Results in Nonlinear Analysis 7 (2024) No. 4, 1–8
https://doi.org/10.31838/rna/2024.07.04.001
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

A reliable algorithm for solving Blasius boundary value problem

Kamel Al-Khaled[1]*, Mahmood Shareef Ajeel[2], Amer Darweesh[1], Hala Al-Khalid[3]

[1]*Department of Mathematics and Statistics, Jordan University of Science and Technology, Irbid 22110, Jordan;* [2]*Department of Material Engineering - College of Engineering, Shatrah University, Thi-Qar 64001, Iraq;* [3]*Department of Mathematics, Western Michigan University, Kalamazoo MI 49008-5248, USA.*

The Blasius equation is a well-known third-order nonlinear ordinary differential equation that can be found in some fluid dynamics boundary layer problems. In this paper, we convert the nonlinear differential equation to an integral equation, this integral equation has a shifted kernel. Our goal is to propose an efficient modification of the standard Adomian decomposition method, combined with the Laplace transform, for solving the Blasius equation. The main impediment to solving the Blasius equation is the absence of the second derivative at zero. Once this derivative has been correctly evaluated, an analytical solution to the Blasius problem can be easily found; as a result, we use our approximate solution to estimate the value of $y''(0)$, also known as the Blasius constant. Understanding the Blasius constant is essential for calculating shear stress at a plate. Furthermore, once this value is determined, we have the initial value problem, which can be solved numerically.

Key words and phrases: Nonlinear differential equations, Blassius equation, Approximate solutions, Adomian decomposition method, laminar flow

Mathematics Subject Classification (2010): 34B15, 30E25, 33F05, 76F40, 74G10.

1. Introduction

Blasius [1] introduced the Blasius equation, a third-order nonlinear two-point boundary value problem, which serves as a cornerstone in understanding the steady flow of viscous incompressible fluids over semi-infinite flat plates. This equation, formulated as:

Email addresses: kamel@just.edu.jo (Kamel AL-Khaled)*; ahdarweesh@just.edu.jo (Amer Darweesh); mahmoodshareef@shu.edu.iq (Mahmood Shareef Ajeel); halakamelmustafa.alkhalid@wmich.edu (Hala Al-Khalid)

Received May 4, 2024; Accepted July 7, 2024; Online August 17, 2024

$$ay'''(t) + y(t)y''(t) = 0, 0 < t < \infty \tag{1}$$

with boundary conditions:

$$y(0) = 0, y'(0) = 0, y'(\infty) = 1, \tag{2}$$

describes the non-dimensional velocity distribution within the laminar boundary layer of such plates. Here, a represents a constant dependent on the fluid and the plate's characteristics, while t signifies the dimensionless distance parameter and $y(t)$ denotes a dimensionless function. This equation encompasses two classical forms, corresponding to $a = 1$ and $a = 2$, respectively.

While exact analytical solutions for the Blasius equation are challenging to obtain across its entire domain due to its nonlinear nature, several approximation methods have been developed. Significantly, researchers have developed practical solution methods such as the variational iteration method [2,3], explicit approximate solutions [4] based on the fixed point theorem [5], and the Adomian decomposition method [6]. Moreover, innovative techniques like the Adomian Mohand transform method [7] have enhanced our understanding and precision in addressing the Blasius problem.

Moreover, recent advancements include novel approaches such as semi-numerical schemes [6], analysis utilizing the Laplace transform [9], and investigations into laminar flow behavior [11, 12]. Homotopy perturbation methods [13] and various other studies [14, 15, 16, 17] have also made significant contributions to this domain. The reader can consult earlier studies that offered comparable approaches to solving nonlinear differential equations, such as [18], and others that offered novel approaches is the RCW method (Rahmanzadeh-Cai-White) [19]. The RCW method is among the most appealing and successful approaches for solving ordinary differential equations numerically.

The main innovation of our research lies in our unique approach to transforming the traditional Blasius equation into an integral equation, which incorporates a shifted kernel. This transformation is significant because it allows us to apply the Laplace transform and the Adomian decomposition method together in a new way. By combining these two methods, we can tackle the problem more effectively, providing clearer insights and potentially more accurate solutions than those derived from standard techniques. This approach has not been explored in previous studies, marking a novel contribution to the literature in this area. The method employed in our research provides a significant advantage in addressing the nonlinear challenges of the Blasius equation. Unlike exact solutions, which are complex and difficult to derive due to the equation's nature, our method offers a reliable approximate solution. It simplifies the process and achieves high accuracy in the results, making it not only effective but also user-friendly for computations. This approach allows for more practical applications and easier integration into further research, enhancing both the utility and accessibility of solving such complex equations.

