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Some results on geometric properties of frames in Banach spaces

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In this paper, we have defined $\mathcal{A}$- cone and related concepts in Banach spaces and prove a result concerning convergence of a sequence in an $\mathcal{A}$- cone. Also, atomic system for a subset of a Banach space is defined and proved that if a Banach space has an atomic system, then every subset of it also has an atomic system.

Key words and phrases: Frame; Atomic System, Cone, $\mathcal{A}$-modulus

Mathematics Subject Classification (2020): 42C15, 46B15.

1. Introduction

Throughout, $\mathcal{H}$ will designate a separable Hilbert space; $\mathcal{B}$ a separable real Banach space; $\mathcal{B}_d$ a Banach space of scalar valued sequences associated with the Banach space $\mathcal{B}$, RO - reconstruction operator; BF - Banach frame; retro BF - retro Banach frame; FO - frame operator and exact BF - exact Banach frame.

Frames were initiated in [2] in the context of non-harmonic Fourier analysis. Frames were reacquainted in [5].

A sequence $\Phi = \{h_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ is termed as a frame for $\mathcal{H}$, if one can find scalars $0 < B_l, B_u < \infty$ satisfying

$$B_l \| y \|^2 \le \sum_{n=1}^{\infty} \left| \langle y, h_n \rangle \right|^2 \le B_u \| y \|^2, \text{for all } y \in \mathcal{H}. \tag{1}$$

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The scalars B_l and B_u in (1) are called frame bounds for Φ. These bounds need not be unique. If $B_l = B_u$, then $\Phi = \{h_n\}_{n=1}^{\infty}$ is said to be a tight frame and if $B_l = B_u = 1$, then $\Phi = \{h_n\}_{n=1}^{\infty}$ is said to be a Parseval frame for $\mathcal{H}$. (1) is termed as the frame inequality for the frame $\Phi = \{h_n\}_{n=1}^{\infty}$. $T : l^2 \to \mathcal{H}$ given by

$$T(\{a_k\}) = \sum_{k=1}^{\infty} a_k h_k, \{a_k\} \in \ell^2$$

is known as the synthesis operator. Also, its adjoint operator $T^* : \mathcal{H} \to l^2$, is termed as the analysis operator and is defined by $T^*(y) = \{\langle y, h_k \rangle\}$, $y \in \mathcal{H}$. If we compose T and T^*, then we obtain the frame operator (FO) $S = TT^* : \mathcal{H} \to \mathcal{H}$ given by

$$S(y) = \sum_{k=1}^{\infty} \langle y, h_k \rangle h_k, \ y \in \mathcal{H}.$$

The FO S is a self adjoint, positive, bounded, and an invertible operator on $\mathcal{H}$. Every vector of the space $\mathcal{H}$ can be expressed as follows:

$$y = SS^{-1}y = \sum_{k=1}^{\infty} \langle S^{-1}y, h_k \rangle h_k = \sum_{k=1}^{\infty} \langle y, Sh_k \rangle h_k, \text{for all } y \in \mathcal{H}.$$

Frames have many applications and uses in various areas of applied mathematics and engineering. Frames were traditionally used in signal and image processing [8, 9, 10, 11]. Today, besides traditional uses, frames are used in sensor networks and packet encoding [16, 17]. Muraleetharan and Thirulogasanthar [18] investigated the invariance of the Fredholm index under perturbations by small norm operators and compact operators. In [6], the topic of K-frame in quaternionic Hilbert spaces was investigated. Jahan, Kumar and Shekhar [13] prove that a cone associated with an exact BF lacks a weakly compact (compact) base, yet it inevitably has an unbounded base and an extremal subset.

Several researchers namely Feichtinger and Gröchenig [7], Casazza et al. [1], Terekhin [19], and Gröchenig [12] extended the idea of frame to Banach spaces in various ways. Atomic decomposition (AD) is one such extension of the notion of frame in Hilbert spaces to Banach spaces. In fact, Coifman and Weiss [4] were the first to suggest the idea of AD for some particular Function spaces. Feichtinger and Gröchenig extended the concept of AD to Banach spaces in [7]. The idea of a BF (BF) for a Banach Space is another extension of frames in Hilbert spaces. Gröchenig [12] was the first to propose the concept of the BF which is defined as follows:

Let $\{b_n\} \subset \mathcal{B}^*$ and $S : \mathcal{B}_d \to \mathcal{B}$ be an operator. The pair $(\{b_n\}, \mathcal{S})$ is called a BF for $\mathcal{B}$ with respect to $\mathcal{B}_d$, if

(1) $b_n(x) \in \mathcal{B}_d, \ \forall \ x \in \mathcal{B}$.
(2) one can find positive numbers $0 < A_l < A_u < \infty$ satisfying

$$A_l \| x \|_{\mathcal{B}} \leq \| \{b_n(x)\} \|_{\mathcal{B}_d} \leq A_u \| x \|_{\mathcal{B}}, \ \ x \in \mathcal{B}. \tag{2}$$

(3) S is a bounded linear operator satisfying $S(\{b_n(x)\}) = x, \ \ x \in \mathcal{B}$.

