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On number of ordered pair of positive integers with given least common multiple

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In this article, we present an expression for the number of ordered pairs of positive integer m with prime factorization $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, where $\alpha_i \in \mathbb{Z}^+$ and en route introduce an identity viz:

$$\prod_{i=1}^{n} (2\alpha_i + 1) = \sum (\alpha_1 + 1)\alpha_2 \ldots \alpha_n + \sum (\alpha_1 + 1)(\alpha_2 + 1)\alpha_3 \ldots \alpha_n + \cdots + \sum (\alpha_1 + 1)(\alpha_2 + 1)\ldots(\alpha_r + 1)\alpha_{r+1} \ldots \alpha_n + \cdots + \alpha_1 \alpha_2 \ldots \alpha_n$$

Moreover, we investigate the asymptotic behavior of the mean and variance of relative number of order pairs with some given least common multiple m. The notion of ordered pair is widely used in the fields of geometry, statistics, computing and programming languages.

Key words and phrases: Ordered pairs, Least Common Multiple (LCM), Counting techniques, Number theoretic functions

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1. Introduction

The study of the multiplicative structure of integers has been a fundamental topic in number theory since ancient times. A central result in this area is the Fundamental Theorem of Arithmetic, also known as the unique prime factorization theorem, which states that every positive integer $m > 1$ can be represented in exactly one way as a product of prime powers [4, 5] in the following fashion

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$$m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} = \prod_{i=1}^{k} p_i^{\alpha_i}$$

where $\alpha_i \in \mathbb{Z}^+$ and $p_1 < p_2 < \cdots < p_k$ are the distinct prime factors of n. This representation is commonly extended to all positive integers, including 1, by the convention that the empty product ($k = 0$) is equal to 1. This theorem has far-reaching consequences and has led to numerous important concepts and results in number theory, such as the notion of the greatest common divisor (GCD) and least common multiple (LCM) of two integers.

2. Some important Properties of lcm of {1,2, . . , n}

The LCM of a set of integers is a fundamental concept with applications in various areas of mathematics and computer science. For instance, in computational geometry, the LCM is used to determine the period of lattice points in space [8]. In cryptography, the LCM plays a crucial role in the construction of certain cryptosystems based on the arithmetic of finite fields [8]. The distribution of LCMs and related arithmetic functions has also been extensively studied in analytic number theory [3, 9, 11]

We know that the Prime Number Theorem (PNT) provides a foundation for understanding the distribution of prime numbers, stating that the number of primes less than or equal to x is asymptotically equivalent to $\frac{x}{\log x}$. Before moving on to our result, we find it pertinent to review some important results involving the asymptotic behaviour of log lcm (1, 2, . . . , n) and its connections with Prime Number Theorem.

We consider the LCM of the first n natural numbers:

$$\text{lcm }(1,2,\ldots,n) = \prod_{p \le n} p^{\lfloor \log_p(n) \rfloor}$$

where p are primes and $\lfloor \log_p(n) \rfloor$ is the highest power of p that divides any number from 1 to n. Taking logarithms on both sides, we get:

$$\log \text{lcm}(1,2,\ldots,n) = \sum_{p \le n} \lfloor \log_p(n) \rfloor \log p$$

Approximating $\lfloor \log_p(n) \rfloor$ by $\log_p(n)$ for large n, we simplify this to:

$$\log \text{lcm}\,(1,2,\ldots,n) \approx \sum_{p \le n} \log_p(n) \log p = \sum_{p \le n} \log n = \log n \cdot \pi(n)$$

where $\pi(n)$ is the number of primes less than or equal to n. Using the Prime Number Theorem, $\pi(n) \sim \frac{n}{\log n}$, so:

$$\log n \cdot \pi(n) \sim \log n \cdot \frac{n}{\log n} = n$$

Thus, we conclude that:

$$\log \text{lcm}(1,2,\ldots,n) \sim n \quad \text{as} \quad n \to \infty \tag{1}$$