Building on these developments, our paper introduces a systematic approach to solving the Blasius problem. We employ the Adomian decomposition method alongside the Laplace Transform, aiming to provide a practical and accurate solution strategy. Through this methodology, we aim to contribute further to the understanding of this fundamental fluid dynamics equation.

2. ADM Linked with Laplace

The main goal in this part is to use the Adomian decomposition technique combined with the Laplace transform. It is important to mention that dealing with integral equations in the presence of the shitting property simplifies the use of the Laplace transform. As a result, our primary concern here is to rewrite Equation (1) as a convolution integral equation. Since $y''(t) \neq 0$, because if so, the boundary conditions are not satisfied, we may divide both sides of equation (1) by $ay''(t)$ to obtain the equivalent form as:

$$\frac{(ay''(t))'}{ay''(t)} = -\frac{y(t)}{a}, t \geq 0,$$

If we integrate both sides of the preceding equation over the interval [0, t], we get

$$a\, y''(t) = \frac{C}{a}\exp\left(-\int_0^t \frac{y(\tau)}{a}\,d\tau\right) t \geq 0,$$

where $C = ay''(0)$ is a constant to be determined later. Next, integrate the above equation and use the condition $y'(0) = 0$, we obtain

$$y'(t) = \frac{C}{a}\int_0^t \exp\left(-\int_0^\xi \frac{y(\tau)}{a}\,d\tau\right) d\xi,\, t \geq 0. \tag{3}$$

Integration by parts of the previous equation and using the given condition $y(0) = 0$, we get a nonlinear Volterra integral equation in convolution type of the form

$$y(t) = \frac{C}{a}\int_0^t (t-\xi)\exp\left(-\int_0^\xi \frac{y(\tau)}{a}\,d\tau\right) d\xi,\, t \geq 0. \tag{4}$$

To obtain a formula for the constant C, we apply the third boundary condition $y'(\infty) = 1$ to the equation (3), which yields

$$C = a \,/ \int_0^\infty \exp\left(-\int_0^t \frac{1}{a}\, y(\tau)\,d\tau\right) dt \tag{5}$$

The nonlinear Volterra integral equation of the first kind (4) includes both the linear term $y(x)$ and the nonlinear function

$$F\left(y(t)\right) = \exp\left(-\int_0^t \frac{y(\tau)}{a}\,d\tau\right) = \sum_{n=0}^{\infty} \mathcal{A}_n(t). \tag{6}$$

The decomposition series can normally represent the linear term $y(t)$ of (4). The Adomian decomposition method (ADM) assumes a series solution for equation (4) in the form [6]

$$y(t) = \sum_{n=0}^{\infty} y_n(t) \tag{7}$$

where the components $y_n(t)$, $n \geq 0$ can be easily computed in a recursive manner. However, the nonlinear function $F(y(t))$ of (4) should be represented by the so-called Adomian polynomials $\mathcal{A}_n$. Adomian [6, 10] developed formulas to generate Adomian polynomials for all types of nonlinearity. Let

$$u(t) = a^{-1}\int_0^t y(\eta)\,d\eta = a^{-1}\left(\int_0^t y_0(\eta)\,d\eta + \int_0^t y_1(\eta)\,d\eta + \int_0^t y_2(\eta)\,d\eta + \ldots\right) = \sum_{n=0}^{\infty} u_n(t).$$

We now have

$$u_0(t) = a^{-1}\int_0^t y_0(\eta)\,d\eta,\, u_1(t) = a^{-1}\int_0^t y_1(\eta)\,d\eta,\, u_2(t) = a^{-1}\int_0^t y_2(\eta)\,d\eta, \ldots$$

So, the the five Adomian polynomials are given by

$$\begin{aligned}
\mathcal{A}_0(t) &= \exp\left(-u_0(t)\right),\\
\mathcal{A}_1(t) &= \exp\left(-u_0(t)\right)\left(-u_1(t)\right),\\
\mathcal{A}_2(t) &= \exp\left(-u_0(t)\right)\left(-u_2(t) + \tfrac{1}{2!}u_1^2(t)\right),\\
\mathcal{A}_3(t) &= \exp\left(-u_0(t)\right)\left(-u_3(t) + u_1(t)u_2(t) - \tfrac{1}{3!}u_1^3(t)\right),\\
\mathcal{A}_4(t) &= \exp\left(-u_0(t)\right)\left(-u_4(t) + \tfrac{1}{2!}u_2^2(t) + u_1(t)u_3(t) - \tfrac{1}{2!}u_1^2(t)u_2(t) + \tfrac{1}{4!}u_1^4(t)\right).
\end{aligned}$$

Analyzing the combination of Laplace transforms with Adomian decomposition involves understanding the kernel determined by the difference $(t-\xi)$ and recalling the Laplace transform of the convolution product.

