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Graded 2*r*-Ideals

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Let G be a group and R be a commutative G-graded ring with nonzero unity. In this article, we establish the concept of graded $2r$-ideals, which lies somewhere between graded r-ideals and graded uniformly pr-ideals. A proper graded ideal P of R is said to be a graded $2r$-ideal of R if whenever $x, y \in h(R)$ such that $xy \in P$, then either $x^2 \in P$ or $y \in zd(R)$, where $zd(R)$ is the set of all zero divisors of R. Several properties of graded $2r$-ideals have been achieved, and various results have been investigated.

Key words and phrases: graded r-ideal, graded pr-ideal, graded uniformly pr-ideal

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1. Introduction

Let G be a group and R be a commutative ring with nonzero unity 1. Then R is called G-graded if $R = \bigoplus_{g\in G} R_g$ with $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$, where R_g is an additive subgroup of R for all $g \in G$. The elements of R_g are called homogeneous of degree g. If $a \in R$, then a can be written uniquely as $a = \sum_{g\in G} a_g$, where a_g is the component of a in R_g and $a_g = 0$ except for finitely many. The additive subgroup R_e is in fact a subring of R and $1 \in R_e$. The set of all homogeneous elements of R is $\bigcup_{g\in G} R_g$ and is denoted by $h(R)$. let I be an ideal of a G-graded ring R. Then I is called a graded ideal if $I = \bigoplus_{g\in G}(I \cap R_g)$, i.e., for $a \in I$, $a = \sum_{g\in G} a_g$, where $a_g \in I$, for all $g \in G$. An ideal of a graded ring is not necessarily a graded ideal. For more terminology, see [6, 7]. let I be a proper graded ideal of R. Then the graded radical of I is

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$Grad(I)$, and is defined to be the set of all $r \in R$ such that for each $g \in G$, there exists a positive integer n_g satisfies $r_g^{n_g} \in I$. One can see that $Grad(I)$ is a graded ideal of R. Also, if $r \in h(R)$, then $r \in Grad(I)$ if and only if $r^n \in I$, for some positive integer n.

A proper graded ideal P of R is said to be graded prime if whenever $x, y \in h(R)$ such that $xy \in P$, then $x \in P$ or $y \in P$ [9]. A proper graded ideal P of R is called a graded r-ideal (graded pr-ideal) if $x, y \in h(R)$ such that $xy \in P$, then $x \in P$ or $y \in zd(R)$ ($x \in Grad(P)$ or $y \in zd(R)$), where $zd(R)$ is the set of all zero divisors of R [2]. Then in [3], a special class of graded pr-ideals that fixing the power of x in the above definition was introduced and examined. A proper graded ideal P of R is called a graded uniformly pr-ideal if there exists a positive integer n such that whenever $x, y \in h(R)$ with $xy \in P$, we have $x^n \in P$ or $y \in zd(R)$. The order of P is the smallest positive integer for which the aforementioned property holds. The next two examples show that a graded r-ideal is not necessarily graded prime:

Example 1.1. *Let R be a graded ring. Then $I = \{0\}$ is a graded r-ideal of R: let $x, y \in h(R)$ such that $xy \in I$ and $y \notin zd(R)$. Then $xy = 0$, and then $x \in Ann(y) = \{r \in R : ry = 0\} = \{0\}$ as $y \notin zd(R)$, which implies that $x = 0 \in I$. On the other hand, I is not necessarily a graded prime ideal of R.*

Example 1.2. *Let R be a graded ring and $0 \neq x \in h(R)$. Then $Ann(x)$ is a graded r-ideal of R: $Ann(x)$ is a graded ideal of R by* ([3], Lemma 2.15). *Let $a, b \in h(R)$ such that $ab \in Ann(x)$ and $b \notin zd(R)$. Then $abx = 0$, and then $ax \in Ann(b) = \{0\}$, which implies that $ax = 0$, and hence $a \in Ann(x)$. On the other hand, $Ann(x)$ is not necessarily a graded prime ideal of R.*

In this article, we follow [4] to establish the concept of graded $2r$-ideals, which lies somewhere between graded r-ideals and graded uniformly pr-ideals. A proper graded ideal P of R is said to be a graded $2r$-ideal of R if whenever $x, y \in h(R)$ such that $xy \in P$, then either $x^2 \in P$ or $y \in zd(R)$. Several properties of graded $2r$-ideals have been achieved, and various results have been investigated.