Indeed more precise estimates of log lcm {1,2, . . .,n} are equivalent to Prime Number Theorem with an error term [6, 2]. Specifically, if lcm (1, ..., n) = $e^{\psi(n)}$, where ψ is the Chebyshev's function, then $\psi(x) = x + o(x)$, as $x \to +\infty$, which is equivalent to Prime Number Theorem [6]. From expression (1), we can say prime number theorem can be phrased as the statement that the least common multiple (lcm) of the first n natural numbers is asymptotic to the exponential of n. Sury in [2] establishes the following two elegant properties of the lcm of first n natural numbers:

$$2^n < lcm\{1,2,3,\cdots, n\} < 4^n \tag{2}$$

and

$$\text{lcm}\ (1,2,3,\cdots n) = \text{lcm}\left(2\binom{n}{2}, 3\binom{n}{3}, \cdots, n\binom{n}{n}\right) \tag{3}$$

The above mentioned few properties of the lcm of {1, 2, . . . n} demonstrate the profound connection between the behavior of primes and the structure of least common multiples among natural numbers up to n.

In this paper, we investigate a particular problem related to the LCM: the enumeration of ordered pairs of positive integers with a given LCM. Specifically, we derive an explicit formula for the number of such ordered pairs, and establish an interesting identity involving symmetric functions of the exponents α_i in the prime factorization. Moreover, we study the asymptotic behavior of the mean and variance of the relative number of ordered pairs with LCM m as $m \to \infty$. Our results not only shed light on the multiplicative structure of integers, but also have potential applications in areas such as cryptography and the analysis of algorithms.

3. Main Results

Theorem 3.1. Let m be a positive integer with prime factorization $m = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}$. Then the total number of ordered pair of positive integers (a, b) with lcm (a, b) equals to m is $\prod_{i=1}^{n}(2\alpha_i + 1)$. Moreover, if $s : \mathbb{R}^n \to \mathbb{R}$ be a function such that $s(t_1, t_2, \dots, t_n) = t_1 t_2 \cdots t_n$, $t_i \in \mathbb{R}$ the following identity holds:

$$\sum_{r=0}^{n} s(\alpha'_{x_1}, \alpha'_{x_2}, \dots \alpha'_{x_r}, \alpha_{x_{r+1}}, \dots, \alpha_{x_n}) = \prod_{i=1}^{n}(2\alpha_i + 1)$$

where $(\alpha_{x_1}, \alpha_{x_2}, \dots, \alpha_{x_n})$, is some permutation of $(\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\alpha'_{x_i} = \alpha_{x_i} + 1$, with $\forall i = 1, 2, \dots, n$.

Proof. For the given positive integer $m = p_1^{\alpha_1}.p_2^{\alpha_2} \dots p_n^{\alpha_n}$, let $f(m)$ denote the number of ordered pairs of positive integers (a, b) with least common multiple lcm (a, b) = m. Let

$$[a = p_1^{\beta_1} p_2^{\beta_2} \dots p_n^{\beta_n} \text{ and } b = p_1^{\gamma_1} p_2^{\gamma_2} \dots p_n^{\gamma_n} \quad \text{where } \beta_i, \gamma_i \in \mathbb{Z}^+]$$

Noting both a and b divide m, and

$$[\text{lcm}(a,b) = p_1^{\max(\beta_1,\gamma_1)} p_2^{\max(\beta_2,\gamma_2)} \dots p_n^{\max(\beta_n,\gamma_n)},]$$

it follows that both β_i and γ_i divide α_i. The possible values of β_i and γ_i are 0, 1, 2, . . ., α_i. Hence the total number of ways to choose both β_i and γ_i are $2\alpha_i + 1$. Thus the total number of ordered pair (a, b) is given by