The convolution of two functions f and g, denoted as $(f * g)(t)$, is expressed as:

$$\mathbf{L}\{(f \star g)(t)\} = \mathbf{L}\left\{\int_0^t f(x-\xi)g(\xi)d\xi\right\} = \mathbf{L}(f)\cdot\mathbf{L}(g)$$

By taking Laplace transforms of both sides of equation (4), and representing them as equation (7) and the nonlinear terms as equation (6), we derive:

$$\mathbf{L}\left[\sum_{n=0}^{\infty} y_n(t)\right] = \frac{C}{a}\mathbf{L}[t]\cdot\mathbf{L}\left[\sum_{n=0}^{\infty}\mathcal{A}_n(t)\right] = \frac{C}{as^2}\mathbf{L}\left[\sum_{n=0}^{\infty}\mathcal{A}_n(t)\right]$$

Assuming $y_0 = 0$ when $a > 0$ and t is small, this method facilitates the utilization of the following recursive relations.

$$\mathbf{L}[y_1(t)] = \frac{C}{as^2}\mathbf{L}[\mathcal{A}_0(t)] = \frac{C}{as^2}\mathbf{L}\left[\exp(-u_0(t))\right] = \frac{C}{as^3}, \tag{8}$$

and,

$$\mathbf{L}[y_{k+1}(t)] = \frac{C}{as^2}\mathbf{L}[\mathcal{A}_k(t)], k \geq 0 \tag{9}$$

In this case, the initial components of the solution $y(t)$ for small values of t are

$$\begin{aligned}
&y_0(t) = 0,\\
&y_1(t) = \mathbf{L}^{-1}\left\{\tfrac{C}{as^3}\right\} = \tfrac{C}{a}\tfrac{t^2}{2!}.\\
&y_2(t) = \mathbf{L}^{-1}\left\{\tfrac{C}{as^2}\mathbf{L}[\mathcal{A}_1(t)]\right\} = \mathbf{L}^{-1}\left\{\tfrac{C}{as^2}\mathbf{L}\left[\exp(-u_0(t))(-u_1(t))\right]\right\} = -\tfrac{C^2}{a^3}\tfrac{t^5}{5!},\\
&y_3(t) = \mathbf{L}^{-1}\left\{\tfrac{C}{as^2}\mathbf{L}[\mathcal{A}_2(t)]\right\} = \mathbf{L}^{-1}\left\{\tfrac{C}{as^2}\mathbf{L}\left[\exp(-u_0(t))\left(-u_2(t) + \tfrac{1}{2!}u_1^2(t)\right)\right]\right\} = \tfrac{11C^3}{a^5}\tfrac{t^8}{8!}\\
&y_4(t) = \mathbf{L}^{-1}\left\{\tfrac{C}{as^2}\mathbf{L}[\mathcal{A}_3(t)]\right\}\mathbf{L}^{-1}\left\{\tfrac{C}{as^2}\mathbf{L}\left[\exp(-u_0(t))\left(-u_3(t) + u_1(t)u_2(t) - \tfrac{1}{3!}u_1^3(t)\right)\right]\right\} = -\tfrac{375C^4}{a^7}\tfrac{t^{11}}{11!}\\
&\vdots
\end{aligned} \tag{10}$$

As a result, the approximate solution depends on the variable t as well as the two constants a, C, and given by

$$y_{app}(t;a,C) = \frac{C}{a}\frac{t^2}{2!} - \frac{C^2}{a^3}\frac{t^5}{5!} + \frac{11C^3}{a^5}\frac{t^8}{8!} - \frac{375C^4}{a^7}\frac{t^{11}}{11!} + \ldots \tag{11}$$

The constant a is predetermined, whereas the constant C can be calculated as shown in Equation (5) and re-written as

$$C\left[\int_0^{\infty}\exp\left(-\int_0^t \frac{y(\xi)}{a}d\xi\right)dt\right] = a. \tag{12}$$

As Adomian polynomials [10] rely on the constant C, we conclude that an approximate value of C can be determined by solving the nonlinear algebraic equation.