2. Graded 2*r*-Ideals

In this section, we introduce and examine the concept of graded $2r$-ideals.

Definition 2.1. *Let R be a graded ring. Then a proper graded ideal P of R is said to be a graded $2r$-ideal of R if whenever $x, y \in h(R)$ such that $xy \in P$, then either $x^2 \in P$ or $y \in zd(R)$.*

Clearly, graded r-ideals are graded $2r$-ideals. However, the next example shows that a graded $2r$-ideal is not necessarily a graded r-ideal:

Example 2.2. *Consider $R = K[x,y]$, where K is a field, and $G = \mathbb{Z}$. Then R is G-graded by $R_n = \bigoplus_{i+j=n, i,j\geq 0} Kx^i y^j$, for all $n \in \mathbb{Z}^+ \cup \{0\}$, and $R_n = 0$, otherwise. Consider the graded ideal $I = \langle xy \rangle$ of R. Then R/I is a G-graded ring by $(R/I)_n = (R_n + I)/I$, for all $n \in \mathbb{Z}$. Consider the graded prime ideals $P = \langle x + I \rangle$ and $Q = \langle y + I \rangle$ of R/I. We show that $zd(R/I) = P \cup Q$. Let $f + I \in zd(R/I)$. Then there exists $g + I \in R/I$ such that $g + I \neq 0 + I$ and $(f + I)(g + I) = 0 + I$, and then $fg \in I$ with $g \notin I$. So, $fg = xyh$, for some $h \in R$, and then x divides fg and y divides fg, which implies that x divides f or x divides g, and y divides f or y divides g. If x divides g and y divides g, then xy divides g, and then $g \in I$, which is a contradiction. So, x divides f or y divides f, which implies that $f + I \in P \cup Q$. Thus, $zd(R) \subseteq P \cup Q$. Let $f + I \in P \cup Q$. Then $f + I \in P$ or $f + I \in Q$. If $f + I \in P$, then $f + I = (x + I)(h + I) = xh + I$, for some $h \in R$, and then $f - xh \in I$ which implies that $f - xh = xyt$, for some $t \in R$, and then $f = xh + xyt$ =, so $yf = xy(h + yt) \in I$, and thus $(y + I)(f + I) = yf + I = 0 + I$ with $y + I \neq 0 + I$ as $y \notin I$, which means that $f + I \in zd(R)$. Similarly, if $f + I \in Q$, then $f + I \in zd(R)$. Hence, $zd(R/I) = P \cup Q$. Now, we show that P^2 is a graded $2r$-ideal of R/I. Let $f + I, g + I \in h(R/I)$ such that $(f + I)(g + I) \in P^2$. Assume that $g + I \notin zd(R/I)$. Then $g + I \notin P$. Since $(f + I)(g + I) \in P^2 \subseteq P$ and P is graded prime, we have $f + I \in P$, which implies that $(f + I)^2 \in P^2$. Hence, P^2 is a graded $2r$-ideal of R/I. On the other hand, P^2 is not a graded r-ideal of R/I since $x + I, x + y + I \in h(R/I)$ such that $(x + I)(x + y + I) = x^2 + xy + I = x^2 + I = (x + I)^2 \in P^2$, $x + I \notin P^2$ and $x + y + I \notin zd(R/I)$.*

Even though the next result is an immediate consequence of the definition of graded $2r$-ideals, it is an important fact since it emphasizes that the components of the graded $2r$-ideals are entirely consisting of zero divisors.

Proposition 2.3. *Let R be a graded ring and P be a graded $2r$-ideal of R. Then $P_g \subseteq zd(R)$, for all $g \in G$.*

Proof. Let $g \in G$ and $x \in P_g$. Then $1, x \in h(R)$ such that $1.x = x \in P$, and then since P is a graded $2r$-ideal and $1^2 = 1 \notin P$, we have that $x \in zd(R)$. Hence, $P_g \subseteq zd(R)$.

