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Nonlinear contraction mapping in probabilistic controlled generalized metric spaces

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This paper presents a new framework referred to as "probabilistic controlled generalized metric spaces," extending the theory of probabilistic metric spaces. The aim is to examine the correlation between this innovative class and the traditional axioms of probabilistic metric spaces. Moreover, the paper delves into proving the existence of fixed points for the $\wp$-probabilistic contraction mapping, even without the presence of the Hausdorff condition. The paper will also feature illustrative examples to underscore the practicality and efficacy of the theories and methodologies presented.

Key words and phrases: Probabilistic controlled generalized metric spaces, Hausdorff condition, nonlinear contractions, Fixed point

Mathematics Subject Classification (2010): 54A05, 54C50, 54E70.

1. Introduction

The inception of probabilistic distance concept is credited to K. Menger [14], a key figure in the advancement of probabilistic metric space theory. Subsequently, numerous probabilistic adaptations of the triangle inequality have emerged (refer to [7, 13, 16, 21, 22, 23]), with the examination of these inequalities playing a pivotal role in the evolution of probabilistic metric space theory. For a more comprehensive grasp of significant advancements in this domain, consulting [17] is recommended.

Among the array of mathematical theories that make such generalized structures intriguing and noteworthy, fixed point theory holds prominence. In recent decades, several renowned mathematicians have established numerous well-known metric fixed point theorems within these structures (see [1, 2, 4, 5, 8, 9, 10, 11, 12, 15, 18, 19, 20] and the references therein).

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In 2016, MBARKI and Naciri [6] introduced an additional abstraction termed probabilistic generalized metric space. They investigated diverse topological and geometrical characteristics of these spaces and illustrated fixed point properties for nonlinear contractions.

This study aims to introduce a novel framework as an extension to the theory of probabilistic generalized metric spaces. We seek to explore the connections between various concepts within this new category termed "probabilistic controlled generalized metric space" and the traditional principles of probabilistic metric space. Additionally, we aim to establish the existence and uniqueness of a fixed point for the $\wp$-probabilistic contraction mapping, even in the absence of the Hausdorff condition. Furthermore, concrete cases are provided to support the validity of our generalizations.

2. Preliminaries

We will start by presenting some fundamental definitions and terminology that will be utilized throughout the investigation.

A non-decreasing function $\Upsilon : [0,1] \to [0,1]$, which exhibits commutativity, and associativity, is designated as a t-norm if it adheres to the subsequent criteria:

(1) $\Upsilon(\gamma,1) = \gamma$, for all $\gamma \in [0,1]$,
(2) $\Upsilon(0,0) = 0$.

Three illustrations of continuous t-norms are presented below:

$$\Upsilon_p(\omega,\ \nu) = \omega\nu,\ \Upsilon_M(\omega,\ \nu) = \min(\omega,\ \nu),\ \text{and}\ \Upsilon_L(\omega,\ \nu) = \max\{\omega + \nu - 1, 0\}.$$

We designate a t-norm Υ as being of H-type [3] if the sequence $\{\Upsilon^n\}_n$ exhibits equicontinuity at $\mathfrak{c} = 1$, meaning

$$\forall \mu \in (0,1) \quad \exists \imath \in (0,1) : \xi > 1 - \imath \Rightarrow \Upsilon^n(\xi) > 1 - \mu \quad \text{for all} \quad n \geq 1.$$

In this context, $\Upsilon^1(\mathfrak{c})$ is defined as $\Upsilon(\mathfrak{c},\mathfrak{c})$, and for every $n \geq 2$, $\Upsilon^n(\mathfrak{c})$ is expressed as $\Upsilon(\mathfrak{c}, \Upsilon^{n-1}(\mathfrak{c}))$.

The t-norm Υ_M exemplifies a simple instance of a t-norm of H-type.

A function $\mathfrak{g} : \mathbb{R}^+ \cup \{\infty\} \to [0,1]$, monotonically non-decreasing, is considered a distance distribution function provided it meets the following criteria

(1) $\mathfrak{g}(0) = 0$ and $\mathfrak{g}(\infty) = 1$,
(2) $\mathfrak{g}$ is left-continuous on $(0,\infty)$.

