*

$$\lim_{n\to+\infty} \mathcal{S}(\omega_n,\omega_n,\omega_{n+1}) = 0$$

and $\{\omega_n\}$ is not a Cauchy sequence. Then there exist $\varepsilon > 0$ and two sequences $\{m(p)\}$ and $\{n(p)\}$ with $m(p) > n(p) > p$ such that

$$\mathcal{S}(\omega_{m(p)},\omega_{m(p)},\omega_{n(p)}) \geq \varepsilon \text{ and } \mathcal{S}(\omega_{m(p)-1},\omega_{m(p)-1},\omega_{n(p)}) < \varepsilon.$$

Also,

(i) $\lim_{n\to+\infty} \mathcal{S}(\omega_{m(p)},\omega_{m(p)},\omega_{n(p)}) = \varepsilon,$

(ii) $\lim_{n\to+\infty} \mathcal{S}(\omega_{m(p)-1},\omega_{m(p)-1},\omega_{n(p)}) = \varepsilon,$

(iii) $\lim_{n\to+\infty} \mathcal{S}(\omega_{m(p)},\omega_{m(p)},\omega_{n(p)-1}) = \varepsilon,$

(iv) $\lim_{n\to+\infty} \mathcal{S}(\omega_{m(p)-1},\omega_{m(p)-1},\omega_{n(p)-1}) = \varepsilon.$

Definition 2.4. [6] *Let $\Gamma : \bar{E} \to \bar{E}$, and for any given $\varsigma \in \mathcal{Z}$ and $L \geq 0$, we say that Γ is almost generalized $\mathcal{Z}_S$-contraction with rational expressions if:*

$$\varsigma(\mathcal{S}(\Gamma\omega,\Gamma\xi,\Gamma z), M(\omega,\xi,z) + LN(\omega,\xi,z)) \geq 0, \tag{2.1}$$

for all $\omega,\xi,z \in \bar{E}$, where,

$$\begin{aligned} M(\omega,\xi,z) = \max\Bigg\{ & \mathcal{S}(\omega,\xi,z), \frac{\mathcal{S}(\xi,\xi,\Gamma\xi)[1+\mathcal{S}(\omega,\omega,\Gamma\omega)]}{1+\mathcal{S}(\omega,\xi,z)}, \frac{\mathcal{S}(\xi,\xi,\Gamma\omega)[1+\mathcal{S}(\omega,\omega,\Gamma\omega)]}{1+\mathcal{S}(\omega,\xi,z)}, \\ & \frac{\mathcal{S}(z,z,\Gamma z)[1+\mathcal{S}(\xi,\xi,\Gamma\xi)]}{1+\mathcal{S}(\omega,\xi,z)}, \frac{\mathcal{S}(z,z,\Gamma z)[1+\mathcal{S}(\omega,\omega,\Gamma\omega)]}{1+\mathcal{S}(\omega,\xi,z)}, \\ & \frac{1}{3}\frac{[\mathcal{S}(z,z,\Gamma\xi)+\mathcal{S}(\xi,\xi,\Gamma z)[1+\mathcal{S}(z,z,\Gamma\omega)]}{1+\mathcal{S}(\omega,\xi,z)} \Bigg\} \end{aligned}$$

$$N(\omega,\xi,z) = \min\left\{ \mathcal{S}(\omega,\omega,\Gamma\omega), \mathcal{S}(\xi,\xi,\Gamma\omega), \mathcal{S}(z,z,\Gamma\omega), \frac{\mathcal{S}(\xi,\xi,\Gamma\omega)[1+\mathcal{S}(\omega,\omega,\Gamma\xi)]}{1+\mathcal{S}(\omega,\xi,z)} \right\}.$$

Theorem 2.1. [6] *Let $\Gamma : \bar{E} \to \bar{E}$, and for any given $\varsigma \in \mathcal{Z}$ and $L \geq 0$, Γ is almost generalized $\mathcal{Z}_S$-contraction with rational expressions, then Γ has a unique fixed point.*

Definition 2.5. [5] *Let $\Gamma : \bar{E} \to \bar{E}$, and for any given $\varsigma \in \mathcal{Z}$ and $L \geq 0$, we say that Γ is almost Suzuki type $\mathcal{Z}_S$-contraction with respect to ς, if*

$$\frac{1}{3}\mathcal{S}(\omega,\omega,\Gamma\omega) < \mathcal{S}(\omega,\xi,z) \text{ implies } \varsigma(\mathcal{S}(\Gamma\omega,\Gamma\xi,\Gamma z), \mathcal{S}(\omega,\xi,z) + LN(\omega,\xi,z)) \geq 0, \tag{2.2}$$

for all $\omega,\xi,z \in \bar{E}$, where,

$$N(\omega,\xi,z) = \min\{\mathcal{S}(\Gamma\omega,\Gamma\omega,\omega), \mathcal{S}(\Gamma\omega,\Gamma\omega,\xi), \mathcal{S}(\Gamma\omega,\Gamma\omega,z)\}.$$

Theorem 2.2. [5] *If $(\bar{E},\mathcal{S})$ is an $\mathcal{S}$ metric space, $\Gamma : \bar{E} \to \bar{E}$, is an almost Suzuki type $\mathcal{Z}_S$-contraction with respect to $\varsigma \in \mathcal{Z}$. Then Γ has a unique fixed point in $\bar{E}$.*

3. Ternary relations

We will use the following notations.

(1) $\bar{E}(\Gamma,\mathcal{R}) = \{\vartheta \in \bar{E} : (\vartheta,\vartheta,\Gamma\vartheta) \in \mathcal{R}\}$, where $\Gamma : \bar{E} \to \bar{E}$,
(2) $\nu(\omega,z,z,\mathcal{R})$, the class of all $\mathcal{S}$-paths in $\mathcal{R}$ from ω to z,
(3) $\mathbb{N}^{**} = \mathbb{N} \cup \{0\}$.

Definition 3.1. [31] *A ternary relation* $\mathcal{R} \subseteq \bar{E}\times\bar{E}\times\bar{E}$, *where* $\bar{E} \neq \emptyset$. *Then* $\mathcal{R}$ *is*:

(i) Reflexive, if $(\varphi,\varphi,\varphi) \in \mathcal{R}$ for all $\varphi \in \bar{E}$;
(ii) Symmetric, if $(\varphi,\xi,z) \in \mathcal{R}$ implies $(\xi,z,\varphi) \in \mathcal{R}$ for all $\varphi,\xi,z \in \bar{E}$;
(iii) Transitive, if $(\varphi,\xi,z) \in \mathcal{R}, (\xi,z,\ell) \in \mathcal{R}$ implies $(\varphi,z,\ell) \in \mathcal{R}$, for all $\varphi,\xi,z,\ell \in \bar{E}$;
(iv) Complete, if $\left[\varphi,\xi,z\right] \in \mathcal{R}$ for all $\varphi,\xi,z \in \bar{E}$.

Definition 3.2. [31] *A ternary relation* $\mathcal{R}$ *is* Γ*-closed, where* Γ *is a selfmap on* $\bar{E}$

$$\text{if } (\omega,\xi,z) \in \mathcal{R} \text{ implies } (\Gamma\omega,\Gamma\xi,\Gamma z) \in \mathcal{R}, \text{ for all } \omega,\xi,z \in \bar{E}.$$

Definition 3.3. [31] *A sequence* $\{\omega_n\}$ *in* $\bar{E}$ *is* $\mathcal{R}$*-preserving if*

$$(\omega_n,\omega_n,\omega_{n+1}) \in \mathcal{R}, \text{ for all } n \in N^{**}.$$

Lemma 3.1. [31] $\mathcal{R}^s = \mathcal{R} \cup \mathcal{R}^{-1}$ *is* Γ *closed when* $\mathcal{R}$ *is* Γ *closed. If* $\mathcal{R}$ *is ternary relation on a nonempty set* $\bar{E}$, *then* $(\omega,\xi,z) \in \mathcal{R}^S$ *if and only if* $[\omega,\xi,z] \in \mathcal{R}$.

Following on the similar lines of [10], we now define $\mathcal{S}$ self closed and $\mathcal{S}$-path in $\mathcal{S}$ metric space as follows.

Definition 3.4. *A ternary relation* $\mathcal{R}$ *is* $\mathcal{S}$ *self closed if there is an* $\mathcal{R}$*- preserving sequence such that* $\omega_n \to^s \omega$ *as* $n \to +\infty$ *then exists a subsequence* $\{\omega_{n_k}\}$ *of* $\{\omega_n\}$ *such that* $[\omega_{n_k},\omega_{n_k},\omega] \in \mathcal{R}$.

Definition 3.5. *Let* ($\bar{E}$, $\mathcal{S}$) *be an* $\mathcal{S}$ *metric space,* $\mathcal{R}$ *is ternary relation defined on* $\bar{E}$, *and let* $\omega,\xi \in \bar{E}$. *Then a finite sequence* $\{\varrho_0,\varrho_1,\ldots,\varrho_l\} \in \bar{E}$ *is called the* $\mathcal{S}$*-path of length l (l is natural number) connecting* ω *to* ξ *in* $\mathcal{R}$ *if* $\varrho_0 = \omega$, $\varrho_l = \xi$ *and* $(\varrho_i,\varrho_{i+1},\varrho_{i+1}) \in \mathcal{R}$ *for all* $i \in \{1,2,\ldots,l-1\}$.

4. Main results

First, we give the following definition.