The numbers A_l and A_u, are listed as frame bounds of the BF $(\{b_n\}, \mathcal{S})$. $S : \mathcal{B}_d \to \mathcal{B}$ is called the reconstruction operator (RO). If $A_l = A_u$, then $(\{b_n\}, \mathcal{S})$ is called a tight frame for $\mathcal{B}$ and if $A_l = A_u = 1$, then $(\{b_n\}, \mathcal{S})$ is called a normalized tight BF for $\mathcal{B}$. The BF $(\{b_n\}, \mathcal{S})$ is termed as an exact BF if there is no reconstruction operator S_0 such that $(\{b_n\}_{i \neq n} S_0)$ $(i \in \mathbb{N})$ is a BF for $\mathcal{B}$.

Next, we give the definition of retro BF introduced by Jain et al. [14]

The pair $(\{g_n\}, \mathcal{T})$ $(\{g_n\} \subset \mathcal{B}, \mathcal{T}: \mathcal{B}^*_d \to \mathcal{B}^*)$ is called a retro BF for $\mathcal{B}^*$ w.r.t. $\mathcal{B}^*_d$, if

(1) $\{y(g_n)\} \in \mathcal{B}^*_d$, for each $y \in \mathcal{B}^*$.
(2) ∃ positive numbers A_1 and A_2 with $0 < A_1 \leq A_2 < \infty$ satisfying

$$A_1 \| y \|_{\mathcal{B}^*} \leq \| \{y(g_n)\} \|_{\mathcal{B}^*_d} \leq A_2 \| y \|_{\mathcal{B}^*}, \ \ y \in \mathcal{B}^*. \tag{3}$$

(3) $\mathcal{T}$ is a bounded linear operator such that $T(\{y(g_n)\}) = y$, $y \in \mathcal{B}^*$.

The numbers A_1 and A_2, are called retro BF bounds of the retro BF ($\{gn\}$, $\mathcal{T}$). $\mathcal{T}: \mathcal{B}^*_d \to \mathcal{B}^*$ is termed as the RO. The tight retro BF is defined as in case of BF. A sequence $\{g_n\} \subset \mathcal{B}$ is called a retro BF sequence if it is a retro BF for $\overline{span}\{g_n\}$.

For further details concerning frames in Banach spaces and allied topics, one may refer to [1, 3, 8, 9, 10, 11, 14, 15].

1.1. Outline of the paper

In this paper, we first recall what we mean by $\mathcal{A}$-subset and $\mathcal{A}$-modulus of $\mathcal{B}$. Examples are given to support their existence. Also, $\mathcal{A}$-cone in a real Banach space is defined and a result concerning the convergence of a sequence in an $\mathcal{A}$-cone satisfying some conditions is proved. Further, we defined atomic system in a Banach space $\mathbb{B}$ and proved that if particular subset $\mathcal{X}$ of $\mathbb{B}$ has an atomic system, then $\mathbb{B}$ has an atomic system.

2. Main Results

Let $\mathcal{B}$ be a real Banach space and $\mathcal{X}$ be any subset of $\mathcal{B}$. Then $\mathcal{X}$ is called an $\mathcal{A}$-subset of $\mathcal{B}$ if $\|u+v\|-1>0, \ \forall \ u,v \in \mathcal{X}$ with $\|u\| \geq 1, \|v\| > 0$. The real number $\eta_{\mathcal{X}}(t)$ defined as

$$\eta_{\mathcal{X}}(t) = \inf \left\{ \| u+v \| -1 : u,v \in \mathcal{X}, \| u \| \geq 1, \| v \| \geq t, t \in [0,\infty) \right\}$$

is called the $\mathcal{A}$-modulus of the set $\mathcal{X}$.

Example 2.1. Consider the Banach space $L^p\,[a, b]$. Then the subset $\mathcal{X} = \left\{ x \in L^p[a,b] : x \geq 0 \right\}$ is an $\mathcal{A}$-subset of $L^p\,[a, b]$ with $\mathcal{A}$-modulus $\eta_{\mathcal{X}}(t) = \left(1+t^p\right)^{1/p} - 1 \ (1 \leq p < \infty, t \in [0,\infty))$.