$$C\left[\sum_{n=0}^{\infty}\int_0^{\infty}\mathcal{A}_n(t;C)dt\right] = a. \tag{13}$$

To estimate the constant C, set

$$\begin{aligned}
w_0(t) &= \frac{1}{a}\int_0^t y_1(\tau)d\tau = \frac{C}{a^2 3!}t^3,\\
w_1(t) &= \frac{1}{a}\int_0^t y_2(\tau)d\tau = -\frac{C^2}{a^4 6!}t^6\\
w_2(t) &= \frac{1}{a}\int_0^t y_3(\tau)d\tau = \frac{11C^3}{a^6 9!}t^9\\
w_3(t) &= \frac{1}{a}\int_0^t y_4(\tau)d\tau = -\frac{C^4}{(6!)^2 a^8}t^{12}.
\end{aligned}$$

The Adomian polynomials can also be expressed as:

$$\begin{aligned}
&\int_0^\infty \mathcal{A}_0(t)dt = \int_0^\infty \exp\left(-w_0(t)\right)dt = \int_0^\infty \exp\left(-\tfrac{C}{3!a^2}t^3\right)dt,\\
&\int_0^\infty \mathcal{A}_1(t)dt = \int_0^\infty -\exp\left(-w_0(t)\right)\left(w_1(t)\right)dt,\\
&\int_0^\infty \mathcal{A}_2(t)dt = \int_0^\infty \exp\left(-w_0(t)\right)\left(-w_2(t)+\tfrac{1}{2!}w_1^2(t)\right)dt,\\
&\int_0^\infty \mathcal{A}_3(t)dt = \int_0^\infty \exp\left(-w_0(t)\right)\left(-w_3(t)+w_1(t)w_2(t)-\tfrac{1}{3!}w_1^3(t)\right)dt,\\
&\vdots
\end{aligned}$$

To evaluate the above integrals, we use the following Lemma.

Lemma 2.1. *The value of the integral* $\int_0^\infty t^n \exp\left(-Mt^3\right)dt$, *in terms in the Gamma functions is* $\frac{1}{3M^{\frac{n+1}{3}}}\Gamma\left(\frac{n+1}{3}\right)$ *for some real constants M, n.*

Proof: Set $x = Mt^3$, then we have $dt = \frac{dx}{3Mt^2}$, and $t = \left(\frac{x}{M}\right)^{1/3}$, therefore the integral reduces to

$$\int_0^\infty t^n \exp\left(-Mt^3\right)dt = \frac{1}{3M^{\frac{n+1}{3}}}\int_0^\infty x^{\frac{n-2}{3}}\exp(-x)dx = \frac{1}{3M^{\frac{n+1}{3}}}\Gamma\left(\frac{n+1}{3}\right).$$

According to the above Lemma, the following integrals are needed to evaluate the constant C.

$$\int_0^\infty \exp\left(-Mt^3\right) = \frac{\Gamma(1/3)}{3M^{1/3}},\ \int_0^\infty t^6 \exp\left(-Mt^3\right) = \frac{\Gamma(7/3)}{3M^{7/3}},$$

and

$$\int_0^\infty t^9 \exp\left(-Mt^3\right) = \frac{\Gamma(10/3)}{3M^{10/3}}.$$

Using the results in the Lemma together with the use of equation (12) we have

$$C\left[\int_0^\infty \mathcal{A}_0(t)dt + \int_0^\infty \mathcal{A}_1(t)dt + \int_0^\infty \mathcal{A}_2(t)dt + \dots\right] = a. \tag{14}$$

Using only one term of the above infinite series, we can approximate the value of the constant C in terms of the constant a, which is $C = \frac{0.484}{a}$. To improve the value of the constant C, we take the first two terms of the above infinite series and obtain a new approximation of the value of C, which is considered more accurate than the previous one, and is given by the value $C = \frac{0.473}{a}$. When determining the value of the constant a (which is usually either 1 or 2, or in between), the value of the constant C is precisely determined, and thus we know the approximate solution to the Blasius equation as given in (11) for small values of t.

3. Applications

In the Blasius equation, understanding the behavior of the function $y(t)$ and its second derivative at zero, $y''(0)$, is crucial for practical applications. In fluid mechanics, the Blasius equation finds its utility in describing the boundary layer that forms near a flat plate as fluid flows past it. This equation, which is both nonlinear and a boundary value problem, offers valuable insights into fluid dynamics, especially concerning viscous incompressible fluids. Solving this equation enables us to determine the shear stress on the plate, which is a critical factor in various engineering applications.