Definition 4.1. *Let* Γ *be a self-map on an* $\mathcal{S}$*-metric space* $\bar{E}$ *with a ternary relation* $\mathcal{R}$, $\varsigma \in \mathcal{Z}$ *and* $L \geq 0$ *such that*

$$\mathcal{S}(\omega,\omega,\Gamma\omega) < 3\mathcal{S}(\omega,\xi,z) \text{ implies } \varsigma(\mathcal{S}(\Gamma\omega,\Gamma\xi,\Gamma z), M_S(\omega,\xi,z)) + LN_S(\omega,\xi,z)) \geq 0, \tag{4.1}$$

for all $\omega,\xi,z \in \bar{E}$, *with* $(\omega,\xi,z) \in \mathcal{R}$, *where*

$$\begin{aligned} M_S(\omega,\xi,z) = \max\Bigg\{ & \mathcal{S}(\omega,\xi,z), \frac{\mathcal{S}(\xi,\xi,\Gamma\xi)[1+\mathcal{S}(\omega,\omega,\Gamma\omega)]}{1+\mathcal{S}(\omega,\xi,z)}, \\ & \frac{\mathcal{S}(\xi,\xi,\Gamma\omega)[1+\mathcal{S}(\omega,\omega,\Gamma\omega)]}{1+\mathcal{S}(\omega,\xi,z)}, \frac{\mathcal{S}(z,z,\Gamma z)[1+\mathcal{S}(\xi,\xi,\Gamma\xi)]}{1+\mathcal{S}(\omega,\xi,z)}, \\ & \frac{\mathcal{S}(z,z,\Gamma z)[1+\mathcal{S}(\omega,\omega,\Gamma\omega)]}{1+\mathcal{S}(\omega,\xi,z)}, \frac{[\mathcal{S}(z,z,\Gamma\xi)+\mathcal{S}(\xi,\xi,\Gamma z)][1+\mathcal{S}(z,z,\Gamma\omega)]}{3[1+\mathcal{S}(\omega,\xi,z)]} \Bigg\} \end{aligned}$$

and

$$N_S(\omega,\xi,z) = \min\left\{ \mathcal{S}(\Gamma\omega,\Gamma\omega,z), \mathcal{S}(\xi,\xi,\Gamma\omega), \mathcal{S}(\Gamma\xi,\Gamma\xi,\Gamma z), \mathcal{S}(z,\Gamma z,\Gamma\xi), \frac{\mathcal{S}(z,z,\Gamma\omega)[1+\mathcal{S}(\omega,\omega,\Gamma\omega)]}{1+\mathcal{S}(\omega,\xi,z)} \right\}.$$

Then Γ *is called Suzuki type* $\mathcal{Z}_{\mathcal{R}_S}$ *contraction mapping.*

Theorem 4.1. *Let $(\bar{E}, \mathcal{S})$ be an $\mathcal{S}$-metric space with a ternary relation $\mathcal{R}$. Let a self map Γ on $\bar{E}$ satisfying subsequent conditions:*

(a) exists $\mathcal{M} \subseteq \bar{E}$ such that $\Gamma\bar{E} \subseteq \mathcal{M}$ and $(\mathcal{M}, \mathcal{S})$ is $\mathcal{R}$-complete;
(b) exist ω_0 such that $(\omega_0, \omega_0, \Gamma\omega_0) \in \mathcal{R}$;
(c) $\bar{E}(\Gamma, \mathcal{R})$ is nonempty;
(d) $\mathcal{R}$ is transitive and $\mathcal{R}$ is Γ closed;
(e) Γ is Suzuki type $\mathcal{Z}_{\mathcal{R}_S}$ contraction;
(f) either Γ is $\mathcal{R}$-continuous or $\mathcal{R}/\mathcal{M}$ is $\mathcal{S}$- self closed provided (4.1) holds for all $\omega, \xi, z \in \bar{E}$ with $(\omega, \xi, z) \in \mathcal{R}$

Then Γ has a fixed point. Moreover, if $\nu(\omega, \xi, z, \mathcal{R}^s)$ non empty then Γ has a unique fixed point.

Proof. Starting by our assumption, $\bar{E}(\Gamma, \mathcal{R}) \neq \emptyset$, let $\omega_0 \in \bar{E}(\Gamma, \mathcal{R})$ and construct the sequence $\{\omega_n\}$ defined as $\omega_{n+1} = \Gamma\omega_n$ for all $n \in \mathbb{N}^{**}$.
Using conditions (b) and (d), we have

$$(\Gamma\omega_0, \Gamma\omega_0, \Gamma^2\omega_0), (\Gamma^2\omega_0, \Gamma^2\omega_0, \Gamma^3\omega_0), \ldots, (\Gamma^n\omega_0, \Gamma^n\omega_0, \Gamma^{n+1}\omega_0) \in \mathcal{R},$$

thus

$$(\omega_n, \omega_n, \omega_{n+1}) \in \mathcal{R},$$

for all $n \in \mathbb{N}^{**}$, hence the sequence $\{\omega_n\}$ is $\mathcal{R}$ preserving sequence. First, we assume that $\omega_m = \omega_{m+1} = \Gamma\omega_m$ for some m, then immediately, ω_m follows as a fixed point of Γ. Next we assume that $\mathcal{S}(\omega_n, \omega_n, \omega_{n+1}) > 0$ for all $n \geq 0$.

Now, we claim that $\lim_{n\to+\infty} \mathcal{S}(\omega_n, \omega_n, \omega_{n+1}) = 0$.

We have $\frac{1}{3}\mathcal{S}(\omega_n, \omega_n, \omega_{n+1}) < \mathcal{S}(\omega_n, \omega_n, \omega_{n+1})$ for all $n \in \mathbb{N}^{**}$ hence from (4.1) and utilizing $\mathcal{R}$ preserving property of ω_n, we have

$$\varsigma(\mathcal{S}(\Gamma\omega_{n-1}, \Gamma\omega_{n-1}, \Gamma\omega_n), M_S(\omega_{n-1}, \omega_{n-1}, \omega_n) + L(N_S(\omega_{n-1}, \omega_{n-1}, \omega_n)) \geq 0, \tag{4.2}$$

where

$$\begin{aligned} M_S(\omega_{n-1}, \omega_{n-1}, \omega_n) &= \max\Bigg\{\mathcal{S}(\omega_{n-1}, \omega_{n-1}, \omega_n), \frac{\mathcal{S}(\omega_{n-1}, \omega_{n-1}, \Gamma\omega_{n-1})[(1 + \mathcal{S}(\omega_{n-1}, \omega_{n-1}, \Gamma\omega_{n-1})]}{1 + \mathcal{S}(\omega_{n-1}, \omega_{n-1}, \omega_n)}, \\ &\quad \frac{\mathcal{S}(\omega_n, \omega_n, \Gamma\omega_n)[1 + \mathcal{S}(\omega_{n-1}, \omega_{n-1}, \Gamma\omega_{n-1})]}{1 + \mathcal{S}(\omega_{n-1}, \omega_{n-1}, \omega_n)}, \\ &\quad \frac{1}{3}\frac{[\mathcal{S}(\omega_n, \omega_n, \Gamma\omega_{n-1}) + \mathcal{S}(\omega_{n-1}, \omega_{n-1}, \Gamma\omega_n)][1 + \mathcal{S}(\omega_n, \omega_n, \Gamma\omega_{n-1})]}{1 + \mathcal{S}(\omega_{n-1}, \omega_{n-1}, \omega_{n+1})}\Bigg\} \\ &= \max\left\{\mathcal{S}(\omega_{n-1}, \omega_{n-1}, \omega_n), \mathcal{S}(\omega_n, \omega_n, \omega_{n+1}), \frac{1}{3}\frac{\mathcal{S}(\omega_{n-1}, \omega_{n-1}, \omega_{n+1})}{1 + \mathcal{S}(\omega_{n-1}, \omega_{n-1}, \omega_n)}\right\} \\ &\leq \max\left\{\mathcal{S}(\omega_{n-1}, \omega_{n-1}, \omega_n), \mathcal{S}(\omega_n, \omega_n, \omega_{n+1}), \frac{1}{3}\frac{[2\mathcal{S}(\omega_{n-1}, \omega_{n-1}, \omega_n) + \mathcal{S}(\omega_n, \omega_n, \omega_{n+1})]}{1 + \mathcal{S}(\omega_{n-1}, \omega_{n-1}, \omega_n)}\right\} \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} N_S(\omega_{n-1}, \omega_{n-1}, \omega_n) = \min\Big\{&\mathcal{S}(\Gamma\omega_{n-1}, \Gamma\omega_{n-1}, \omega_n), \mathcal{S}(\omega_{n-1}, \omega_{n-1}, \Gamma\omega_{n-1}), \mathcal{S}(\Gamma\omega_{n-1}, \Gamma\omega_{n-1}, \Gamma\omega_n), \mathcal{S}(\omega_n, \Gamma\omega_n, \Gamma\omega_{n-1}), \\ &\frac{\mathcal{S}(\omega_n, \omega_n, \Gamma\omega_{n-1})(1 + \mathcal{S}(\omega_{n-1}, \omega_{n-1}, \Gamma\omega_{n-1})]}{1 + \mathcal{S}(\omega_{n-1}, \omega_{n-1}, \omega_n)}\Big\} = 0 \end{aligned} \tag{4.4}$$

If

$$\mathcal{S}(\omega_{n-1}, \omega_{n-1}, \omega_n) < \mathcal{S}(\omega_n, \omega_n, \omega_{n+1}),$$

then from (4.3), we have

$$M_S(\omega_{n-1},\omega_{n-1},\omega_n)=\max\left\{\mathcal{S}(\omega_{n-1},\omega_{n-1},\omega_n),\mathcal{S}(\omega_n,\omega_n,\omega_{n+1}),\ \frac{1}{3}\frac{[2\mathcal{S}(\omega_{n-1},\omega_{n-1},\omega_n)+\mathcal{S}(\omega_n,\omega_n,\omega_{n+1})]}{1+\mathcal{S}(\omega_{n-1},\omega_{n-1},\omega_n)}\right\}=\mathcal{S}(\omega_n,\omega_n,\omega_{n+1}). \tag{4.5}$$

Therefore, from (4.2), (4.3), (4.4) and (4.5), we have

$$\varsigma(\mathcal{S}(\omega_n,\omega_n,\omega_{n+1}),\mathcal{S}(\omega_n,\omega_n,\omega_{n+1}))+L(0)\geq 0$$

this implies

$$\varsigma(\mathcal{S}(\omega_n,\omega_n,\omega_{n+1}),\mathcal{S}(\omega_n,\omega_n,\omega_{n+1}))\geq 0,$$

it is a contradiction, thus

$$\mathcal{S}(\omega_n,\omega_n,\omega_{n+1})<\mathcal{S}(\omega_{n-1},\omega_{n-1},\omega_n).$$

Similarly, we can prove that

$$\mathcal{S}(\omega_{n-1},\omega_{n-1},\omega_n)<\mathcal{S}(\omega_{n-2},\omega_{n-2},\omega_{n-1}).$$

Combining above, we get

$$\mathcal{S}(\omega_n,\omega_n,\omega_{n+1})<\mathcal{S}(\omega_{n-1},\omega_{n-1},\omega_n),$$

for all $n\in\mathbb{N}^{**}$ and

$$\varsigma(\mathcal{S}(\omega_n,\omega_n,\omega_{n+1}),\mathcal{S}(\omega_{n-1},\omega_{n-1},\omega_n))\geq 0. \tag{4.6}$$