Example 2.2. Consider the Banach space $\mathcal{C}[a,b]$ of continuous functions on $[a,b]$. Then, for each $t' \in [a,b]$, the subset $\mathcal{X} = \left\{ f \in \mathcal{C}[a,b] : f(t') = \| f \| \right\}$ is an $\mathcal{A}$-subset of $\mathcal{C}[a,b]$ with $\mathcal{A}$-modulus $\eta_{\mathcal{X}}(t) = t, \ t \in [0,\infty)$.

Recall that the Rademacher functions $R_p(x)$ on $[a,b]$ are given by

$$\mathcal{R}_p(x) = sign\left(\sin(2^p \pi x)\right), \forall \ p \in \mathbb{N} \tag{4}$$

Example 2.3. Let $\{a_n\}$ be a sequence with $a_n = \pm 1, \ n \in \mathbb{N}$. Then $\mathcal{X} = \{a_n R_n : R_n$ are as defined in (4)$\}$ is an $\mathcal{A}$-subset of $L^p\,[a, b]$ ($1 < p \leq \infty$) However, one may verify that $\mathcal{X}$ is not an $\mathcal{A}$-subset of $L^1[a, b]$.

Definition 2.4. A subset $\mathcal{K}$ of a real Banach space $\mathcal{B}$ is called an $\mathcal{A}$-cone if $\mathcal{K}$ is an $\mathcal{A}$-subset of $\mathcal{B}$ and $\mathcal{K}$ is a closed subset of $\mathcal{B}$ satisfying $\mathcal{K}+\mathcal{K} \subset \mathcal{K}, \ \ \lambda\mathcal{K} \subset \mathcal{K} \ (\lambda \geq 0)$ and $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$.

Example 2.5. Let $\{\phi_n\}$ be a sequence of unit vectors in the real Hilbert space ℓ^2 and let $\mathcal{K} = \left\{ u \in \ell^2 : \phi_n(u) \geq 0, \forall n \in \mathbb{N} \right\}$ be a cone associated with $\{\phi_n\}$. Then $\mathcal{K}$ is an $\mathcal{A}$-cone and its $\mathcal{A}$-modulus is given by

$$\mathcal{M}_k(t) = \sqrt{1+t^2} - 1, \ \forall \ t \in [0,\infty).$$

Next, we prove a result related to the convergence of a sequence in an $\mathcal{A}$-cone satisfying certain conditions.

Theorem 2.6. Let $\mathcal{K}$ be an $\mathcal{A}$-cone in $\mathcal{B}$. If $\{g_n\}$ is any sequence in $\mathcal{K}$ such that, for an increasing sequence $\{k_n\}$ in $\mathbb{N}$, $\sup_{1 \leq n < \infty} \left\| \sum_{i=1}^{k_n} g_i \right\| \leq \alpha < \infty$, then $\lim_{k \to \infty} \sum_{i=1}^{k} g_i = g_0 \left(\in \mathcal{K} \right)$.

Proof. By hypothesis

$$\left\|\frac{u}{\|u\|}+\frac{v}{\|u\|}\right\|-1>0,\quad \forall\ u,v\in\mathcal{K}\setminus\{0\},$$

This gives

$$\|u+v\|>\|u\|,\quad \text{for all}\ \ u,v\in\mathcal{K}\setminus\{0\}.$$

Write

$$u=\sum_{i=1}^{k}g_i\ \ \text{and}\ \ v=g_{k+1},\ \forall\ k\in\mathbb{N}.$$

Then $\left\{\sum_{i=1}^{k}g_i\right\}_{k=1}^{\infty}$ is a non-decreasing sequence in $\mathcal{K}$. Therefore, there exists a $\beta<\alpha$ such that

$$\beta=\lim_{k\to\infty}\left\|\sum_{i=1}^{k}g_i\right\| \tag{5}$$

Let if possible $\beta\to\infty$. Then, one can find an $\epsilon>0$ and an increasing sequence $\{p_n\}$ in $\mathbb{N}$ satisfying

$$\left\|\sum_{i=p_k+1}^{p_{k+1}}g_i\right\|\geq\epsilon,\ \forall\ n\in\mathbb{N} \tag{6}$$