The collection of distance distribution functions is denoted by Δ^+. Additionally,

$$\mathcal{D}^+ = \{\mathfrak{g} \in \Delta^+ : \lim_{t \to \infty} \mathfrak{g}(t) = 1\}.$$

An exemplary instance of a distance distribution function within $\mathcal{D}^+$ is the Heaviside step function. It's defined as follows

$$\mathbb{H}(\gamma) = \begin{cases} 0 & \text{if} \quad \gamma \leq 0, \\ 1 & \text{if} \quad \gamma > 0. \end{cases}$$

2.1. Probabilistic controlled generalized metric space

This section serves to introduce a novel category of generalized probabilistic metric spaces termed probabilistic controlled generalized metric spaces. Additionally, we delve into the examination of diverse topological and geometric attributes inherent in these spaces.

Definition 2.1. *A probabilistic controlled generalized metric space (abbreviated as pcgms) is defined as a quadruple* $(\mathcal{W}, \mathfrak{I}, \Upsilon, \upsilon)$ *where* $\mathcal{W}$ *is a nonempty set,* $\mathfrak{I}$ *is a function from* $\mathcal{W} \times \mathcal{W}$ *into* Δ^+, Υ *is a t-norm,* υ *is a mapping from* $\mathbb{R}^+$ *into* $[1,\infty)$, *and the following conditions are met: for all* $\mathfrak{p}, \mathfrak{q} \in \mathcal{W}$ *and for all distinct points* $\mathfrak{a}, \mathfrak{b} \in \mathcal{W}$, *each distinct from* $\mathfrak{p}$ *and* $\mathfrak{q}$.

(*i*) $\mathfrak{I}_{\mathfrak{pp}} = \mathbb{H}$;

(*ii*) $\mathfrak{I}_{\mathfrak{pq}} = \mathbb{H} \Rightarrow \mathfrak{p} = \mathfrak{q}$;

(*iii*) $\mathfrak{I}_{\mathfrak{pq}} = \mathfrak{I}_{\mathfrak{qp}}$;

(*iv*) $\mathfrak{I}_{\mathfrak{pq}}(\mathfrak{s} + \upsilon(\mathfrak{r})\mathfrak{r} + \mathfrak{t}) \geq \Upsilon\left(\mathfrak{I}_{\mathfrak{pa}}(\mathfrak{s}), \Upsilon\left(\mathfrak{I}_{\mathfrak{ab}}(\mathfrak{r}), \mathfrak{I}_{\mathfrak{bq}}(\mathfrak{t})\right)\right)$ *for all* $\mathfrak{s},\ \mathfrak{r},\ \mathfrak{t} > 0$.

Recall that a probabilistic generalized metric space is defined as a triple $(\mathcal{W}, \mathfrak{I}, \Upsilon)$ satisfying conditions (*i*) through (*iii*), along with the following inequality

$$(vi)\ \mathfrak{I}_{\mathfrak{kq}}(\mathfrak{s} + \mathfrak{r} + \mathfrak{t}) \geq \Upsilon\left(\mathfrak{I}_{\mathfrak{ka}}(\mathfrak{s}), \Upsilon\left(\mathfrak{I}_{\mathfrak{ab}}(\mathfrak{r}), \mathfrak{I}_{\mathfrak{bq}}(\mathfrak{t})\right)\right) \text{ (Quadrilateral inequality)}$$

This holds true for all distinct $\mathfrak{k}, \mathfrak{q}, \mathfrak{a}, \mathfrak{b} \in \mathcal{W}$ and $\mathfrak{s},\ \mathfrak{r},\ \mathfrak{t} > 0$.

Remark 2.2. *Each probabilistic generalized metric space can be seen as a probabilistic controlled generalized metric space when* $\upsilon(\gamma) = 1$ *for all* $\gamma \geq 0$. *However, the reverse assertion may not hold true. The subsequent example elucidates this point.*