Hence, $\{\mathcal{S}(\omega_n,\omega_n,\omega_{n+1})\}$ is non-increasing sequence of non-negative real numbers, which is convergent and hence there exists $r\geq 0$ such that

$$\lim_{n\to+\infty}\mathcal{S}(\omega_n,\omega_n,\omega_{n+1})=r.$$

Assume that $r>0$, then from (4.6) and property of (ς3), we have

$$0\leq\limsup_{n\to+\infty}\varsigma(\mathcal{S}(\omega_n,\omega_n,\omega_{n+1}),\mathcal{S}(\omega_{n-1},\omega_{n-1},\omega_n))<0,$$

it is a contradiction. Therefore $r=0$, so,

$$\lim_{n\to+\infty}\mathcal{S}(\omega_n,\omega_n,\omega_{n+1})=0. \tag{4.7}$$

Now, we wish to show that $\{\omega_n\}$ is a Cauchy sequence. On contrary, if possible suppose that $\{\omega_n\}$ is not a Cauchy sequence, then by Lemma 2.1, there exist $\varepsilon>0$ and sub sequences $\{m_p\}$ and $\{n_p\}$ of positive integers such that

$$\lim_{p\to+\infty}\{\mathcal{S}(\omega_{m_p},\omega_{m_p},\omega_{n_p}),\mathcal{S}(\omega_{m_{p-1}},\omega_{m_{p-1}},\omega_{n_p-1}),\mathcal{S}(\omega_{m_p},\omega_{m_p},\omega_{n_p-1}),\mathcal{S}(\omega_{m_{p-1}},\omega_{m_{p-1}},\omega_{n_p})\}=\varepsilon. \tag{4.8}$$

Now, if possible suppose there exists a $p\geq p^*$ such that

$$\frac{1}{3}\mathcal{S}(\omega_{m_{p-1}},\omega_{m_{p-1}},\omega_{m_p})\geq\mathcal{S}(\omega_{m_{p-1}},\omega_{m_{p-1}},\omega_{n_p-1}).$$

Taking limits as $p\to+\infty$ and owing Lemma 2.1, we obtain $\varepsilon\leq 0$, it is a contradiction. Therefore

$$\frac{1}{3}\mathcal{S}(\omega_{m_{p-1}},\omega_{m_{p-1}},\omega_{m_p})<\mathcal{S}(\omega_{m_{p-1}},\omega_{m_{p-1}},\omega_{n_p}),$$

for all $p\geq p^*$. Now, we have

$$M_S(\omega_{m_{p-1}},\omega_{m_{p-1}},\omega_{n_p}) = \max\Bigg\{\mathcal{S}(\omega_{m_{p-1}},\omega_{m_{p-1}},\omega_{n_p}), \frac{\mathcal{S}(\omega_{m_{p-1}},\omega_{m_{p-1}},\Gamma\omega_{m_{p-1}})\left[1+\mathcal{S}(\omega_{m_{p-1}},\omega_{m_{p-1}},\Gamma\omega_{m_{p-1}})\right]}{1+\mathcal{S}(\omega_{m_{p-1}},\omega_{m_{p-1}},\omega_{n_p})}$$
$$\frac{\mathcal{S}(\omega_{m_{p-1}},\omega_{m_{p-1}},\Gamma\omega_{m_{p-1}})\left[1+\mathcal{S}(\omega_{m_{p-1}},\omega_{m_{p-1}},\Gamma\omega_{m_{p-1}})\right]}{1+\mathcal{S}(\omega_{m_{p-1}},\omega_{m_{p-1}},\omega_{n_p})},$$
$$\frac{\mathcal{S}(\omega_{n_p},\omega_{n_p},\Gamma\omega_{n_p})\left[1+\mathcal{S}(\omega_{m_{p-1}},\omega_{m_{p-1}},\Gamma\omega_{m_{p-1}})\right]}{1+\mathcal{S}(\omega_{m_{p-1}},\omega_{m_{p-1}},\omega_{n_p})}$$
$$\frac{\mathcal{S}(\omega_{n_p},\omega_{n_p},\Gamma\omega_{n_p})\left[1+\mathcal{S}(\omega_{m_{p-1}},\omega_{m_{p-1}},\Gamma\omega_{m_{p-1}})\right]}{1+\mathcal{S}(\omega_{m_{p-1}},\omega_{m_{p-1}},\omega_{n_p})},$$
$$\frac{1}{3}\frac{\left[\mathcal{S}(\omega_{n_p},\omega_{n_p},\Gamma\omega_{m_{p-1}})+\mathcal{S}(\omega_{m_{p-1}},\omega_{m_{p-1}},\Gamma\omega_{n_p})\left[1+\mathcal{S}(\omega_{n_p},\omega_{n_p},\Gamma\omega_{m_{p-1}})\right]\right]}{1+\mathcal{S}(\omega_{n_p-1},\omega_{m_{p-1}},\omega_{n_p})}\Bigg\}$$

Taking limits as $p \to +\infty$, using (4.8), we have

$$\lim_{p\to+\infty} M_S(\omega_{m_{p-1}},\omega_{m_{p-1}},\Gamma\omega_{n_p}) = \max\left\{\varepsilon,0,0,0,\frac{2\varepsilon(1+\varepsilon)}{3(1+\varepsilon)}\right\} = \varepsilon. \tag{4.9}$$

Also,

$$N_S(\omega_{m_{p-1}},\omega_{m_{p-1}},\omega_{n_p}) = \min\Bigg\{\mathcal{S}(\Gamma\omega_{m_{p-1}},\Gamma\omega_{m_{p-1}},\omega_{n_p}),\mathcal{S}(\omega_{m_{p-1}},\omega_{m_{p-1}},\Gamma\omega_{m_{p-1}}),$$
$$\mathcal{S}(\Gamma\omega_{m_{p-1}},\Gamma\omega_{n_p},\Gamma\omega_{n_p}),\mathcal{S}(\omega_{n_p},\Gamma\omega_{n_p},\Gamma\omega_{m_{p-1}})$$
$$\frac{\mathcal{S}(\omega_{n_p},\omega_{n_p},\Gamma\omega_{m_{p-1}})\left[1+\mathcal{S}(\omega_{m_{p-1}},\omega_{m_{p-1}},\Gamma\omega_{m_{p-1}})\right]}{1+\mathcal{S}(\omega_{m_{p-1}},\omega_{m_{p-1}},\omega_{n_p})}\Bigg\} = 0. \tag{4.10}$$

Thus using (4.1) with $\omega = \omega_{m_{p-1}}, \xi = \omega_{m_{p-1}}, z = \omega_{n_p}$, utilizing (4.9), (4.10) and condition (ς3), we deduce

$$0 \le \limsup_{p\to+\infty} \varsigma(\mathcal{S}(\omega_{m_p},\omega_{m_p},\omega_{n_{p+1}}), M_S(\omega_{m_p-1},\omega_{m_p-1},\omega_{n_p}) + LN_S(\omega_{m_p-1},\omega_{m_p-1},\omega_{n_p})) < 0,$$

it is a contradiction. Hence $\{\omega_n\}$ is a Cauchy sequence in $\bar{E}$. Since

$$\{\omega_n\} \subseteq \Gamma\bar{E} \subseteq \mathcal{M},$$

we conclude that $\{\omega_n\}$ is an $\mathcal{R}$-preserving Cauchy sequence in $\mathcal{M}$.

Owing to ($\mathcal{M}$, $\mathcal{S}$) in $\mathcal{R}$-complete, there exists $q \in \mathcal{M}$ satisfying $\omega_n \xrightarrow{\mathcal{S}} q$.

Firstly, we suppose that Γ is $\mathcal{R}$-continuous, then

$$q = \lim_{n\to+\infty} \omega_{n+1} = \lim_{n\to+\infty} \Gamma\omega_n = \Gamma \lim_{n\to+\infty} \omega_n = \Gamma q.$$

Again, in view of our assumption $\mathcal{R}/\mathcal{M}$ is $\mathcal{S}$-self closed, $\{\omega_n\}$ is an $\mathcal{R}$-preserving sequence and

$$\lim_{n\to+\infty} \omega_n \xrightarrow{\mathcal{S}} q$$

then exists sub sequence $\{\omega_{n_p}\}$ of $\{\omega_n\}$ with

$$[\omega_{n_p},\omega_{n_p},q] \in (\mathcal{R}/\mathcal{M}). \tag{4.11}$$

We now assert that

$$\frac{1}{3}\mathcal{S}(\omega_{n_p},\omega_{n_p},\omega_{n_{p+1}}) < \mathcal{S}(\omega_{n_p},\omega_{n_p},q),$$

for all p. On contrary, if

$$\frac{1}{3}\mathcal{S}(\omega_{n_p},\omega_{n_p},\omega_{n_{p+1}}) \geq \mathcal{S}(\omega_{n_p},\omega_{n_p},q),$$

for some p, then we have

$$3\mathcal{S}(\omega_{n_p},\omega_{n_p},q) \leq \mathcal{S}(\omega_{n_p},\omega_{n_p},\omega_{n_{p+1}}) \leq 2\mathcal{S}(\omega_{n_p},\omega_{n_p},q) + \mathcal{S}(\omega_{n_{p+1}},\omega_{n_{p+1}},q),$$

so,

$$\mathcal{S}(\omega_{n_p},\omega_{n_p},q) \leq \mathcal{S}(\omega_{n_{p+1}},\omega_{n_{p+1}},q),$$

this is a contradiction. Therefore,

$$\frac{1}{3}\mathcal{S}(\omega_{n_p},\omega_{n_p},\Gamma\omega_{n_p}) < \mathcal{S}(\omega_{n_p},\omega_{n_p},q),$$

using (4.1), we have

$$0 \leq \varsigma(\mathcal{S}(\Gamma\omega_{n_p},\Gamma\omega_{n_p},\Gamma q), M_S(\omega_{n_p},\omega_{n_p},q) + LN_S(\omega_{n_p},\omega_{n_p},q)). \tag{4.12}$$