However, numerically solving the Blasius equation poses challenges, particularly in assessing the function $y'(t)$ at infinity, denoted as $y'(\infty)$. One approach to overcoming this challenge involves transforming the problem into an initial value problem. This transformation allows us to find the second derivative at the origin, $y''(0)$, also known as the Blasius constant. This constant is essential for calculating the shear stress on the plate due to fluid movement. Once determined, numerical techniques can be employed to solve the initial value problem effectively.

In Table 1, we present both analytical solutions and their derivatives alongside corresponding numerical solutions for various values of the constant a. It's important to emphasize that our primary objective is to ascertain $y''(0)$. The precise value of the second derivative at zero, along with its upper and lower limits, has been extensively investigated in prior studies [4]. We have provided a comparison between our method and two other methods in Table 2 to ensure the calculations of the validity and effectiveness of the obtained solution in this paper. One of the compared methods is called RCW; the specifics of this method are found in [22], which is a novel approach with a good idea and provides accurate solutions. The analytic solution from [23] serves as the second solution utilized in Table 2. The three approaches provide the same accuracy, as can be seen by looking at the numbers in Table 2. However, our method stands out due to its ease of use and straightforward computations. Furthermore, we are able to alter the value of the constant a at any moment without rebuilding the solution.

Table 1: Analytical results for $y(t)$ and their derivatives for different values of the constant a.

Estimated values of $y(t)$, $y'(t)$, $y''(t)$ when a = 1, 1.4, 1.7, 2.0			
when $a = 1.0$	$y(t)$	$y'(t)$	$y''(t)$
$t = 0.0$	0	0	0.472884
$t = 0.5$	0.0590519	0.235859	0.468246
$t = 1.0$	0.2346053	0.463788	0.437175
$t = 2.0$	0.8925522	0.820933	0.251662
when $a = 1.4$	$y(t)$	$y'(t)$	$y''(t)$
$t = 0.0$	0	0	0.399657
$t = 0.5$	0.0499419	0.199677	0.398447
$t = 1.0$	0.1993462	0.397257	0.390134
$t = 2.0$	0.7845731	0.763567	0.332373
when $a = 1.7$	$y(t)$	$y'(t)$	$y''(t)$
$t = 0.0$	0	0	0.362682
$t = 0.5$	0.0453283	0.181272	0.362125
$t = 1.0$	0.1811193	0.361574	0.358271
$t = 2.0$	0.7184524	0.708411	0.330047
when $a = 2.0$	$y(t)$	$y'(t)$	$y''(t)$
$t = 0.0$	0	0	0.3343766
$t = 0.5$	0.0417934	0.167152	0.3340861
$t = 1.0$	0.1670722	0.333797	0.3320652
$t = 2.0$	0.6500164	0.659752	0.3168332

Table 2: The value of $y(t)$ when $a = 2$ compared with other methods.

t	Our Method	RCW [22]	Analytical solution [23]
0.4	0.0266515	0.0266	0.0266
0.8	0.1061321	0.1061	0.1061
1.2	0.2379002	0.2379	0.2379
1.6	0.4203821	0.4203	0.4203
2.0	0.6500130	0.6500	0.6500

4. Conclusions

Boundary conditions at infinity are a general problem for numerical solution methods. In this paper, we used a combination of the Laplace Transform and the Adomian decomposition method to obtain a closed form solution to the Blasius equation. We also examined an analytical approximate solution for the nonlinear Blasius equation, which is applicable to both turbulent and laminar flow. Table 1 shows the results of $y''(0)$, which were established in this study, for various values of a. Additionally, our solution was compared to the other solutions, and Table 2 indicates that our method exhibits excellent agreement with other techniques. Our solution's accuracy and reliability are guaranteed because the results are completely consistent with those obtained using other powerful methods.

References

[1] H. Blasius, Grenzschichten in Flüssigkeiten mit kleiner Reibung, *Z. Math. Phys.* **56**: 1–37 (1908).

[2] J. He, Approximate analytical solution of Blasius' equation. *Communi. in Nonlinear Sci. and Numerical Simul.* **3(4)**: 260–263 (1998).