Example 2.3. *Consider* $\mathcal{W} = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}\}$; *and* $\upsilon : \mathbb{R}^+ \to [1, +\infty)$ *where* $\forall \gamma \in \mathbb{R}^+; \upsilon(\gamma) = \gamma^3 + 1$. *Define* $\mathfrak{I} : \mathcal{W} \times \mathcal{W} \to \Delta^+$ *as follows*

$$\mathfrak{I}_{\rho\tau}(\varsigma) = \begin{cases} \mathbb{H}(\varsigma) & \text{if} \quad \rho = \tau, \\ \mathbb{H}(\varsigma - 10) & \text{if} \quad \rho, \tau \in \{\mathfrak{a}, \mathfrak{d}\} \text{ and } \rho \neq \tau, \\ \mathbb{H}(\varsigma - 2) & \text{otherwise.} \end{cases}$$

It is straightforward to confirm that $(\mathcal{W}, \mathfrak{I}, \Upsilon_M, \upsilon)$ *forms a (pcgms), however, it does not constitute a probabilistic generalized metric space due to the following reason:*

$$\begin{aligned} \mathfrak{I}_{\mathfrak{ad}}(9) &= 0 \\ &< 1 \\ &= \Upsilon_M(\mathfrak{I}_{\mathfrak{ab}}(3), \Upsilon_M(\mathfrak{I}_{\mathfrak{bc}}(3), \mathfrak{I}_{\mathfrak{cd}}(3))). \end{aligned}$$

Definition 2.4. *Consider* $(\mathcal{W}, \mathfrak{I})$ *as a probabilistic semimetric space (i.e., satisfying conditions* (*i*), (*ii*), *and* (*iii*) *of Definition 2.1). For any* ς *in* $\mathcal{W}$ *and* $\gamma > 0$, *the strong* γ*-neighborhood of* ς *is defined as the set*

$$\mathbb{U}_\varsigma(\gamma) = \{\tau \in \mathcal{W} :\ \mathfrak{I}_{\varsigma\tau}(\gamma) > 1 - \gamma\}.$$

The collection of strong neighborhoods at ς *is denoted as* $\Omega_\varsigma = \{\mathbb{U}_\varsigma(\gamma) :\ \gamma > 0\}$, *and the strong neighborhood system for* $\mathcal{W}$ *is the union* $\Omega = \bigcup_{\varsigma \in \mathcal{W}} \Omega_\varsigma$.

The sequence convergence is described as follows

Definition 2.5. *Let* $\{\mathfrak{x}_n\}$ *be a sequence in a probabilistic semimetric space* $(\mathcal{W}, \mathfrak{I})$. *Then*

1. *The sequence* $\{\mathfrak{x}_m\}$ *is said to converge to* $\mathfrak{x} \in \mathcal{W}$ *if for every* $\eta > 0$, *there exists a positive integer* J_η *such that* $\mathfrak{I}_{\mathfrak{x}_m\mathfrak{x}}(\eta) > 1 - \eta$ *whenever* $m \geq J_\eta$.

2. *The sequence* $\{\mathfrak{x}_m\}$ *is termed a Cauchy sequence if for every* $\eta > 0$, *there exists a positive integer* J_η *such that* $n,\ m \geq J_\eta$ *implies* $F_{\mathfrak{x}_n\mathfrak{x}_m}(\eta) > 1 - \eta$.

3. $(\mathcal{W}, \mathfrak{I})$ *is regarded as complete if every Cauchy sequence has a limit.*

As a probabilistic generalized metric space is a probabilistic controlled generalized metric space, the following deduction can be drawn from [6]

Remark 2.6. *Let* $(\mathcal{W}, \mathfrak{I}, \Upsilon, \upsilon)$ *be a probabilistic controlled generalized metric space. Despite* Υ *being continuous, the following assertions generally hold*

- *The notions of convergent sequence and Cauchy sequence are decoupled in* $(\mathcal{W},\mathfrak{I},\Upsilon,\upsilon)$.
- *In general,* $\mathfrak{I}$ *is not a continuous functional.*
- *Generally,* $(\mathcal{W},\mathfrak{I},\Upsilon,\upsilon)$ *equipped with the topology* Ω *is not a Hausdorff topological space.*

2.2. $\wp$*-probabilistic contraction in probabilistic controlled generalized metric space*

This section examines the $\wp$-probabilistic contraction definition and present some lemmas crucial for subsequent analysis.

Lemma 2.7 ([6]). *Let* $\{\mathfrak{y}_m\}$ *be a sequence in a probabilistic semimetric space* $(\mathcal{W}, \mathfrak{I})$ *and* $\mathfrak{y} \in \mathcal{W}$.