Now

$$M_S(\omega_{n_p},\omega_{n_p},q) = \max\left\{\mathcal{S}(\omega_{n_p},\omega_{n_p},q), \frac{\mathcal{S}(\omega_{n_p},\omega_{n_p},\Gamma\omega_{n_p})\left[1+\mathcal{S}(\omega_{n_p},\omega_{n_p},\Gamma\omega_{n_p})\right]}{1+\mathcal{S}(\omega_{n_p},\omega_{n_p},q)},\right.$$
$$\frac{\mathcal{S}(\omega_{n_p},\omega_{n_p},\Gamma\omega_{n_p})\left[1+\mathcal{S}(\omega_{n_p},\omega_{n_p},\Gamma\omega_{n_p})\right]}{1+\mathcal{S}(\omega_{n_p},\omega_{n_p},q)}, \frac{\mathcal{S}(q,q,\Gamma q)\left[1+\mathcal{S}(\omega_{n_p},\omega_{n_p},\Gamma\omega_{n_p})\right]}{1+\mathcal{S}(\omega_{n_p},\omega_{n_p},q)},$$
$$\left.\frac{\mathcal{S}(q,q,\Gamma q)\left[1+\mathcal{S}(\omega_{n_p},\omega_{n_p},\Gamma\omega_{n_p})\right]}{1+\mathcal{S}(\omega_{n_p},\omega_{n_p},q)}, \frac{1}{3}\frac{\left[\mathcal{S}(q,q,\Gamma\omega_{n_p})+\mathcal{S}(\omega_{n_p},\omega_{n_p},\Gamma q)\right]\left[1+\mathcal{S}(q,q,\Gamma\omega_{n_p})\right]}{1+\mathcal{S}(\omega_{n_p},\omega_{n_p},q)}\right\}$$

letting $p \to +\infty$ and employing (4.7), we have

$$\lim_{p\to+\infty} M_S(\omega_{n_p},\omega_{n_p},q) = \mathcal{S}(q,q,\Gamma q) \tag{4.13}$$

and

$$\lim_{p\to+\infty} N_S(\omega_{n_p},\omega_{n_p},q) = \lim_{p\to+\infty}\min\left\{\mathcal{S}(\Gamma\omega_{n_p},\Gamma\omega_{n_p},q), \mathcal{S}(\omega_{n_p},\omega_{n_p},\Gamma\omega_{n_p}), \mathcal{S}(\Gamma\omega_{n_p},\Gamma\omega_{n_p},\Gamma q),\right.$$
$$\left.\mathcal{S}(q,\Gamma q,\Gamma\omega_{n_p}), \frac{\mathcal{S}(q,q,\Gamma\omega_{n_p})[1+\mathcal{S}(\omega_{n_p},\omega_{n_p},\Gamma\omega_{n_p})]}{1+\mathcal{S}(\omega_{n_p},\omega_{n_p},q)}\right\} = 0. \tag{4.14}$$

Thus in view of conditions (4.12), (4.13), (4.14) and (ς3), we derive

$$0 \leq \limsup_{p\to+\infty}\ \varsigma(\mathcal{S}(\omega_{n_{p+1}},\omega_{n_{p+1}},\Gamma q), M_S(\omega_{n_p},\omega_{n_p},q) + LN_S(\omega_{n_p},\omega_{n_p},q)) < 0,$$

this is a contradiction. Hence $\mathcal{S}(q,q,\Gamma q) = 0$ implies $q = \Gamma q$.

To prove uniqueness, let r^*, ϑ^* be two fixed points of Γ such that $r^* \neq \vartheta^*$.

Since by our assumption, we have

$$\nu(r^*,\vartheta^*,\vartheta^*,\mathcal{R}^{\mathcal{S}}) \neq \phi$$

then there exists an $\mathcal{S}$-path say $(\varrho_0, \ldots, \varrho_1)$ of length l on $\mathcal{R}^{\mathcal{S}}$ from r^* to ϑ^* so that $\varrho_0 = r^*, \varrho_l = \vartheta^*$ and $[\varrho_i,\varrho_{i+1},\varrho_{i+1}] \in \mathcal{R}^{\mathcal{S}}$ for $i \in 0,1,2,\ldots,l-1$, which implies by Lemma 3.1, we get $[\varrho_i,\varrho_{i+1},\varrho_{i+1}] \in \mathcal{R}$, as $\mathcal{R}$ is transitive, we conclude $[\varrho_0,\varrho_l,\varrho_l] \in \mathcal{R}$. Thus inview of (4.1), we have

$$\frac{1}{3}\mathcal{S}(\varrho_0,\varrho_0,\Gamma\varrho_0)=\frac{1}{3}\mathcal{S}(\varrho_0,\varrho_0,\varrho_l)<\mathcal{S}(\varrho_0,\varrho_l,\varrho_l)$$

hence from (4.1), we have

$$\varsigma(\mathcal{S}(\Gamma\varrho_0,\Gamma\varrho_l,\Gamma\varrho_l),M_S(\varrho_0,\varrho_l,\varrho_l)+N_S(\varrho_0,\varrho_l,\varrho_l)), \quad (4.15)$$

where,

$$M_S(\varrho_0,\varrho_l,\varrho_l)=max\left\{\mathcal{S}(\varrho_0,\varrho_l,\varrho_l),\frac{\mathcal{S}(\varrho_l,\varrho_l,\Gamma\varrho_l)[1+\mathcal{S}(\varrho_0,\varrho_0,\Gamma\varrho_0)]}{1+\mathcal{S}(\varrho_0,\varrho_l,\varrho_l)},\right.$$
$$\frac{\mathcal{S}(\varrho_l,\varrho_l,\Gamma\varrho_0)[1+\mathcal{S}(\varrho_0,\varrho_0,\Gamma\varrho_0)]}{1+\mathcal{S}(\varrho_0,\varrho_l,\varrho_l)},\frac{\mathcal{S}(\varrho_l,\varrho_l,\Gamma\varrho_l)[1+\mathcal{S}(\varrho_l,\varrho_l,\Gamma\varrho_l)]}{1+\mathcal{S}(\varrho_0,\varrho_l,\varrho_l)},$$
$$\left.\frac{\mathcal{S}(\varrho_l,\varrho_l,\Gamma\varrho_l)[1+\mathcal{S}(\varrho_0,\varrho_0,\Gamma\varrho_0)]}{1+\mathcal{S}(\varrho_0,\varrho_0,\varrho_0)},\frac{1}{3}\frac{[\mathcal{S}(\varrho_l,\varrho_l,\Gamma\varrho_l)+\mathcal{S}(\varrho_l,\varrho_l,\Gamma\varrho_l)][1+\mathcal{S}(\varrho_l,\varrho_l,\Gamma\varrho_0)]}{1+\mathcal{S}(\varrho_0,\varrho_l,\varrho_l)}\right\}$$
$$=\max\left\{\mathcal{S}(r^*,\vartheta^*,\vartheta^*),0,\frac{\mathcal{S}(\vartheta^*,\vartheta^*,r^*)}{1+\mathcal{S}(r^*,\vartheta^*,\vartheta^*)},0,0,0\right\}=\mathcal{S}(r^*,\vartheta^*,\vartheta^*) \quad (4.16)$$

and

$$N_S(\varrho_0,\varrho_l,\varrho_l)=\min\left\{\mathcal{S}(\Gamma\varrho_0,\Gamma\varrho_0,\varrho_l),\mathcal{S}(\varrho_l,\varrho_l,\Gamma\varrho_0),\mathcal{S}(\Gamma\varrho_l,\Gamma\varrho_l,\Gamma\varrho_l)\right.$$
$$\left.\mathcal{S}(\varrho_l,\Gamma\varrho_l,\Gamma\varrho_l),\frac{\mathcal{S}(\varrho_l,\varrho_l,\Gamma\varrho_0)[1+\mathcal{S}(\varrho_0,\varrho_0,\Gamma\varrho_0)]}{1+\mathcal{S}(\varrho_0,\varrho_l,\varrho_l)}\right\}=0, \quad (4.17)$$

thus from (4.15), (4.16) and (4.17), we have

$$0\le\varsigma(\mathcal{S}(r^*,\vartheta^*,\vartheta^*),\mathcal{S}(r^*,\vartheta^*,\vartheta^*))<\mathcal{S}(r^*,\vartheta^*,\vartheta^*)-\mathcal{S}(r^*,\vartheta^*,\vartheta^*)=0,$$

a contradiction. Hence, $r^*=\vartheta^*$.

The following example supports our result.

Example 4.1. Let $\bar{E}$ = [0, 9), we define $\mathcal{S}:\bar{E}^3\to[0,+\infty)$ by

$$\mathcal{S}(\omega,\xi,z)=\begin{cases}0, & if\ \omega=\xi=z,\\ \max\{\omega,\xi,z\}, & if \omega\ne\xi\ne z.\end{cases}$$

Consider a ternary relation on $\bar{E}$ as
$\mathcal{R}$ ={(1, 2, 8), (1, 7, 2), (1, 3, 7), (1, 1, 1), (0, 0, 1), (8, 8, 0), (2, 2, 1), (7, 7, 1), (3, 3, 1), (3, 3, 3), (7, 7, 7), (8, 4, 1), (2, 3, 1), (3, 3, 2)}.

We define Γ on $\bar{E}$ by $\Gamma\omega=\begin{cases}1, & if \omega\in[0,1],\\ 7, & if \omega\in(1,3],\\ 3, & if \omega\in(3,7],\\ 2, & if \omega\in(7,9).\end{cases}$

Let $\mathcal{M}=[0,7]\subseteq[0,9)$, then clearly, $\Gamma\bar{E}=\{1,2,3,7\}\subseteq\mathcal{M}\subseteq\bar{E}$. Evidently, Γ is discontinuous. Also, $\mathcal{R}$ is Γ-closed and transitive.

For ω = 1, Γ ω = 1, we have (1, 1, 1) ∈ $\mathcal{R}$ implies $\bar{E}$ (Γ,$\mathcal{R}$) ≠ ∅.