$$f(m) = \prod_{i=1}^{n}(2\alpha_i + 1) \tag{4}$$

Next, we count the number of ordered pair of positive integer (a, b) with $\text{lcm}(a,b) = m = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}$ in an alternative way. Before proceeding to how we can achieve it in general form, let us consider the proposed scheme for the case $n = 3$ i.e, $m = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$. Suppose ($a$, b) is any ordered pair of positive integers such that lcm (a, b) = m. Choose the value of a, which has maximum possible exponent, so that

$$a = p_1^{\alpha_1} p_2^{x} p_3^{y} \text{ where } x = 0,1,2,\dots,\alpha_2 - 1 \text{ and } y = 0,1,2,\dots,\alpha_3 - 1 \tag{5}$$

It obviously follows from the above equation that a can be chosen in $\alpha_2 \times \alpha_3$ ways so that the corresponding b can be chosen as

$$b = p_1^{z}.p_2^{\alpha_2}.p_3^{\alpha_3} \text{ where } z = 0,1,2,\dots,\alpha_1 \tag{6}$$

giving us $(\alpha_1 + 1)$ choices for b. All such eight possibilities have been summarized in the following Table 1. Let A denotes the exponent of p_i that is one less than maximum possible, which for p_i will be $\alpha_i - 1$ and M denotes the maximum exponent, which for p_i is α_i, $i = 1, 2, 3$.

Table 1: All Possibilities of p_i

Exponent of p_1	Exponent of p_2	Exponent of p_3	No. of ordered pair with given LCM
M	M	A	$\alpha_3 (\alpha_1 + 1) (\alpha_2 + 1)$
A	M	M	$\alpha_1 (\alpha_2 + 1) (\alpha_3 + 1)$
M	A	M	$(\alpha_1 + 1) \alpha_2 (\alpha_3 + 1)$
A	M	A	$\alpha_1 (\alpha_2 + 1) \alpha_3$
A	A	M	$\alpha_1\alpha_2 (\alpha_3 + 1)$
M	M	A	$(\alpha_1 + 1) \alpha_2\alpha_3$
M	M	M	$(\alpha_1 + 1) (\alpha_2 + 1) (\alpha_2 + 1)$
A	A	A	$\alpha_1\alpha_2\alpha_3$

In general, suppose

$$m = p_1^{\alpha_1}.p_2^{\alpha_2} \dots p_n^{\alpha_n}$$

Let $a, b \in \mathbb{Z}^+$ such that lcm $(a, b) = m$ with a containing maximum possible exponents in the primes $p_{i_{x_1}}, p_{i_{x_2}}, \dots, p_{i_{x_n}}$. Then the number of such ordered pairs (a, b) is given by

$$(\alpha_{x_1} + 1)(\alpha_{x_2} + 1)\cdots(\alpha_{x_r} + 1).\alpha_{x_{r+1}} \cdots \alpha_{x_n} \tag{7}$$

where $(\alpha_{x_1} \dots \alpha_{x_n})$ is some permutation of $(\alpha_1, \dots, \alpha_n)$. By taking

$$\alpha_{x_i} + 1 = \alpha'_{x_i}$$

the above expression (7) can be written as

$$(\alpha_{x_1} + 1)(\alpha_{x_2} + 1)\cdots(\alpha_{x_r} + 1).\alpha_{x_{r+1}} \cdots \alpha_{x_n} = \alpha'_{x_1}.\alpha'_{x_2} \dots \alpha'_{x_r}.\alpha_{x_{r+1}} \dots \alpha_{x_n} \tag{8}$$

Let us define a function $s : \mathbb{R}^n \to \mathbb{R}$, defined by

$$s(t_1, t_2, \dots, t_n) := t_1 t_2 \cdots t_n$$

Then the expression (8) become

$$s(\alpha'_{x_1}, \alpha'_{x_2}, \dots, \alpha'_{x_r}, \alpha_{x_{r+1}}, \dots, \alpha_{x_n}) = \alpha'_{x_1}.\alpha'_{x_2} \dots \alpha'_{x_r}.\alpha_{x_{r+1}} \dots \alpha_{x_n}$$