(1) $\{\mathfrak{y}_m\}$ *is convergent to* $\mathfrak{y}$ *if either:*
- $\lim_{m\to\infty} \mathfrak{I}_{\mathfrak{y}_m\mathfrak{y}}(\gamma)=1$ *for all* $\gamma>0$, *or*
- *for every* $\xi > 0$ *and* $\sigma \in (0,1)$, *there exists a positive integer* $J(\xi, \sigma)$ *such that* $\mathfrak{I}_{\mathfrak{y}_m\mathfrak{y}}(\xi)>1-\sigma$ *whenever* $m \geq J(\xi, \sigma)$.

(2) $\{\mathfrak{y}_m\}$ *is a Cauchy sequence if either:*
- $\lim_{m,n\to\infty} \mathfrak{I}_{\mathfrak{y}_m\mathfrak{y}_n}(\gamma)=1$ *for all* $\gamma>0$, *or*
- *for every* $\xi > 0$ *and* $\sigma \in (0,1)$, *there exists a positive integer* $J(\xi, \sigma)$ *such that* $\mathfrak{I}_{\mathfrak{y}_m\mathfrak{y}_n}(\xi)>1-\sigma$ *whenever* n, $m \geq J(\xi, \sigma)$.

The symbol Ψ denotes the collection of all functions $\wp:[0,\infty)\to[0,\infty)$ satisfying

$$0<\wp(r)<r \text{ and } \lim_{n\to\infty}\wp^n(r)=0 \text{ for each } r>0.$$

Definition 2.8 ([6]). *Let* $\wp:[0,\infty)\to[0,\infty)$ *be a function such that* $\wp(\gamma)<\gamma$ *for* $\gamma>0$, *and* $\mathfrak{f}$ *be a self-map of a probabilistic semimetric space* $(\mathcal{W}, \mathfrak{I})$. *We say that* $\mathfrak{f}$ *is a* $\wp$*-probabilistic contraction if*

$$\mathfrak{I}_{\mathfrak{f}\mathfrak{p}\mathfrak{f}\mathfrak{q}}(\wp(\xi)) \geq \mathfrak{I}_{\mathfrak{p}\mathfrak{q}}(\xi), \quad \forall \mathfrak{p},\mathfrak{q}\in\mathcal{W} \text{ and } \xi>0. \tag{1}$$

Lemma 2.9 ([6]). *Let* $\mathfrak{g}$ *be a distance distribution function in* $\mathcal{D}^+$, *if there exists* $\wp\in\Psi$ *for which*

$$\mathfrak{g}(\wp(\sigma))\geq\mathfrak{g}(\sigma) \textit{ for all } \sigma>0,$$

then $\mathfrak{g} = \mathbb{H}$.

Lemma 2.10 ([7]). *Let* $\wp\in\Psi$, $\{\mathfrak{g}_n\}$ *be a sequence of elements from* Δ^+, *and* $\mathfrak{g} \in \mathcal{D}^+$. *If for all* $\gamma > 0$ *and all* $n \in \mathbb{N}$

$$\mathfrak{g}_n(\wp^n(\gamma))\geq\mathfrak{g}(\gamma).$$

Then $\lim_{n\to+\infty}\mathfrak{g}_n(\gamma)=1$ *for each* $\gamma>0$.

3. Main Results

Let's start by establishing two fundamental lemmas crucial for our main theorem's proof.

Lemma 3.1. *Let* $\wp\in\Psi$, $\mathfrak{f}$ *be a self-map of a probabilistic semimetric space* $(\mathcal{W}, \mathfrak{I})$ *where* $Ran(\mathfrak{I})\subset\mathcal{D}^+$, *and let* $\{\mathfrak{y}_n\}$ *be a sequence of elements from* $\mathcal{W}$ *defined by* $\mathfrak{y}_0\in\mathcal{W}$ *and* $\mathfrak{f}\mathfrak{y}_n=\mathfrak{y}_{n+1}$ *such that* $\mathfrak{y}_n\neq\mathfrak{y}_{n-1}$ *for all* $n\in\mathbb{N}^*$. *Suppose that*

$$\mathfrak{I}_{\mathfrak{y}_{n+1}\mathfrak{y}_n}(\wp(\zeta))\geq\mathfrak{I}_{\mathfrak{y}_n\mathfrak{y}_{n-1}}(\zeta) \;\; \forall n\in\mathbb{N}^* \text{ and } \zeta>0. \tag{2}$$

Then $\mathfrak{y}_n\neq\mathfrak{y}_m$ *for all distinct* $n,m \in \mathbb{N}$.