If $\{\omega_n\}$ is any $\mathcal{R}$- preserving sequence with $\omega_n\xrightarrow{\mathcal{S}}\omega$,

$$(\omega_n,\omega_{n+1},\omega_{n+1})\in\mathcal{R}/\mathcal{M}$$

there exists $n\in\mathcal{N}^{**}$ with ω_n = { 1, 2, 3, 7} for all $n\ge N^{**}$. Now, we define $\varsigma:[0,+\infty)\times[0,+\infty)\to\mathbb{R}$ by

$$\varsigma(l,\mathfrak{m})=\frac{4}{5}\mathfrak{m}-l,$$

for all $l,\mathfrak{m}\in[0,+\infty)$ and $L\geq 10$. We have to verify the inequality when

$$(\omega,\xi,z)\in\{(1,2,8),(1,7,2),(1,3,7),(8,8,0),(7,7,1),(3,3,1),(8,4,1),(2,3,1)\},$$

since in remaining cases, we have

$$\mathcal{S}(\Gamma\omega,\Gamma\xi,\Gamma z)=0,$$

we have

$$\frac{1}{3}(\omega,\omega,\Gamma\omega)<\mathcal{S}(\omega,\xi,z),$$

this implies

$$\varsigma(\mathcal{S}(\Gamma\omega,\Gamma\xi,\Gamma z),M_s(\omega,\xi,z)+L(N_s(\omega,\xi,z))=\frac{4}{5}(M_s(\omega,\xi,z)+L(N_s(\omega,\xi,z)))-\mathcal{S}(\Gamma\omega,\Gamma\xi,\Gamma z)\geq 0$$

and we have

$$\mathcal{S}(\Gamma\omega,\Gamma\xi,\Gamma z)\leq\frac{4}{5}(M_s(\omega,\xi,z)+L(N_s(\omega,\xi,z))).$$

Case (1): When $\omega = 1$, $\xi = 2$, $z = 8$, we have

$$\frac{1}{3}(1,1,\Gamma 1)=0<\mathcal{S}(\omega,\xi,z)=8,$$

now, from (4.1), we have

$$\varsigma(\mathcal{S}(\Gamma\omega,\Gamma\xi,\Gamma z),M_s(\omega,\xi,z)+L(N_s(\omega,\xi,z))=\frac{4}{5}M_s(\omega,\xi,z)+L(N_s(\omega,\xi,z)))-\mathcal{S}(\Gamma\omega,\Gamma\xi,\Gamma z)=\frac{293}{45}.$$

Case (2): When $\omega =1$, $\xi = 7$, $z = 2$, we have

$$\frac{1}{3}(1,1,\Gamma 1)=0<\mathcal{S}(\omega,\xi,z)=7,$$

from (4.1), we have

$$\varsigma(\mathcal{S}(\Gamma\omega,\Gamma\xi,\Gamma z),M_s(\omega,\xi,z)+L(N_s(\omega,\xi,z))=\frac{4}{5}M_s(\omega,\xi,z)+L(N_s(\omega,\xi,z)))-\mathcal{S}(\Gamma\omega,\Gamma\xi,\Gamma z)=\frac{3}{5}.$$

Case (3): When $\omega = 1$, $\xi = 3$, $z = 7$, we have

$$\frac{1}{3}(1,1,\Gamma 1)=0<\mathcal{S}(\omega,\xi,z)=7,$$

from (4.1), we have

$$\varsigma(\mathcal{S}(\Gamma\omega,\Gamma\xi,\Gamma z),M_S(\omega,\xi,z)+L(N_s(\omega,\xi,z))=\frac{4}{5}M_S(\omega,\xi,z)+L(N_s(\omega,\xi,z)))-\mathcal{S}(\Gamma\omega,\Gamma\xi,\Gamma z)=\frac{28}{5}.$$

Case (4): When $\omega = 8$, $\xi = 8$, $z = 0$, we have

$$\frac{1}{3}(8,8,\Gamma 8)=\frac{8}{3}<\mathcal{S}(\omega,\xi,z)=8,$$

from (4.1), we have

$$\varsigma(\mathcal{S}(\Gamma\omega,\Gamma\xi,\Gamma z),M_S(\omega,\xi,z)+L(N_s(\omega,\xi,z))=\frac{4}{5}(M_S(\omega,\xi,z)+L(N_s(\omega,\xi,z)))-\mathcal{S}(\Gamma\omega,\Gamma\xi,\Gamma z)=\frac{102}{5}.$$

Case (5): When $\omega = 7$, $\xi = 7$, $z = 1$ we have

$$\frac{1}{3}(7,7,\Gamma 7)=\frac{7}{3}<\mathcal{S}(\omega,\xi,z)=7,$$

from (4.1), we have

$$\varsigma(\mathcal{S}(\Gamma\omega,\Gamma\xi,\Gamma z),M_S(\omega,\xi,z)+L(N_s(\omega,\xi,z))=\frac{4}{5}(M_S(\omega,\xi,z)+L(N_s(\omega,\xi,z)))-\mathcal{S}(\Gamma\omega,\Gamma\xi,\Gamma z)=\frac{161}{5},$$

Case (6): When ω = 3, ξ = 3, z = 1, we have

$$\frac{1}{3}(7,7,\Gamma 7)=\frac{7}{3}<\mathcal{S}(\omega,\xi,z)=3,$$

from (4.1), we have

$$\varsigma(\mathcal{S}(\Gamma\omega,\Gamma\xi,\Gamma z),M_S(\omega,\xi,z)+L(N_s(\omega,\xi,z))=\frac{4}{5}(M_S(\omega,\xi,z)+L(N_s(\omega,\xi,z)))-\mathcal{S}(\Gamma\omega,\Gamma\xi,\Gamma z)=\frac{141}{5}.$$

Case (7): When ω = 8, ξ = 4, z = 1, we have

$$\frac{1}{3}(8,8,\Gamma 8)=\frac{8}{3}<\mathcal{S}(\omega,\xi,z)=8,$$

from (4.1), we have

$$\varsigma(\mathcal{S}(\Gamma\omega,\Gamma\xi,\Gamma z),M_S(\omega,\xi,z)+L(N_s(\omega,\xi,z))=\frac{4}{5}(M_S(\omega,\xi,z)+L(N_s(\omega,\xi,z)))-\mathcal{S}(\Gamma\omega,\Gamma\xi,\Gamma z)=\frac{97}{5}.$$

Case (8) : When ω = 2, ξ = 3, z = 1, we have

$$\frac{1}{3}(7,7,\Gamma 7)=\frac{7}{3}<\mathcal{S}(\omega,\xi,z)=3,$$

from (4.1), we have

$$\varsigma(\mathcal{S}(\Gamma\omega,\Gamma\xi,\Gamma z),M_S(\omega,\xi,z)+L(N_s(\omega,\xi,z))=\frac{4}{5}(M_S(\omega,\xi,z)+L(N_s(\omega,\xi,z)))-\mathcal{S}(\Gamma\omega,\Gamma\xi,\Gamma z)=\frac{101}{5}.$$

Thus all the hypotheses of Theorem (4.1) are verified. Here 1 is the unique fixed point of Γ. Here it is worth noting that the mapping Γ neither satisfies the contractive condition (2.1) nor adheres to contractive condition (2.2), when ω = 0, ξ = 1, z = 2, we have

$$\mathcal{S}(\Gamma 0,\Gamma 1,\Gamma 2)=7,$$

$M(\omega,\xi,z)=3$ *and* $N(\omega,\xi,z)=0.$ Also, when ω = 0, ξ = 1, z = 2,

$$\frac{1}{3}\mathcal{S}(\Gamma 0,\Gamma 0,0)=\frac{1}{3}<\mathcal{S}(0,1,2),$$

but $\mathcal{S}(\Gamma 0,\Gamma 1,\Gamma 2)=7,$ $\mathcal{S}(0,1,2)=2$ and

$$L\mathcal{N}(\omega,\xi,z)=L\min\{\mathcal{S}(\Gamma 0,\Gamma 0,1),\mathcal{S}(\Gamma 0,\Gamma 0,1),\mathcal{S}(\Gamma 0,\Gamma 0,2)\}=0,$$

hence by virtue of condition (ii) of ς, there does not exists any $\varsigma\in\mathcal{Z}$ such that equations (2.1) and (2.2) are satisfied, hence $\mathcal{S}$ is not almost generalized $\mathcal{Z}_S$ contraction with rational expressions and almost Suzuki-type $\mathcal{Z}_S$ contraction. Hence Theorem 2.1 and Theorem 2.2 cannot be applied to this example. Hence, we can conclude that our results are more general than the results due to Babu et al., [5,6].

5. Results for some Suzuki-type contraction mappings

Priyabarta et al., [22] introduced θ_s-admissible mapping with respect to η_s.

Definition 5.1. Let $(\bar{E},\mathcal{S})$ be an $\mathcal{S}$-metric space $\Gamma:\bar{E}\to\bar{E},$ and $\theta_s,\ \eta_s:\bar{E}^3\to[0,+\infty).$ Then Γ is an θ_s admissible with respect to η_s if $\omega,\xi,z\in\bar{E},$

$$\theta_s(\omega,\xi,z)\geq\eta_s(\omega,\xi,z)\ \textit{implies}\ \theta_s(\Gamma\omega,\Gamma\xi,\Gamma z)\geq\eta_s(\Gamma\omega,\Gamma\xi,\Gamma z).$$

Note that if $\eta_s(\omega,\xi,z)=1,$ then Γ is θ_s admissible and if $\theta_s(\omega,\xi,z)=1,$ then Γ is θ_s-sub admissible mapping.

We now define triangular θ_s admissible with respect to η_s.

Definition 5.2. Let $(\bar{E},\mathcal{S})$ be an $\mathcal{S}$-metric space, $\Gamma:\bar{E}\to\bar{E},$ $\theta_s,\ \eta_s:\bar{E}^3\to[0,+\infty).$ then Γ is an θ_s triangular admissible with respect to η_s if for all $\omega,\xi,z\in\bar{E},$ we have

(i) $\theta_s(\omega,\xi,z)\geq\eta_s(\omega,\xi,z)$ *implies* $\theta_s(\Gamma\omega,\Gamma\xi,\Gamma z)\geq\eta_s(\Gamma\omega,\Gamma\xi,\Gamma z),$

(ii) $\theta_s(\omega,\xi,z)\geq\eta_s(\omega,\xi,z),$ $\theta_s(\xi,z,u)\geq\eta_s(\xi,z,u)$ implies $\theta_s(\omega,z,u)\geq\eta_s(\omega,z,u),$ for any $u\in\bar{E}.$

When $\eta_s(\omega,\xi,z) = 1$, we say that Γ is triangular θ_s-admissible mapping, when $\theta(\omega,\xi,z)=1$, then Γ is triangular η_s subadmissible.