For a graded ideal P of R and $a \in h(R)$, $(P : a) = \{r \in R : ra \in P\}$ is a graded ideal of R containing P ([3], Lemma 2.15).

Proposition 2.4. *Let R be a graded ring and P be a graded $2r$-ideal of R. Then for every $a \in h(R)$, either $a^2 \in P$ or $((P : a))_g \subseteq zd(R)$, for all $g \in G$.*

Proof. Let $a \in h(R)$ such that $a^2 \notin P$. Assume that $g \in G$ and $b \in ((P : a))_g$. Then $a,b \in h(R)$ such that $ab \in P$, and then since P is a graded $2r$-ideal, we have that $b \in zd(R)$. Hence, $((P : a))_g \subseteq zd(R)$.

Theorem 2.5. Let R be a graded ring.

(1) If P is a graded $2r$-ideal of R, *then $Grad(P)$* is a graded r-ideal of R.
(2) If P and Q are graded $2r$-ideals of R, then $P \cap Q$ is a graded $2r$-ideal of R.
(3) If P and Q are graded r-ideals of R, then PQ is a graded $2r$-ideal of R.
(4) Let P and Q be proper graded ideals of R such that $P + Q = R$. If PQ is a graded $2r$-ideal of R, then P and Q are graded $2r$-ideals of R.
(5) Let P and Q be proper graded ideals of R such that $P + Q = R$. If $P \cap Q$ is a graded $2r$-ideal of R, then P and Q are graded $2r$-ideals of R.
(6) Every maximal graded $2r$-ideal of R is a graded prime ideal of R.

Proof. (1) Let $x, y \in h(R)$ such that $xy \in Grad(P)$. Then $x^n y^n = (xy)^n \in P$, for some positive integer n, and then either $(x^n)^2 \in P$ or $y^n \in zd(R)$ as P is a graded $2r$-ideal. If $(x^n)^2 \in P$, then $x^{2n} \in P$, which implies that $x \in Grad(P)$. If $y^n \in zd(R)$, then $y \in zd(R)$. Hence, $Grad(P)$ is a graded r-ideal of R.

(2) Let $x, y \in h(R)$ such that $xy \in P \cap Q$ and $y \notin zd(R)$. Then $xy \in P$ and $xy \in Q$, and then $x^2 \in P$ and $x^2 \in Q$, which implies that $x^2 \in P \cap Q$. Hence, $P \cap Q$ is a graded $2r$-ideal of R.
(3) Let $x, y \in h(R)$ such that $xy \in PQ$ and $y \notin zd(R)$. Then $xy \in P$ and $xy \in Q$, and then $x \in P$ and $x \in Q$, which implies that $x^2 = x.x \in PQ$. Hence, PQ is a graded $2r$-ideal of R.
(4) Since $P + Q = R$, $1 = x + y$, for some $x \in P$ and $y \in Q$, and then as $1 \in R_e$, $1 = 1_e = (x + y)_e = x_e + y_e$. Note that, as P and Q are graded ideals, $x_e \in P$ and $y_e \in Q$. Let a, $b \in h(R)$ such that $ab \in P$ and $b \notin zd(R)$. Then $ay_e b \in PQ$, and then $a^2 y_e^2 \in PQ \subseteq P$, which implies that $a^2 = a^2.1 = a^2(x_e + y_e)^2 = a^2 x_e^2 + 2a^2 x_e y_e + a^2 y_e^2 \in P$. Hence, P is a graded $2r$-ideal of R. Similarly, Q is a graded $2r$-ideal of R.
(5) Since $P + Q = R$, $P \cap Q = PQ$, and then the result holds from (4).
(6) Let P be a maximal graded $2r$-ideal of R. Assume that a, $b \in h(R)$ such that $ab \in P$ and $a \notin P$. Then $Grad(P)$ is a graded r-ideal of R by (1), and then $Grad(P)$ is a graded $2r$-ideal of R, and so by maximality of P, $P = Grad(P)$ is a graded r-ideal of R, which implies that $(P : a)$ is a graded r-ideal of R, and then again by the maximality of P, $P = (P : a)$, and thus $b \in (P : a) = P$. Hence, P is a graded prime ideal of R.