Note that $\beta\neq 0$ and so for any δ with $0<\delta<\beta$ there exists a positive integer say $m(\epsilon)$ such that

$$\left\|\sum_{i=1}^{p_k}g_i\right\|\geq\beta-\delta,\ \forall\ k\geq m$$

Therefore, we obtain

$$a_1=\frac{\epsilon}{\beta}\leq\frac{\left\|\sum_{i=p_k+1}^{p_{k+1}}g_i\right\|}{\left\|\sum_{i=1}^{p_k}g_i\right\|}=a_2.$$

This gives

$$\left\|\sum_{i=1}^{p_{m+1}}g_i\right\|=\left\|\sum_{i=1}^{p_m}g_i\right\|\left\|\frac{\sum_{i=1}^{p_m}g_i}{\left\|\sum_{i=1}^{p_m}g_i\right\|}+\frac{\sum_{i=p_m+1}^{p_{m+1}}g_i}{\left\|\sum_{i=1}^{p_m}g_i\right\|}\right\|\geq(\beta-\delta)\left[1+\mathcal{M}_{\mathcal{K}}\left\|\frac{\sum_{i=p_m+1}^{p_{m+1}}g_i}{\left\|\sum_{i=1}^{p_m}g_i\right\|}\right\|\right]\geq(\beta-\delta)\left(1+\mathcal{M}_{\mathcal{K}}\left(\frac{\epsilon}{\beta}\right)\right)$$

Since $\mathcal{M}_{\mathcal{K}}\left(\frac{\epsilon}{\beta}\right)>0$ for sufficiently small $\delta>0$, $(\beta-\delta)\left(1+\mathcal{M}_{\mathcal{K}}\left(\frac{\epsilon}{\beta}\right)\right)>\beta$.

This is a contradiction. So $\lim_{k\to\infty}\sum_{i=1}^{k}g_i<\infty$.

Since $\mathcal{K}$ is closed, there exists an element say $g_0\in\mathcal{K}$ such that $\lim_{k\to\infty}\sum_{i=1}^{k}g_i=g_0$.

Corollary 2.7. Let $\mathcal{K}$ be an $\mathcal{A}$-cone in a real Banach space $\mathcal{B}$. If $\{g_n\}$ and $\{h_n\}$ are sequences in $\mathcal{K}$ satisfying

$$\lim_{n\to\infty}\left(\sum_{j=1}^{n}g_j+h_n\right)=g, \tag{7}$$

then $lim_{n\to\infty}\sum_{i=1}^{n}g_i=f\in\mathcal{K}$.

Moreover, if $\|g\|=\|f\|$, then $g=f$.

Proof. In view of Theorem 2.6, it is enough to prove that

$$\left\|\sum_{j=1}^{n} g_j\right\| \leq \| g \|, \quad \text{for all} \quad n \in \mathbb{N}.$$

Let if possible for some p, $\left\|\sum_{j-1}^{p} g_j\right\| \geq \| g \|$. Then, using a property of $\mathcal{A}$-cone, for $\sum_{j=1}^{p} g_j$ and $\sum_{j=p+1}^{n} g_j + h_n$ in $\mathcal{K}$, we have

$$\left\|\sum_{j=1}^{n} g_j + h_n\right\| \geq \left\|\sum_{j=1}^{p} g_j\right\| > \|g\|, (n > p).$$

This contradicts (7).

Suppose that $\|f\| = \|g\|$. Let $\epsilon > 0$ be such that, for some increasing sequence $\{k_n\}$ of natural numbers, $\| h_{k_j} \| > \epsilon, \forall\, j \in \mathbb{N}$. Then for a suitably chosen δ $(0 < \delta < \| g \|)$ and a positive integer $m(\delta)$, we have

$$\left\|\sum_{j=1}^{k_n} g_j + h_{k_n}\right\| \geq \left(\| g \| - \delta\right)\left(1 + \mathcal{K}_{\mathcal{X}}\left(\frac{\epsilon}{\| \delta \|}\right)\right) > \| f \|, \ \forall\ n \geq m(\delta).$$

Since $\|f\| = \|g\|$ and by (7), we compute

$$\| f \| = \| g \| = \lim_{n \to \infty} \left\|\sum_{j=1}^{k_n} g_j + y_{k_n}\right\| > \| f \|.$$

This is a contradiction.