Definition 5.3. *Let* $(\bar{E},\mathcal{S})$ *be an S-metric space,* $\Gamma:\bar{E}\to\bar{E}$ *and* $\theta_s,\eta_s:\bar{E}^3\to[0,+\infty)$. *Then* Γ *is* $\theta_s-\eta_s$ *Suzuki-type* $\mathcal{Z}_S$ *contraction mapping if there exist* $L \geq 0$ *and* $\varsigma \in \mathcal{Z}$ *such that for all* $\omega,\xi,z\in\bar{E}$, $\theta_s(\omega,\xi,z)\geq\eta_s(\omega,\xi,z)$ *and*

$$\frac{1}{3}\mathcal{S}(\omega,\omega,\Gamma\omega)<\mathcal{S}(\omega,\xi,z) \text{ implies } \varsigma(\mathcal{S}(\Gamma\omega,\Gamma\xi,\Gamma z),M_S(\omega,\xi,z)+LN_S(\omega,\xi,z))\geq 0, \tag{5.1}$$

where $M_S(\omega, \xi, z)$ *and* $N_S(\omega, \xi, z)$ *are defined as in Definition 4.1.*

Corollary 5.1. *Let* $\Gamma:\bar{E}\to\bar{E}$, *and* $\theta_s,\eta_s:\bar{E}^3\to[0,+\infty)$. *be two mappings on an S-metric space* $\bar{E}$. *Suppose that* Γ *is* $\theta_s-\eta_s$ *Suzuki-type* $\mathcal{Z}_S$ *contraction mapping satisfying the following conditions:*

(i) let $\omega_0\in\bar{E}$ *such that* $\theta_s(\omega_0,\Gamma\omega_0,\Gamma\omega_0)\geq\eta_s(\omega_0,\Gamma\omega_0,\Gamma\omega_0)$,

(ii) Γ *is triangular* θ_s *admissible mapping with respect to* η_s,

(iii) if $\{\omega_n\}$ *is a sequence in* $\bar{E}$ *such that* $\theta_s(\omega_n,\omega_n,\Gamma\omega_n)\geq\eta_s(\omega_n,\omega_n,\Gamma\omega_n)$, $n\in\mathbb{N}^{**}$ *and* $\omega_n\to q$ *as* $n\to+\infty$ *there exists* $\{\omega_{n(p)}\}$ *of* $\{\omega_n\}$ *such that* $\theta_s(\omega_{n(p)},\omega_{n(p)},z)\geq\eta_s(\omega_{n(p)},\omega_{n(p)},z)$, *for all* $p\in\mathbb{N}^{**}$.

Then Γ *has a fixed point in* $\bar{E}$. *In addition, if for any two fixed points* p, q *of* Γ *such that* $\theta_s(p,q,q)\geq\eta_s(p,q,q)$, *then* Γ *has a unique fixed point.*

Proof. Define $\mathcal{R}$ on $\bar{E}$ as $(\omega, \xi, z) \in \mathcal{R}$ if and only if $\theta_s(\omega,\xi,z)\geq\eta_s(\omega,\xi,z)$.
We now have the following observations.

(i) Let $\omega_0\in\bar{E}$ such that

$\theta_s(\omega_0,\omega_0,\Gamma\omega_0)\geq\eta_s(\omega_0,\omega_0,\Gamma\omega_0)$ implies $(\omega_0,\omega_0,\Gamma\omega_0)\in\mathcal{R}$ and $\bar{E}(\Gamma,\mathcal{R})\neq\phi$.

(ii) If $(\omega, \xi, z) \in \mathcal{R}$, then $\theta_s(\omega,\xi,z)\geq\eta_s(\omega,\xi,z)$. As Γ is θ_s triangular admissible map with respect to η_s, we have $\theta_s(\Gamma\omega,\Gamma\xi,\Gamma z)\geq\eta_s(\Gamma\omega,\Gamma\xi,\Gamma z)$ then $(\Gamma\omega,\Gamma\xi,\Gamma z)\in\mathcal{R}$, thus $\mathcal{R}$ is Γ closed.

(iii) If $(\omega, \xi, z) \in \mathcal{R}$, $(\xi, z, u) \in \mathcal{R}$ then $\theta_s(\omega,\xi,z)\geq\eta_s(\omega,\xi,z)$ and $\theta_s(\xi,z,u)\geq\eta_s(\xi,z,u)$, , since Γ is triangular θ_s admissible with respectη_s, we have $\theta_s(\xi,z,u)\geq\eta_s(\xi,z,u)$, therefore $\mathcal{R}$ is transitive.

(iv) If $(\omega, \xi, z) \in \mathcal{R}$ then $\theta_s(\xi,z,u)\geq\eta_s(\xi,z,u)$, since Γ is almost $\theta_s-\eta_s$ Suzuki type $\mathcal{Z}_S$-contraction then Γ is Suzuki type $\mathcal{Z}_{\mathcal{R}_S}$ contraction.

(v) From assumed condition *(iii)*, we have $(\omega_n,\omega_n,\Gamma\omega_n)\in\mathcal{R}$, for all $n\in\mathbb{N}^{**}$ and $lim_{n\to+\infty}\omega_n=q$, then there exists subsequence $\{\omega_{n(p)}\}$ of $\{\omega_n\}$ such that

$$(\omega_{n(p)},\omega_{n(p)},q)\in\mathcal{R},$$

for all $p\in\mathbb{N}^{**}$. Hence the assumptions of Theorem 4.1 are satisfied, Γ has a fixed point in $\bar{E}$. Also, if for any two fixed points p, q of Γ such that $\theta_s(p,q,q)\geq\eta_s(p,q,q)$, then $\nu(p,q,q,\mathcal{R})\neq\emptyset$. Therefore, by Theorem 4.1 it follows that Γ has a unique fixed point. □

6. Application to nonlinear matrix equations

In this section, we utilize our research findings to establish a conclusion about the existence of solutions for a nonlinear matrix equation attributed with a ternary relation.

Let the set $\mathcal{M}(n)$ encompasses all square matrices of order of $n \times n$. Let $\mathcal{H}(n)$ represents the set of Hermitian matrices, i.e., matrices that are equal to their conjugate transpose, the set $\wp(n)$ refers to the set of positive definite matrices, while $\mathfrak{K}(n)$ represents the set of positive semi-definite matrices, which have non-negative eigenvalues.

For $\Lambda \in \mathcal{M}(n)$, we denote the singular values of Λ by $sv(\Lambda)$ (Singular values are the absolute values of eigen values of a matrix) and sum of all singular values by $sv^+(\Lambda)$ and $\Lambda \succeq \prod$ signifies that $\Lambda - \prod \in \mathfrak{K}(n)$. i.e., $sv^+(\Lambda) = \|\Lambda\|_{tr}$, where $\|\cdot\|_{tr}$ denotes the trace norm. On $\mathcal{H}(n)$, we define $\Lambda \succ \prod$ signifies that $\Lambda - \prod \in \wp(n)$.

Lemma 6.1. [25] *If $\Lambda \succeq 0$ and $\Pi \succeq 0$ are $n \times n$ matrices, then*

$$0 \preceq tra(\Lambda\Pi) \preceq \|\Lambda\| tra(\Pi).$$

We now obtain positive definite solution to the following non-linear matrix equation (NME)

$$\Phi = \mathcal{U} + \sum_{i=1}^{n} \mathcal{D}_i^* \mathcal{F}(\Phi) \mathcal{D}_i, \tag{6.1}$$

where $\mathcal{U}$ is Hermitian matrix, $\mathcal{D}_i^*$ is conjugate transpose of $\mathcal{D}_i$ and $\mathcal{F} : \mathbf{H}(n) \to \wp(n)$ is an order-preserving mapping such that $\mathcal{F}_0 = 0$, where $\mathbf{H}(n)$, $\wp(n)$ stands the set of Hermitian matrices and set of positive definite matrices respectively.

Theorem 6.1. Consider NME (6.1) with the following conditions;

(i) *there exists $\mathcal{U} \in \wp(n)$ such that $\mathcal{U} + \sum_{i=1}^{n} \mathcal{D}_i^* \mathcal{F}(\mathcal{U}) \mathcal{D}_i \succ 0$,*

(ii) *for all $\Phi, \Pi, \Omega \in \wp(n)$ with $\Phi \preceq \Pi \preceq \Omega$ implies*

$$\sum_{i=1}^{n} \mathcal{D}_i^* \mathcal{F}(\Phi) \mathcal{D}_i \preceq \sum_{i=1}^{n} \mathcal{D}_i^* \mathcal{F}(\Pi) \mathcal{D}_i \preceq \sum_{i=1}^{n} \mathcal{D}_i^* \mathcal{F}\, \Omega\, \mathcal{D}_i,$$

(iii) *for all $\Phi, \Pi, \Omega \in \wp(n)$ with $\Phi \preceq \Pi \preceq \Omega$ implies*

$$\frac{2}{3} \|\Phi - \mathcal{F}\Phi\|_{tr} \le \|\Phi - \Omega\|_{tr} + \|\Pi - \Omega\|_{tr},$$

(iv) $\sum_{i=1}^{n} \mathcal{D}_i \mathcal{D}_i^* \prec \gamma I_n$, *$\gamma$ a positive number*

(v) *there exist $k \in (0, 1)$ and $L \ge 0$ such that for all $\Phi, \Pi, \Omega \in \wp(n)$ with $\Phi \preceq \Pi \preceq \Omega$, the following inequality holds*

$$\|\mathcal{F}\Phi - \mathcal{F}\Omega\|_{tr} \le \frac{k}{2\gamma}\left[(M_s(\Phi,\Pi,\Omega)) + LN_s(\Phi,\Pi,\Omega)\right],$$

and

$$\|\mathcal{F}\Pi - \mathcal{F}\Omega\|_{tr} \le \frac{k}{2\gamma}\left[(M_s(\Phi,\Pi,\Omega)) + LN_s(\Phi,\Pi,\Omega)\right],$$

where

$$\begin{aligned} M_s(\Phi,\Pi,\Omega) = \max\Bigg\{ & \|\Phi - \Omega\|_{tr} + \|\Pi - \Omega\|_{tr}, \frac{2\|\Pi - \Gamma\Pi\|_{tr}(1 + 2\|\Phi - \Gamma\Phi\|_{tr})}{1 + \|\Phi - \Omega\|_{tr} + \|\Pi - \Omega\|_{tr}}, \\ & \frac{2\|\Pi - \Gamma\Phi\|_{tr})(1 + 2\|\Phi - \Gamma\Phi\|_{tr})}{1 + \|\Phi - \Omega\|_{tr} + \|\Pi - \Omega\|_{tr}}, \frac{2\|\Omega - \Gamma\Omega\|_{tr}(1 + 2\|\Pi - \Gamma\Pi\|_{tr})}{1 + \|\Phi - \Omega\|_{tr} + \|\Pi - \Omega\|_{tr}}, \\ & \frac{2\|\Omega - \Gamma\Omega\|_{tr}(1 + 2\|\Phi - \Gamma\Phi)\|_{tr}}{1 + \|\Phi - \Omega\|_{tr} + \|\Pi - \Omega\|_{tr}}, \frac{2(\|\Omega - \Gamma\Pi\|_{tr} + \|\Pi - \Gamma\Omega\|_{tr})(1 + 2\|\Omega - \Gamma\Phi)\|_{tr})}{3(1 + \|\Phi - \Omega\|_{tr} + \|\Pi - \Omega\|_{tr})}, \\ N_s(\Phi,\Pi,\Omega) = \min\{ & 2\|\Gamma\Phi - \Omega\|_{tr}, 2\|\Pi - \Gamma\Omega\|_{tr}, 2\|\Gamma\Pi - \Gamma\Omega\|_{tr}, \\ & \|\Omega - \Gamma\Pi\|_{tr} + \|\Gamma\Omega - \Gamma\Pi\|_{tr}, \frac{2\|\Omega - \Gamma\Phi\|_{tr}[1 + 2\|\Phi - \Gamma\Phi\|_{tr}]}{1 + \|\Phi - \Omega\|_{tr} + \|\Pi - \Omega\|_{tr}} \Bigg\}. \end{aligned}$$