Clearly, if P is a graded prime ideal of R with $P \cap h(R) \subseteq zd(R)$, then P is a graded r-ideal of R, and so P is a graded $2r$-ideal of R. In the next result, we discuss the case when the graded $2r$-ideals of R are all graded prime. Recall that a commutative graded ring R with unity is said to be a graded domain if R has no homogeneous zero divisors. Obviously, if R is a domain and R is graded, then R is a graded domain. However, a graded domain is not necessarily domain ([1], Example 2.4). The next proposition shows that the graded $2r$ ideals of R are all graded prime if and only if R is a graded domain.

Proposition 2.6. *Let R be a graded ring. Then the followings statements are equivalent:*

(1) *Every graded $2r$-ideal of R is a graded prime ideal of R.*
(2) *R is a graded domain.*
(3) *$\{0\}$ is the only graded $2r$-ideal of R.*

Proof. (1) $\Rightarrow$ (2): Since $\{0\}$ is a graded $2r$-ideal of R, $\{0\}$ is a graded prime ideal of R, and then R is a graded domain.

(2) $\Rightarrow$ (3): Let P be a graded $2r$-ideal of R and $a \in P$. Then for any $g \in G$ $a_g \in P_g \subseteq zd(R) = \{0\}$ by Proposition 2.3. So, $a_g = 0$, for all $g \in G$, which implies that $a = 0$. Hence, $P = \{0\}$.

(3) $\Rightarrow$ (1): Let $x \in h(R) - \{0\}$. Then $Ann(x)$ is a graded $2r$-ideal of R, and then $Ann(x) = \{0\}$. Thus, x is a regular element, and hence R is a graded domain. So, $\{0\}$ is a graded prime ideal of R which is the only graded $2r$-ideal of R.

A proper graded ideal P of R is said to be graded primary if whenever $x, y \in h(R)$ such that $xy \in P$, then either $x \in P$ or $y \in Grad(P)$[8]. Clearly, if P is a graded primary ideal of R with $P \bigcap h(R) \subseteq zd(R)$, then P is a graded r-ideal of R, and so P is a graded $2r$-ideal of R. In the next result, we discuss the case when the graded $2r$-ideals of R are all graded primary. Recall that a graded ring R is an HUN-ring if every homogeneous element of R is either unit or nilpotent. Indeed, R is an HUN-ring if and only if $nil(R)$ is a graded maximal ideal of R. The next theorem shows that the graded $2r$ ideals of R are all graded primary if and only if R is a graded domain or an HUN-ring.

Theorem 2.7. *Let R be a graded ring. Then every graded $2r$-ideal of R is a graded primary ideal of R if and only if R is a graded domain or an HUN-ring.*

Proof. Suppose that every graded $2r$-ideal of R is a graded primary ideal of R. Then $\{0\}$ is a graded primary ideal of R, and then $nil(R) = zd(R)$. Assume that R is neither graded domain nor HUN-ring. Let M be a graded maximal ideal of R. Then there exists $a \in M - nil(R)$, and then $a_g \notin nil(R)$, for some $g \in G$. Note that as M is a graded ideal, $a_g \in M$. Consider $0 \neq b \in nil(R)$, so $b_g \in nil(R)$ as $nil(R)$ is a graded ideal. Let k be the smallest positive integer such that $b_g^k = 0$. We show that $I = \langle a_g b_g^{k-1} \rangle$ is a graded $2r$-ideal of R. Let $x, y \in h(R)$ such that $xy \in I$ and $y \notin zd(R)$. Then $x^2y^2 = 0$, and then as $\{0\}$ is a graded r-ideal, we have $x^2 = 0 \in I$. Hence, I is a graded $2r$-ideal of R, and so I is a graded primary ideal of R with $Grad(I) = nil(R)$. Now, $a_g b_g^{k-1} \in I$ with $a_g \notin nil(R)$, so $b_g^{k-1} \in I$, which implies that $b_g^{k-1}(1 - ra_g) = 0$, for some $r \in R$, which means that $1 - ra_g \in zd(R) = nil(R) \subseteq M$, so $1 \in M$, which is a contradiction. Hence, R is a graded domain or an HUN-ring. Conversely, if R is a graded domain, then $\{0\}$ is the only graded $2r$-ideal of R, which is graded primary. If R is an HUN-ring, then every proper graded ideal of R is graded primary. In particular, every graded $2r$-ideal of R is graded primary.