[3] J.-H. He, A simple perturbation approach to Blasius equation, *Appl. Math. Comput.* **140**: 217 (2003). https://doi.org/10.1016/S0096-3003(02)00189-3

[4] A. Khanfer, L. Bougofa, S. Bougoufa, Analytic Approximate Solution of the Extended Blasius Equation with Temperature-Dependent Viscosity, *J of Nonlinear Math. Phys.* **30**: 287–302 (2023). https://doi.org/10.1007/s44198-022-00084-3

[5] S. Mziou, L. Bougoffa, R.C. Rach. Generalized Blasius equation, uniqueness and analytical approximation solution, *TJMM.* **10(2)**: 95–111 (2018).

[6] G. Adomian, A review of the decomposition method and some recent results for nonlinear equations, *Computers Math. Appl.* **21(5)**: 101–127 (1991).

[7] R. Khandelwal, Y. Khandelwal, Solution of Blasius Equation Concerning with Mohand Transform, *Int. J. Appl. Comput. Math.* **6**: 128 (2020). https://doi.org/10.1007/s40819-020-00871-w

[8] L. Bougoffa, Wazwaz, A.-M, New approximate solutions of the Blasius equation, *Inter. J of Numerical Methods for Heat and Fluid Flow.* **25(7)**: 1590–1599 (2015). https://doi.org/10.1108/HFF-08-2014-0263

[9] H. Aminikhah, Analytical Approximation to the Solution of Nonlinear Blasius' Viscous Flow Equation by LTNHPM, *International Scholarly Research Network ISRN Mathematical Analysis*. Volume 2012, Article ID 957473, 10 pages.

[10] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer Academic, 1994.

[11] M.M. Rahman, S. Khan, M.A. Akbar, Numerical and analytical solutions of new Blasius equation for turbulent flow. *Heliyon.* **9**: e14319 (2023).

[12] A. Asaithambi, Numerical Solution of the Blasius Equation with Crocco-Wang Transformation, *J of Applied Fluid Mech.* **9(5)**: 2595–2603 (2016).

[13] F.M. Allan, M.I. Syam, On the analytic solutions of the nonhomogeneous Blasius problem, *J. Comput. Appl. Math.* **182**: 362 (2005). https://doi.org/10.1016/j.cam.2004.12.017

[14] J. Parlange, R. Braddock, G. Sander, Analytical approximations to the solution of the Blasius equation, *Acta Mech.* **38**: 119 (1981). https://doi.org/10.1007/BF01351467

[15] J. Zheng, X. Han, Z. Wang, C. Li, J. Zhang, A globally convergent and closed analytical solution of the Blasius equation with beneficial applications, *AIP Adv.* **7**: 065311 (2017).

[16] D. Durante, E. Rossi, A. Colagrossi, G. Graziani, Numerical simulations of the transition from laminar to chaotic behaviour of the planar vortex flow past a circular cylinder, *Commun. Nonlinear Sci. Numer. Simul.* **48**: 18–38 (2017). https://doi.org/10.1016/j.cnsns.2016.12.013

[17] A. Jafarimoghaddam, M. Soler, A. Simorgh, The optimal decomposition method (ODM) for nonlinear problems, *J. Comput. Sci.* **62**: 101690 (2022). https://doi.org/10.1016/j.jocs.2022.101690

[18] K. Al-Khaled, M.S. Ajeel, I. Abu-Irwaq, H.K. Al-Khalid, An Efficient Approximate Method for Solving Bratu's Boundary Value Problem. *Inter. J of Elect. and Comp. Eng.* **14(5)**: 5738–5743 (2024).

[19] M. Rahmanzadeh, T. Asadi, M. Atashafrooz, A Numerical Algorithm based on the RCW method to solve a set of First-Order Ordinary Differential Equations, *J of the Serbian Soc. for Comput. Mechan.* **14(1)**: 63–74 (2020). https://doi.org/10.24874/jsscm.2020.14.01.06

[20] S. Abbasbandy, A numerical solution of Blasius equation by Adomian's decomposition method and comparison with homotopy perturbation method, Chaos, *Solitons & Fractals*. **31(1)**: 257–260. (2007). https://doi.org/10.1016/j.chaos.2005.10.071

[21] A. Ebaid, N. Al-Armani, A New Approach for a Class of the Blasius Problem via a Transformation and Adomian's Method, *Abstract and Appl. Analy*. Volume 2013, Article ID 753049, 8 pages http://doi.org/10.1155/2013/753049

[22] M. Rahmanzadeh, T. Asadi, M. Atashafrooz, The Development and Application of the RCW Method for the Solution of the Blasius Problem, *J of Appl. and Comput. Mechan.* **6(1)**: 105–111 (2020).

[23] F.M. White, Fluid Mechanics, McGraw-Hill (1986).