Since r varies from 0 to n, the required count of all expressions of form is given by

$$\sum_{r=0}^{n} s(\alpha'_{x_1}, \alpha'_{x_2}, \dots \alpha'_{x_r}, \alpha_{x_{r+1}}, \dots, \alpha_{x_n}) \tag{9}$$

Thus, the expression (9) gives us the number of ordered pair of positive integer (a, b) with lcm (a, b) $= m$ where $m = p_1^{\alpha_1}.p_2^{\alpha_2} \dots p_n^{\alpha_n}$.

Now in order to establish the identity (1), we need to show that the number of ordered pair of positive integer is given $m = p_1^{\alpha_1}.p_2^{\alpha_2} \dots p_n^{\alpha_n}$.

Therefore, by combining the expressions (4) and (9), we get

$$\sum_{r=0}^{n} s(\alpha'_{x_1}, \alpha'_{x_2}, \dots \alpha'_{x_r}, \alpha_{x_{r+1}}, \dots, \alpha_{x_n}) = \prod_{i=1}^{n} (2\alpha_i + 1) \tag{10}$$

Using symmetric expressions, the LHS of the equation (10) can be written as

$$\prod_{i=1}^{n}(2\alpha_i+1)=\Sigma(\alpha_1+1)\alpha_2\ldots\alpha_n+\Sigma(\alpha_1+1)(\alpha_2+1)\alpha_3\ldots\alpha_n+\cdots+ \\ +\Sigma(\alpha_1+1)(\alpha_2+1)\ldots(\alpha_r+1)\alpha_{r+1}\ldots\alpha_n+\cdots+\alpha_1\alpha_2\ldots\alpha_n \tag{11}$$

which establishes the theorem. □

For the demonstration of the proposed identity, we shall present an illustrative example.

Example 3.2. *Let* $m=72=2^3\cdot 3^2$, *then* $\alpha_1=3,\alpha_2=2$ *and number of distinct primes n = 2. Using expression (11), L.H.S of equation (10) is given by*

$$\begin{aligned}&\sum_{r=0}^{n}s(\alpha'_{x_1},\alpha'_{x_2},\ldots\alpha'_{x_r},\alpha_{x_{r+1}},\ldots,\alpha_{x_n})\\ &=\alpha_1\alpha_2+\alpha_1(\alpha_2+1)+(\alpha_1+1)\alpha_2+(\alpha_1+1)(\alpha_2+1)\\ &=3\cdot 2+3\cdot 3+4\cdot 2+4\cdot 3\\ &=6+9+8+12\\ &=35.\end{aligned}$$

Also, R.H.S of the equation (10) becomes

$$\prod_{i=1}^{n}(2\alpha_i+1)=(2\cdot 3+1)(2\cdot 2+1)=7\times 5=35.$$

Remark 3.3. *It is worth mentioning that by taking* $\alpha_1=\alpha_2=\cdots=\alpha_n=\alpha$, *the expression (11) yields*

$$(2\alpha+1)^n=\sum_{r=0}^{n}\binom{n}{r}(\alpha+1)^r\alpha^{n-r}$$

which boils down to the Binomial expansion of $(\alpha+\beta)^n$ *with* $\beta=\alpha+1$. *For more identities of this type, one may refer to [1, 10].*

4. Analysis and Properties of *f*(*m*)

In this section, we investigate the asymptotic behaviour of the mean and variance of relative number of order pairs with least common multiple m.

Let $S_m=\{1, 2, ..., m\}$ and $L_m=\{(x, y) : \text{lcm}(x, y)=m \text{ and } x, y\in S_m\}$.