Let R be a G-graded ring and I be a graded ideal of R. Then R/I is a G-graded ring by $(R/I)_g = (R_g + I)/I$, for all $g \in G$ [7].

Proposition 2.8. *Let $Q \subseteq P$ be two graded ideals of R. Then P/Q is a graded $2r$-ideal of R/Q if and only if for every $x, y \in h(R)$ with $xy \in P$, we have $x^2 \in P$* or $(Q : y) \neq Q$.

Proof. Suppose that P/Q is a graded $2r$-ideal of R/Q. Let $x, y \in h(R)$ with $xy \in P$ and $x^2 \notin P$. Then $(x + Q)(y + Q) \in P/Q$ and $(x + Q)^2 \notin P/Q$, and then $y + Q \in zd(R/Q)$, which implies that there exists $a \notin Q$ with $(a + Q)(y + Q) = 0 + Q$, so $ay \in Q$, which means that $a \in (Q : y)$. Hence, $(Q : y) \neq Q$. Conversely, let $x + Q, y + Q \in h(R/Q)$ such that $(x + Q)(y + Q) \in P/Q$ and $(x + Q)^2 \notin P/Q$. Then $xy \in P$ and $x^2 \notin P$, and then $(Q : y) \neq Q$. So, there exists $a \in (Q : y) - Q$, which implies that $(a + Q)(y + Q) = ay + Q = 0 + Q$. Thus, $y + Q \in zd(R/Q)$, and hence P/Q is a graded $2r$-ideal of R/Q.

Corollary 2.9. *Let $Q \subseteq P$ be two graded ideals of R. If Q is a graded r-ideal of R and P/Q is a graded $2r$-ideal of R/Q, then P is a graded $2r$-ideal of R.*

Proof. Let $x, y \in h(R)$ such that $xy \in P$ and $x^2 \notin P$. Then by Proposition 2.8, $(Q : y) \neq Q$, and then there exists $a \in (Q : y) - Q$, which implies that $a_g \notin Q$, for some $g \in G$, but $a_g \in (Q : y)$ as $(Q : y)$ is a graded ideal. Now, Q is a graded r-ideal with $a_g y \in Q$ and $a_g \notin Q$, so $y \in zd(R)$. Hence, P is a graded $2r$-ideal of R.

Let R and S be two G-graded rings. Then a ring homomorphism $f: R \to S$ is said to be a graded ring homomorphism if $f(R_g) \subseteq S_g$, for all $g \in G$. Moreover, if f is a graded ring epimorphism, then $f(R_g) = S_g$, for all $g \in G$ [7].

Proposition 2.10. *Let $f: R \to S$ be a graded ring homomorphism. Then $f^{-1}(P)$ is a graded $2r$-ideal of R, for every graded $2r$-ideal P of S, if and only if $f(x)$ is a regular element in S, for every homogeneous regular element x in R.*

Proof. Suppose that $f^{-1}(P)$ is a graded $2r$-ideal of R, for every graded $2r$-ideal P of S. Let x be a homogeneous regular element in R. Assume that $f(x)$ is not regular in S. Then $f(x) \in zd(S)$, and then there exists $s \in S - \{0\}$ such that $f(x)s = 0$. Since $s \neq 0$, $s_h \neq 0$, for some $h \in G$. On the other hand, $\sum_{g\in G} f(x)s_g = f(x)\left(\sum_{g\in G} s_g\right) = f(x)s = 0$, but $\{0\}$ is a graded ideal and $f(x)s_g$ is a homogeneous element, for all $g \in G$, so we have $f(x)s_g = 0$, for all $g \in G$. In particular, $f(x)s_h = 0$. So, $f(x) \in I = Ann(s_h)$. Now, I is a graded $2r$-ideal of S, so $f^{-1}(I)$ is a graded $2r$-ideal of R with $x \in f^{-1}(I)$. Since x is homogeneous in R, $x \in R_g$, for some $g \in G$, and then by Proposition 2.3, $x = x_g \in (f^{-1}(I))_g \subseteq zd(R)$, which is a contradiction. Hence, $f(x)$ is regular in S. Conversely, let P be a graded $2r$-ideal of S. Assume that $x, y \in h(R)$ such that $xy \in f^{-1}(P)$ and $y \notin zd(R)$. Then y is a homogeneous regular element in R, and then $f(y)$ is regular in S, so $f(y) \notin zd(S)$. Now, $f(x), f(y) \in h(S)$ such that $f(x)f(y) = f(xy) \in P$, and then $f(x^2) = (f(x))^2 \in P$, so $x^2 \in f^{-1}(P)$. Hence, $f^{-1}(P)$ is a graded $2r$-ideal of R.