Then (6.1) has a solution $\bar{\Phi}$. *In addition, the iteration*

$$\Phi_n = \mathcal{U} + \sum_{i=1}^{n} \mathcal{D}_i^* \mathcal{F}(\Phi_{n-1})\mathcal{D}_i,$$

where $\Phi_0 \in \wp(n)$ *satisfies* $\Phi_0 \preceq \mathcal{U} + \sum_{i=1}^{n} \mathcal{D}_i^* \mathcal{F}(\Phi_{n-1})\mathcal{D}_i$, *converges in the sense of trace norm* $\|\cdot\|_{tr}$, *to the solution of (6.1).*

Proof. First we define mapping $\Gamma : \wp(n) \to \wp(n)$ by

$$\Gamma\Phi = \mathcal{U} + \sum_{i=1}^{n} \mathcal{D}_i^* \mathcal{F}(\Phi)\mathcal{D}_i,$$

for all $\Phi \in \wp(n)$. We define

$$\mathcal{R} = \{(\Phi, \Pi, \Omega) \in \wp(n) \times \wp(n) \times \wp(n) : \Phi \preceq \Pi \preceq \Omega\}.$$

The solution of a matrix equation (6.1) will be subsequently the fixed point of Γ. Clearly, Γ is well defined on ⪯, Γ is closed, since

$$\mathcal{U} + \sum_{i=1}^{n} \mathcal{D}_i^* \mathcal{F}(\Phi)\mathcal{D}_i \succ 0,$$

$\mathcal{U} \preceq \mathcal{U} \preceq \Gamma\mathcal{U}$ and hence $(\mathcal{U}, \mathcal{U}, \Gamma\mathcal{U}) \in \mathcal{R}$ this implies $\wp(n)(\Gamma, \mathcal{R}) \neq \varnothing$. Define $\mathcal{S} : \wp(n) \times \wp(n) \times \wp(n) \to \mathbb{R}^+$ by

$$\mathcal{S}(\Phi, \Pi, \Omega) = \|\Phi - \Omega\|_{tr} + \|\Pi - \Omega\|_{tr},$$

for all $\Phi, \Pi, \Omega \in \wp(n)$. Then $(\wp(n), \mathcal{S})$ is an $\mathcal{S}$ metric space with respect to ternary relation $\mathcal{R}$.

Let $(\Phi, \Pi, \Omega) \in \mathcal{R}^* = \{(\Phi, \Pi, \Omega) \in \mathcal{R}, \Gamma\Phi \neq \Gamma\Pi \neq \Gamma\Omega\}$. By assumptions (*ii*), (*iii*) and (*iv*), we have

$$\begin{aligned}
\mathcal{S}(\Gamma\Phi, \Gamma\Pi, \Gamma\Omega) &= \|\Gamma\Phi - \Gamma\Omega\|_{tr} + \|\Gamma\Pi - \Gamma\Omega\|_{tr} \\
&= \|\sum_{i=1}^{n} \mathcal{D}_i^* (\mathcal{F}(\Phi) - \mathcal{F}(\Omega))\mathcal{D}_i\|_{tr} + \|\sum_{i=1}^{n} \mathcal{D}_i^* (\mathcal{F}(\Pi) - \mathcal{F}(\Omega))\mathcal{D}_i\|_{tr} \\
&\leq \|\sum_{i=1}^{n} \mathcal{D}_i^* \mathcal{D}_i (\mathcal{F}(\Phi) - \mathcal{F}(\Omega))\|_{tr} + \|\sum_{i=1}^{n} \mathcal{D}_i^* \mathcal{D}_i (\mathcal{F}(\Pi) - \mathcal{F}(\Omega))\|_{tr} \\
&\leq \|\sum_{i=1}^{n} \mathcal{D}_i^* \mathcal{D}_i\| \|(\mathcal{F}(\Phi) - \mathcal{F}(\Omega))\|_{tr} + \|\sum_{i=1}^{n} \mathcal{D}_i^* \mathcal{D}_i\| \|(\mathcal{F}(\Pi) - \mathcal{F}(\Omega))\|_{tr} \\
&\leq \|\sum_{i=1}^{n} \mathcal{D}_i^* \mathcal{D}_i\| \frac{k}{2\gamma}\left[M_s(\Phi, \Pi, \Omega)) + LN_s(\Phi, \Pi, \Omega)\right] \\
&\quad + \|\sum_{i=1}^{n} \mathcal{D}_i^* \mathcal{D}_i\| \frac{k}{2\gamma}\left[M_s(\Phi, \Pi, \Omega)) + LN_s(\Phi, \Pi, \Omega)\right] \\
&\leq k\left[(M_s(\Phi, \Pi, \Omega)) + LN_s(\Phi, \Pi, \Omega)\right],
\end{aligned}$$

this implies

$$0 < k\left[(M_s(\Phi, \Pi, \Omega)) + LN_s(\Phi, \Pi, \Omega)\right] - \mathcal{S}(\Gamma\Phi, \Gamma\Pi, \Gamma\Omega).$$

Hence by considering $\varsigma(t, s) = ks - t,\ k \in (0,1)$, we get

$$0 \leq \varsigma(\mathcal{S}(\Phi, \Pi, \Omega), (M_s(\Phi, \Pi, \Omega)) + LN_s(\Phi, \Pi, \Omega)).$$

In view of existence of greatest lower bound and least upper bound of for all $\Phi, \Pi, \Omega \in \wp(n)$, we have $\nu(\Phi, \Pi, \Omega, \mathcal{R})$ is nonempty. Thus by Theorem 6.1 it can be deduced that there exists $\mathfrak{F}^* \in \wp(n)$ such that $\Gamma(\mathfrak{F}^*) = \mathfrak{F}^*$ holds. Hence the matrix equation (6.1) has a solution. Thus on using Theorem 4.1, Γ has a unique fixed point, and hence we conclude that (6.1) has a unique solution in $\wp(n)$.

Example 6.1. *Consider NME* (21) *for* $i = 3$, $n = 4$, $k = 0.4$, $\gamma = 158.1$ *and* $L = 2$ *with an order-preserving continuous mapping* $\mathcal{F} : \wp(n) \to \wp(n)$ *by* $\mathcal{F}\Phi = 3\Phi^{\frac{1}{2}}$ *with* $\mathcal{F}(0) = 0$ *i.e,*

$$\Phi = \mathcal{U} + \mathcal{D}_1^* 3\Phi^{\frac{1}{2}} \mathcal{D}_1 + \mathcal{D}_2^* 3\Phi^{\frac{1}{2}} \mathcal{D}_2 + \mathcal{D}_3^* 3\Phi^{\frac{1}{2}} \mathcal{D}_3,$$

where

$$\mathcal{U} = \begin{bmatrix} 9.0020010412 & 8.0000013812 & 12.000001735 & 0.000002082 \\ 2.0012013812 & 0.0020018742 & 0.000002360 & 0.000002846 \\ 13.000001735 & 6.0000023607 & 10.002002984 & 0.000003605 \\ 4.0000020825 & 0.0000028461 & 3.001136094 & 0.002004374 \end{bmatrix}$$

$$\mathcal{D}_1 = \begin{bmatrix} 5.009001 & 0.015412 & 4.0184125 & 0.0251667 \\ 0.120034 & 3.5010123 & 2.0020345 & 0.1800123 \\ 0.1410654 & 0.0038345 & 0.0052234 & 0.0066345 \\ 0.0125567 & 0.0192347 & 0.0318548 & 0.2091987 \end{bmatrix}$$

$$\mathcal{D}_2 = \begin{bmatrix} 3.0020001 & 0.1800125 & 0.50102341 & 2.0154021 \\ 1.0000005 & 0.0132234 & 0.0159234 & 1.01920981 \\ 2.0046234 & 4.0062123 & 0.0092986 & 0.20911234 \\ 0.03852234 & 0.0251456 & 0.0184987 & 0.00792345 \end{bmatrix}$$

$$\mathcal{D}_3 = \begin{bmatrix} 2.2100105 & 4.00302342 & 7.1070678 & 0.0140345 \\ 7.0095456 & 0.00152098 & 3.00361234 & 0.01461235 \\ 0.00134561 & 0.01345678 & 0.00662345 & 0.00967891 \\ 0.31883456 & 0.07973987 & 0.01599867 & 0.00532134 \end{bmatrix}$$

To verify all the hypotheses of Theorem 6.1, we use the following iteration for $\mathcal{F}(\Phi) = \Phi_{n-1}$ *i.e.,*

$$\Phi_n = \mathcal{U} + \mathcal{D}_1^* 3\Phi_{n-1}^{\frac{1}{2}} \mathcal{D}_1 + \mathcal{D}_2^* 3\Phi_{n-1}^{\frac{1}{2}} \mathcal{D}_2 + \mathcal{D}_3^* 3\Phi_{n-1}^{\frac{1}{2}} \mathcal{D}_3$$

We now start with the following three initial values

$$\Phi_0 = \begin{bmatrix} 3.237200104166 & 0.25060138885 & 0.29900173588 & 0.1250000208250 \\ 1.25000138885 & 4.20200187490 & 0.213000236074 & 0.28900284610 \\ 1.11000173588 & 0.35000236074 & 2.00200298535 & 0.00000360941 \\ 6.08000208250 & 0.008032846106 & 0.27111360941 & 5.10200437210 \end{bmatrix}$$

$$\Pi_0 = \begin{bmatrix} 1.00077577436 & 0.00387817630 & 0.00977314110 & 0.00146356780 \\ 0.00878176300 & 0.00416351410 & 2.00523511220 & 0.00148455210 \\ 0.00506929770 & 0.00112202140 & 1.00164133970 & 1.00971391711 \\ 0.00146356781 & 0.00455216912 & 0.00897139172 & 1.000051590732 \end{bmatrix}$$

$$\Omega_0 = \begin{bmatrix} 2.4517101 & 0.29662345 & 0.05618790 & 0.2667987 \\ 0.3180130 & 1.0952345 & 0.2204987 & 0.62518965 \\ 0.0551989 & 0.21713456 & 3.62892874 & 0.06328903 \\ 0.1262456 & 0.4560789 & 0.0633543 & 5.6826897 \end{bmatrix}$$

After 20 iterations the following solution is obtained.

$$\Phi = \Phi_{20} = \begin{bmatrix} 6.6309 & 2.4505 & 4.8127 & 0.8351 \\ 2.4452 & 2.9417 & 2.8794 & 0.4750 \\ 4.8150 & 2.8854 & 6.3703 & 0.1492 \\ 0.8389 & 0.4746 & 0.1522 & 0.4472 \end{bmatrix}$$

Numerical calculations of Example 6.1 as shown in the following Table 1.