Then obviously $L_m\subset S_m\times S_m$ so that $f(m)\le m^2$. Hence

$$0<\frac{f(m)}{m^2}\le 1$$

Noting the formula for $f(m)$, we see that f is neither monotonically increasing nor monotonically decreasing. However we have the following theorem.

Theorem 4.1. Let $m\in\mathbb{Z}^+$ and $f(m)$ denote the number of ordered pairs of positive integers (a, b) with lcm $(a, b)=m$. Then we have

$$\lim_{m\to\infty}\frac{f(m)}{m^2}=0$$

Proof. To prove the above claim, we exploit the relation between $f(m)$ and the number of divisors of m^2. Let $m=\prod_{i=1}^{k}p_i^{\alpha_i}$. Then $m^2=\prod_{i=1}^{k}p_i^{2\alpha_i}$. Therefore, numbers of divisors of m^2 is given by

$$\tau(m^2)=\prod_{i=1}^{m}(2\alpha_i+1)$$

Therefore, using Theorem (3.1), we have

$$f(m) = \tau(m^2)$$

Next, we show that the number of divisors of m^2 can at the most be $2m$–1. For, if i divides m^2, so does m^2/i and for $i = 1, 2, \dots, m-1, i \neq \frac{m^2}{i}$. This shows that the number of divisors of m^2 can at most be $2m$–1. Therefore

$$f(m) \leq 2m{-}1$$

Hence,

$$\lim_{m\to\infty} \frac{f(m)}{m^2} = 0.$$

□

To have an idea of the variance of the sequence $\left\langle \frac{f(m)}{m^2} \right\rangle$, we use Cauchy's theorem on limits [12], which states that it if a sequence $\langle a_m \rangle$ converge to l, so does the sequence of arithmetic means given by

$$y_m = \frac{a_1 + a_2 + \cdots + a_m}{m}$$

Initially, we compute the mean of first m terms. Let $a_m = \frac{f(m)}{m^2}$. Then as established in the above Theorem (4.1), we have $\lim_{m\to\infty} a_m = 0$. Hence,

$$\frac{a_1 + a_2 + \ldots + a_m}{m} = 0$$

Therefore the limiting mean $\bar{a}$ of the sequence $\langle a_m \rangle$ is zero.

The limiting variance of the sequence is given by

$$\begin{aligned} V &= \lim_{m\to\infty} \frac{\sum_{i=0}^{m} (a_i - \bar{a})^m}{m} \\ &= \lim_{m\to\infty} \frac{a_1^2 + a_2^2 + \cdots + a_m^2}{m} \\ &= \lim_{m\to\infty} a_m^2 \\ &= 0 \end{aligned}$$

Remark 4.2. *It is worth noting that that f(m) can be linked to apparently two disparate counting problems. First, f(m) can also be used to compute the number of ways of making a rectangle of area* m^2 *with integer sides. Since for every divisor i of* m^2*, we have a rectangle of sides i and* $\frac{m^2}{i}$*, we conclude there are total of*

$$\frac{\tau(m^2) + 1}{2}$$

rectangular arrangements.

Second, and perhaps more interestingly, given an integer resistance m the number of ways of picking up two resistances x and y which produce a combined resistance of m is given by

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{m}$$

The above equation can be written as

$$XY = m^2$$

where $X = x - m$ *and* $Y = y - m$*. This facilitates an one-one correspondence between solutions (X, Y) and and the solutions (x, y) of the parallel resistance equation, so that clearly, there are total of* $\tau(m^2)$ *solutions, which is nothing but f(m).*

The variation of $f(n)/n^2$ with respect to n is depicted in Figure 1. Moreover, the variation of moving average and moving variance of $f(n)/n^2$ with respect to n is respectively shown in Figures 2 and 3.