From now onwards, by $\mathbb{B}$, we mean a separable Banach space. Next, we define an atomic system for a Banach space $\mathbb{B}$ as follows:

Let $\mathbb{B}$ be a Banach Space and and $\mathcal{X}$ be a subset of $\mathbb{B}$. Then $\left(\{x_n\}, \{f_n\}\right)$, $\{x_n\} \subseteq \mathcal{X}, \{f_n\} \subseteq \mathcal{X}^*$ is called an atomic system for $\mathcal{X}$ with respect to $\mathcal{X}_d$ if

(1) $\{f_n(x)\} \in \mathcal{X}_d$, $x \in \mathcal{X}$

(2) $\exists$ positive constants A and B such that

$$A \| x \|_{\mathcal{X}} \leq \| \{f_n(x)\} \|_{\mathcal{X}_d} \leq B \| x \|_{\mathcal{X}}, x \in \mathcal{X}$$

(3) $x = \sum_{n=1}^{\infty} f_n(x) x_n$, $x \in \mathcal{X}$

Let $\mathbb{B}$ be a Banach Space. Define

$$\mathcal{X} = \left\{x \in \mathbb{B} : f(x) = c, \text{ where } f \in \mathcal{X}^* \text{ and } c \neq f(0) = 0\right\}.$$

Then $\mathcal{X}$ is a Banach Space. Also, define

$$\mathcal{X}_d = \left\{\{f_n(x)\} : x \in \mathcal{X} \text{ and} \{f_n\} \in \mathcal{X}^*\right\}.$$

Then $\mathcal{X}_d$ is also a Banach Space.

In the following result, we prove that if $\Psi = \left(\{x_n\}, \{f_n\}\right)$ is an atomic system for $\mathcal{X}$, then ψ is also an atomic system for $\mathbb{B}$. More precisely, we prove that

Theorem 2.8. If $\Psi = \left(\{x_n\}, \{f_n\}\right)$ is an Atomic System for $\mathcal{X}$ with respect to $\mathcal{X}_d$, then ψ is an Atomic System for $\mathbb{B}$ with respect to $\mathbb{B}_d = \left\{\{f_n(x)\} : x \in \mathbb{B},\ f_n^{'s} \in \mathcal{X}^*\right\}$.

Proof. Suppose first that $\Psi = \left(\{x_n\}, \{f_n\}\right)$ is an Atomic System for $\mathcal{X}$ with respect to $\mathcal{X}_d$. Then

$$\{f_n(x)\} \in \mathcal{X}_d,\ x \in \mathcal{X}. \tag{8}$$

∃ a positive constants A and B such that

$$A \| x \|_{\mathcal{X}} \leq \| \{f_n(x)\} \|_{\mathcal{X}_d} \leq B \| x \|_{\mathcal{X}}, x \in \mathcal{X} \tag{9}$$

and

$$x = \sum_{n=1}^{\infty} f_n(x) x_n, \; x \in \mathcal{X}. \tag{10}$$

Let $y \in \mathbb{B}$ be any element such that $f(y) \neq 0$. Then

$$\frac{c}{f(y)}.y \in \mathcal{X}.$$

So

$$\left\{ f_n\left(\frac{c.y}{f(y)} \right) \right\} \in \mathcal{X}_d \subset \mathbb{B}_d.$$

i.e

$$\{f_n(y)\} \in \mathbb{B}_d . A \left\| \frac{c.y}{f(y)} \right\| \leq \left\| \{f_n(y)\} \right\|_{\mathbb{B}_d} \frac{|c|}{|f(y)|} \leq B \frac{|c|}{|f(y)|} \|y\|, y \in \mathbb{B}$$

i.e

$$A \| y \|_{\mathbb{B}} \leq \| \{f_n(y)\} \|_{\mathbb{B}_d} \leq B \| y \|_{\mathbb{B}}, y \in \mathbb{B}$$

$$\frac{c}{f(y)} y = \sum_{n=1}^{\infty} f_n \left(\frac{c}{f(y)} y \right) x_n, y \in \mathbb{B}$$

or

$$y = \sum_{n=1}^{\infty} f_n(y) x_n, y \in \mathbb{B}.$$

Also, if $y \in \mathbb{B}$ such that $f(y) = 0$, then, by definition of $\mathcal{X}$, $y + y_0 \in \mathcal{X}$, for any $y_0 \in \mathcal{X}$ such that y_0 satisfy (8),(9), and (10).
Since

$$\{f_n(y + y_0)\} = \{f_n(y_0)\} \in \mathcal{X}_d,$$

We have

$$A \|y + y_0\|_{\mathcal{X}} \leq \left\| \{f_n(y_0)\} \right\|_{\mathcal{X}_d} \leq B \|y + y_0\|_{\mathcal{X}}.$$

Thus, for every $y \in \mathbb{B}$, ∃ a $y_0 \in \mathcal{X}$ such that $y = y_0$.
Hence, $\Psi = (\{x_n\}, \{f_n\})$ is an Atomic System for $\mathbb{B}$.

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