Table 1. Numerical calculations of Example 2

Initial value	$\mathcal{F}(\Phi_0)$	*Iteration number*	*CPU (sec.)*	*Error*
Φ_0	$\Phi_0^{\frac{1}{2}}$	21	0.032896	2.209*e*–03
Π_0	$\Pi_0^{\frac{1}{2}}$	22	0.032234	2.295*e*–03
ω_0	$\Omega_0^{\frac{1}{2}}$	21	0.032769	7.18*e*–03

In figure 1, we illustrate the convergence phenomenon through a visual representation.

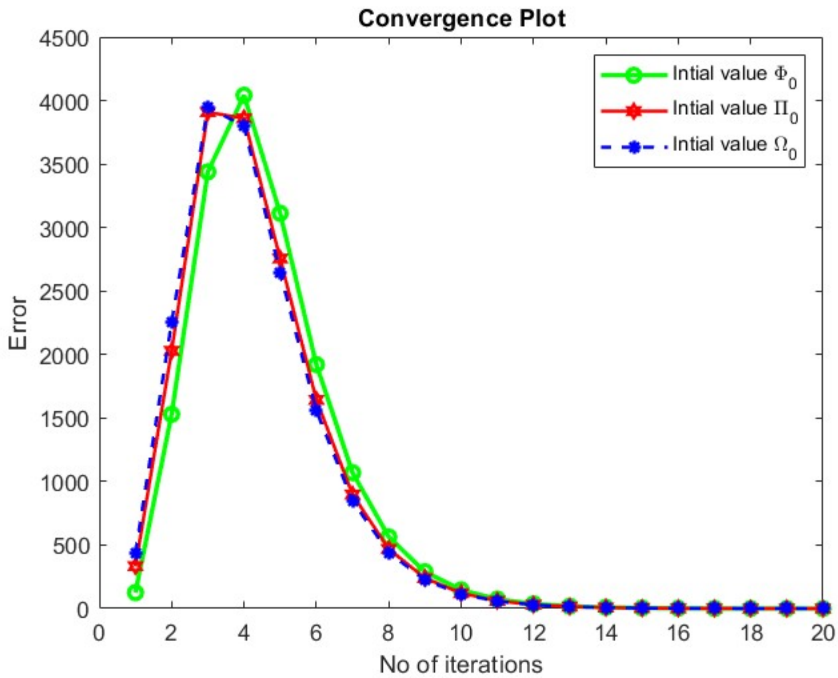

Figure 1: Graph of convergence behaviour

Conclusion

This study presents novel fixed point theorems for Suzuki-type $\mathcal{Z}_{\mathcal{R}_S}$ contraction mappings in S-metric spaces, which do not necessarily derive from a standard metric. As a result, more general conclusions are drawn compared to existing literature. Our findings are applied to demonstrate the existence of solutions for nonlinear matrix equations. Additionally, we provide a numerical example to illustrate the practical implementation of our results.

A key aspect of our approach is the use of weaker conditions, such as $\mathcal{R}$-completeness on subspaces instead of full-space completeness and $\mathcal{R}$-continuity rather than standard continuity. We also explore the property that $\mathcal{R}|_{\mathcal{M}}$ is $\mathcal{S}$ self closed. These contraction conditions reduce classical forms when the universal relation is considered. Our results offer a detailed framework for further research into S-metric spaces equipped with ternary relations.

There remain several intriguing directions for future research. For instance, readers could explore the study of unique and non-unique fixed points, as well as fixed circles, e. g [13, 18, 20, 21, 30] using ternary relations in S metric spaces.

Acknowledgments

The authors S. Aljohani, A. Aloqaily and N. Mlaiki would like to thank Prince Sultan University for paying the publication fees for this work through TAS LAB.

Conflicts of interest

The authors declare no conflicts of interest.

References

[1] A. Alam, M. Imdad, Relation theoretic metrical coincidence theorems, *Filomat*, 31 (2017), 4421–4439.

[2] M. A. Alghamdi, O. S. Gulyaz, E. Karapınar, E. A note on extended Z-contraction, *Mathematics*, 8(2020), 195.

[3] M. Asadi, M. Azhini, E. Karapnar, H. Monfrared, Simulation Function over M-Metric Spaces, *East Asian Math. J.* 33(5), (2017). 559–570.

[4] M. Asadi, M. Gabeleh, C. Vetro, A New Approach to the Generalization of Darbo's Fixed Point Problem by Using Simultation, Functions With Application to Integral Equations, *Results Math* 74(86), (2019), 15p.

[5] G. V. R. Babu, S. P. Durga, G. Srichandana, Fixed points of almost Suzuki type Z_S contractionsin S metric spaces, *Mat. Vesn.*, 74(2022), 130–140.

[6] G. V. R. Babu, S. P. Durga, G. Srichandana, Fixed points of almost generalized Z_S contractions with rational expressions in $\mathcal{S}$ metric spaces, *J. Math. Comput. Sci.*, 11 (2021), 914–937.

[7] B. K. Dass, S. Gupta, An extension of Banach contraction principle through rational expression, *Indian J. Pure Appl. Math.*, 6 (1975), 1455–1458.

[8] T. Došenović, S. Radenović, A. Rezvani, S. Sedghi, Coincidence point theorems in S metric spaces using inegral type of contraction, *Sci. Bull., Ser. A, Appl. Math. Phys., Politeh. Univ. Buchar.*, 79 (2017), 145–158.

[9] U. C. Gairola, K. Deepak, Suzuki type fixed point theorems in S metric space, *Inter. J. Math. Ana. Appl.*, 5 (2017), 277–289.

[10] M. Hasanuzzaman, M. Imdad, Relation theoretic metrical fixed point results for Suzuki type Z_R contraction with an application, *AIMS Math.*, 5 (2020), 2071–2087.

[11] N. T. Hieu, N. T. Ly, N. V. Dung, A generalization of Ćirić quasi-contractions for maps on S-metric spaces, *Thai J. Math.*, 13 (2015), 369–380.

[12] D. S. Jaggi, Some unique fixed point theorems, *Indian J. Pure Appl. Math.*, 8 (1977), 223–230.

[13] M. Joshi, A. Tomar, T. Abdeljawad, On fixed points, their geometry and application to satellite web coupling problem in S-metric spaces, *AIMS Math.*, 8 (2023), 4407–4441.

[14] F. Khojasteh, S. Shukla, S. Radenović, A new approach to the study fixed point theorems via simulation functions, *Filomat*, 29 (2015), 1189–1194.

[15] P. Kumam, D. Gopal, L. Budhiyi, A new fixed point theorem under Suzuki type Z-contraction mappings, *J. Math. Anal.*, 8 (2017), 113–119.

[16] R. M. Kumar, B. Singh, A novel approach to G metric spaces by using ternary relation, *Internat. J. of Pure and Appl. Math. Sci.*, 14 (2021), 29–38.

[17] N. Mlaiki, N. Y. Ozgur, N. Taş, New fixed point theorems on an S metric space via simulation functions, *Mathematics*, 583, (2019), 13 p.

[18] N. Mlaiki, U. Celik, N. Y. Tas, N.Ozgur and A. Mukheimer, Wardowski type contractions and the fixed-circle problem on S-metric spaces, *J. Math.*, 2018, Art. ID 9127486, 9 pp.

[19] H. Monfared , M. Asadi, A. Farajzadeh, New Generalization of Darbo's fixed Point Theorem via α-admissible Simulation Functions with Application, *Sahad Communications in Mathematical Analysis* 17(2)(2020), 161–17,

[20] N. Y. Ozgur, N. Taş, Some fixed-circle theorems on metric spaces, *Bull. Malays. Math. Sci. Soc.*, 42(2019), 1433–1449.

[21] N. Y. Ozgur, N. Taş Fixed-circle problem on S-metric spaces with a geometric viewpoint, *Facta Univ., Ser. Math. Inf.*, 34(2019), 459–472.

[22] N. Priyobarta, Y. Rohen, S. Thounaojam, S. Radenović, Some remarks on α-admissibility in S-metric spaces, *J. Inequal. Appl.*, 14 (2022), 16 p.

[23] A. C. M. Ran, M. C. B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, *Proc. Amer. Math. Soc.*, 132 (2004), 1435–1443.

[24] A. Z. Rezazgui, A. A. Tallafha, W. Shatanawi, Common fixed point results via $Av\alpha$–contractions with a pair and two pairs of self-mappings in the frame of an extended quasi b-metric space, *AIMS Math.*, 8 (2023), 7225–7241.

[25] K. Sawangsup, W. Sintunavarat, On modified Z-contractions and an iterative scheme for solving nonlinear matrix equations, *J. Fixed Point Theory Appl.*, 20 (2018), 80, 19p.
[26] S. Sedghi, N. Shobe, A. Aliouche, A generalization of fixed point theorems in S-metric spaces, *Mat. Vesn.*, 64 (2012), 258–266.
[27] S. Sedghi, M. Rezaee, M. Mahdi, T. Došenović, S. Radenović, Common fixed point theorems for contractive mappings satisfying Φ-maps in S-metric spaces, *Acta Univ. Sapientiae, Math.*, 8 (2016), 298–311.
[28] W. Shatanawi, T. A. M. Shatnawi, New fixed point results in controlled metric type spaces based on new contractive conditions, *AIMS Math.*, 8 (2023), 9314–9330.
[29] T. Suzuki, A new type of fixed point theorem in metric spaces, Nonlinear Anal., 71 (2009), 5313–5317.
[30] A. Tomar, N. Tas, M. Joshi, On interpolative type multiple fixed points their geometry and applications on S-Metric spaces, *Appl. Math. E-Notes*, (23) (2023), 243–259.
[31] L. Wangwe, Common Fixed Point Theorem for multivalued mappings using ternary relation in G metric Space with an application.