Recall that for a ring R, $S \subseteq R$ is said to be essential in R if $S \cap I \neq \{0\}$, for every nonzero ideal I of R.

Corollary 2.11. *Let R be a graded ring such that R_e is essential in R. If P is a graded $2r$-ideal of R, then P_e is an $2r$-ideal of R_e.*

Proof. Define $f: R_e \to R$ by $f(x) = x$. Then f is a graded ring homomorphism. Let $x \in R_e$ be a regular element. Assume that $f(x)$ is not regular in R. Then there exists $r \in R - \{0\}$ such that $rx = rf(x) = 0$, so $r \in Ann_R(x)$, and then $Ann_R(x)$ is a nonzero ideal of R. Thus $R_e \cap Ann_R(x) \neq \{0\}$, which implies that there exists $t \in R_e - \{0\}$ such that $tx = 0$, which is a contradiction. So, $f(x)$ is regular in R. Hence, by Proposition 2.10, $f^{-1}(P) = P \cap R_e = P_e$ is an $2r$-ideal of R_e.

Proposition 2.12. *Let $f: R \to S$ be a graded ring epimorphism such that $f(zd(R)) \subseteq zd(S)$. If P is a graded $2r$-ideal of R and $Ker(f) \subseteq P$, then $f(P)$ is a graded $2r$-ideal of S.*

Proof. Let $s,t \in h(S)$ such that $st \in f(P)$. Then there exist $x, y \in h(R)$ such that $f(x) = s$ and $f(y) = t$, and then $f(xy) = f(x)f(y) = st \in f(P)$, which implies that $xy \in P$ as $Ker(f) \subseteq P$. So, either $x^2 \in P$ or $y \in zd(R)$, and then either $s^2 = (f(x))^2 = f(x^2) \in f(P)$ or $t = f(y) \in f(zd(R)) \subseteq zd(S)$. Hence, $f(P)$ is a graded $2r$-ideal of S.

Let R and S be two G-graded rings. Then $R \times S$ is a G-graded ring by $(R \times S)_g = R_g \times S_g$, for all $g \in G$. A graded ring R is said to be a cross product if R_g contains a unit, for all $g \in G$[7].

Proposition 2.13. *Let R and S be two G-graded rings such that R and S are cross products. Assume that P and Q are two graded ideals of R and S respectively. Then $P \times Q$ is a graded $2r$-ideal of $R \times S$ if and only if P is a graded $2r$-ideal of R and $Q = S$ or Q is a graded $2r$-ideal of S and $P = R$ or P and Q are graded $2r$-ideals of R and S respectively.*

Proof. Suppose that $P \times Q$ is a graded $2r$-ideal of $R \times S$. Then $P \times Q$ should be proper, and then $P \neq R$ or $Q \neq S$. Assume that $P \neq R$. Let $x, y \in h(R)$ such that $xy \in P$ and $y \notin zd(R)$. Then $y \in R_h$, for some $h \in G$. Since S is a cross product, S_h contains a unit, say s. Now, $(x,0),(y,s) \in h(R \times S)$ such that $(x,0)(y,s) = (xy,0) \in P \times Q$ and $(y,s) \notin zd(R \times S)$, so $(x,0)^2 = (x^2,0) \in P \times Q$, and then $x^2 \in P$. Hence, P is a graded $2r$-ideal of R. Similarly, if $Q \neq S$, then Q is a graded $2r$-ideal of S. Conversely, suppose that P and Q are graded $2r$-ideals of R and S respectively. Let $(x,t),(y,s) \in h(R \times S)$ such that $(x,t)(y,s) = (xy,ts) \in P \times Q$ and $(y,s) \notin zd(R \times S)$. Then $x, y \in h(R)$, $t, s \in h(S)$ such that $xy \in P$, $ts \in Q$, $y \notin zd(R)$ and $s \notin zd(S)$. Thus, $x^2 \in P$ and $t^2 \in Q$, and then $(x,t)^2 = (x^2,t^2) \in P \times Q$. Hence, $P \times Q$ is a graded $2r$-ideal of $R \times S$. Similarly for the other cases.