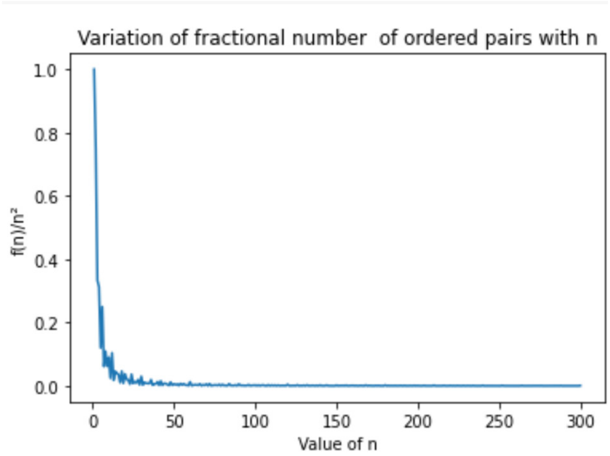

Figure 1: Variation of $f(n)/n^2$ with n.

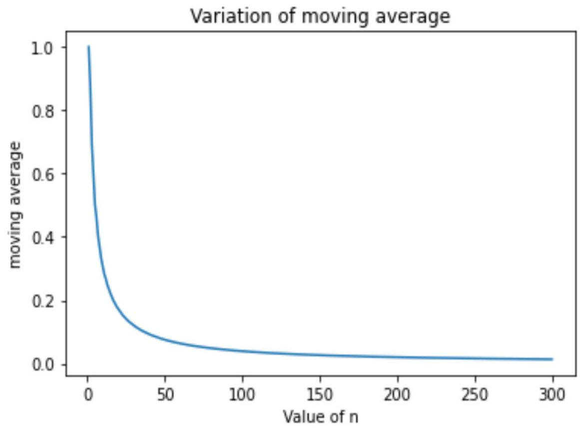

Figure 2: Variation of moving average of $f(n)/n^2$ with n.

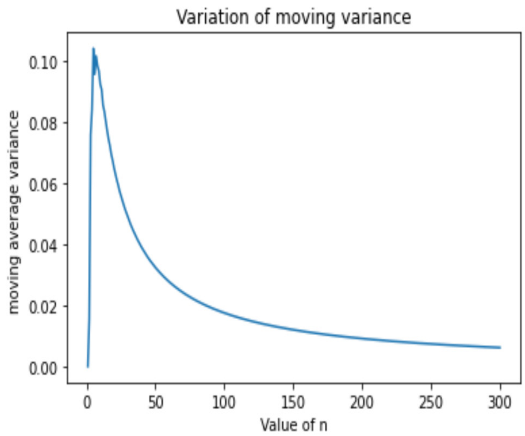

Figure 3: Variation of moving variance of $f(n)/n^2$ with n

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Appendix: Python Code for Simulation

The graphical and computational study of the sequence $f(n)/n^2$ has been carried out through the python code presented as follows.

Listing 1: Python Code for Simulation

```
import numpy as np
import matplotlib.pyplot as plt

#code for plotting f(n)/n^2 wrt n
y = np.array([])
limit = 301
for k in range(1, limit):
    c = 0
    for i in range(1, k+1):
        for j in range(1, k+1):
            if np.lcm(i, j) == k:
                c = c + 1
    y = np.append(y, c/(k**2))

x = np.arange(1, limit)
plt.plot(x, y)
plt.title("Variation of fractional number of ordered pairs with n")
plt.xlabel("Value of n")
plt.ylabel("f(n)/ n ")
plt.show()

# code for plotting moving average
w = []
for i in range(1, limit):
    w.append(np.average(y[:i]))

plt.plot(x, w)
plt.title("Variation of moving average")
plt.xlabel("Value of n")
plt.ylabel("moving average")
plt.show()

# code for plotting moving variance
vary = []
for i in range(1, limit):
    vary.append(np.var(y[:i]))

plt.plot(x, vary)
plt.title("Variation of moving variance")
plt.xlabel("Value of n")
plt.ylabel("moving average variance ")
plt.show()
```