Let R be a G-graded ring. Assume that M is a left unitary R-module. Then M is said to be G-graded if $M = \bigoplus_{g\in G} M_g$ with $R_g M_h \subseteq M_{gh}$, for all $g, h \in G$, where M_g is an additive subgroup of M, for all $g \in G$. The elements of M_g are called homogeneous of degree g. It is clear that M_g is an R_e-submodule of M, for all $g \in G$. We assume that $h(M) = \bigcup_{g\in G} M_g$. Let N be an R-submodule of a graded R-module M. Then N is said to be a graded R-submodule if $N = \bigoplus_{g\in G}(N \cap M_g)$, i.e., for $x \in N$, $x = \sum_{g\in G} x_g$, where $x_g \in N$, for all $g \in G$. It is known that an R-submodule of a graded R-module is not necessarily graded. The idealization $R(+)M = \{(r,m) : r \in R, m \in M\}$ of M is a commutative ring with componentwise addition and multiplication; $(x,m_1)+(y,m_2) = (x+y, m_1+m_2)$ and $(x,m_1)(y,m_2) = (xy, xm_2 + ym_1)$, for each $x, y \in R$ and $m_1, m_2 \in M$. Let G be an abelian group and M be a G-graded R-module. Then $x = R(+)M$ is G-graded by $x_g = R_g(+)M_g$, for all $g \in G$. If P is a graded ideal of R and N is a graded R-submodule of M, then $P(+)N$ is a graded ideal of $R(+)M$ provided that $PM \subseteq N$ [10]. Note that $zd(R(+)M) = \{(x,m) : x \in zd(R) \cup zd(M),\ m \in M\}$, where $zd(M) = \{x \in R : xm = 0 \text{ for some } 0 \neq m \in M\}$ ([5], Theorem 3.5).

Proposition 2.14. *Let G be an abelian group, M be a G-graded R-module and P be a graded ideal of R. Then $P(+)M$ is a graded $2r$-ideal of $R(+)M$ if and only if for every $x, y \in h(R)$ with $xy \in P$, we have $x^2 \in P$ or $y \in zd(R) \cup zd(M)$.*

Proof. Suppose that $P(+)M$ is a graded $2r$-ideal of $R(+)M$. Let $x, y \in h(R)$ with $xy \in P$ and $x^2 \notin P$. Then $(x,0)(y,0) = (xy,0) \in P(+)M$ and $(x,0)^2 = (x^2,0) \notin P(+)M$. Hence, $(y,0) \in zd((R(+)M))$, which gives that $y \in zd(R) \cup zd(M)$. Conversely, let $(x,m),(y,t) \in h((R(+)M)$ such that $(x,m)(y,t) = (xy, xt+ym) \in P(+)M$ and $(y,t) \notin zd(R(+)M)$. Then $xy \in P$ and $y \notin zd(R) \cup zd(M)$. Hence, $x^2 \in P$, and then $(x,m)^2 = (x^2, 2xm) \in P(+)M$. Thus, $P(+)M$ is a graded $2r$-ideal of $R(+)M$.

Corollary 2.15. *Let G be an abelian group and M be a G-graded R-module. If P is a graded $2r$-ideal of R, then $P(+)M$ is a graded $2r$-ideal of $R(+)M$.*

Corollary 2.16. *Let G be an abelian group and M be a G-graded R-module such that $zd(M) \subseteq zd(R)$. Then P is a graded $2r$-ideal of R if and only if $P(+)M$ is a graded $2r$-ideal of $R(+)M$.*

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