Proof. Suppose there exist n, $m \in \mathbb{N}$ such that $\mathfrak{y}_n=\mathfrak{y}_m$ and $n < m$. In this situation, our objective is to prove that $\mathfrak{y}_{n+1}=\mathfrak{y}_n$. Firstly, let's establish through induction that for every $n \in \mathbb{N}$ and $\zeta > 0$, we have

$$\Im_{\mathfrak{y}_{n+j}\mathfrak{y}_{n+j+1}}(\wp(\zeta)) \geq \Im_{\mathfrak{y}_n\mathfrak{y}_{n+1}}(\zeta) \text{ for all } j \geq 1. \tag{3}$$

From Equation (2), we have $\Im_{\mathfrak{y}_{n+1}\mathfrak{y}_{n+2}}(\wp(\zeta)) \geq \Im_{\mathfrak{y}_n\mathfrak{y}_{n+1}}(\zeta)$, thus Equation (3) is valid for $j = 1$. Assume Equation (3) holds true for $j \geq 1$.

$$\Im_{\mathfrak{y}_{n+j+1}\mathfrak{y}_{n+j+2}}(\wp(\zeta)) \geq \Im_{\mathfrak{y}_{n+j}\mathfrak{y}_{n+j+1}}(\zeta) \geq \Im_{\mathfrak{y}_{n+j}\mathfrak{y}_{n+j+1}}(\wp(\zeta)) \geq \Im_{\mathfrak{y}_n\mathfrak{y}_{n+1}}(\zeta).$$

Thus, Equation (3) holds for every integer $j \geq 1$. Substituting $j = m - n$, we find

$$\Im_{\mathfrak{y}_n \mathfrak{f}\mathfrak{y}_n}(\wp(\zeta)) = \Im_{\mathfrak{y}_{n+(m-n)}\mathfrak{y}_{n+(m-n)+1}}(\wp(\zeta)) \geq \Im_{\mathfrak{y}_n \mathfrak{f}\mathfrak{y}_n}(\zeta) \text{ for each } \zeta > 0.$$

According to Lemma 2.9 and condition (ii) specified in Definition 2.1, this leads to the conclusion that $\mathfrak{y}_n = \mathfrak{y}_{n+1}$.

Lemma 3.2. *Let* $\wp \in \Psi$, $\mathfrak{f}$ *denotes a self-map of a probabilistic controlled generalized metric space* $(\mathcal{W}, \Im, \Upsilon, \upsilon)$ *equipped with a t-norm* Υ *of H-type, where* $Ran(\Im) \subset \mathcal{D}^+$ *and* υ *is an increasing mapping. Let* $\{\mathfrak{y}_n\}$ *be a sequence of elements from* $\mathcal{W}$ *defined by* $\mathfrak{y}_0 \in \mathcal{W}$ *and* $\mathfrak{f}\mathfrak{y}_n = \mathfrak{y}_{n+1}$ *such that* $\mathfrak{y}_i \neq \mathfrak{y}_j$ *for all distinct* $i, j \in \mathbb{N}$. *Suppose that*

$$\Im_{\mathfrak{y}_{n+1}\mathfrak{y}_{m+1}}(\wp(\gamma)) \geq \Im_{\mathfrak{y}_n\mathfrak{y}_m}(\gamma), \ \forall m, n \in \mathbb{N} \text{ and } \gamma > 0, \tag{4}$$

then the sequence $\{\mathfrak{y}_n\}$ *is Cauchy.*

Proof. To prove the sequence $\{\mathfrak{y}_n\}$ is Cauchy. To illustrate this, consider the sequence $\{\mathbb{G}_n\}$ consisting of elements from Δ^+ defined as follows

$$\mathbb{G}_n = \Im_{\mathfrak{y}_n\mathfrak{y}_{n+1}} \text{ for all } n \in \mathbb{N}.$$

By referring to (4), it's evident that for every $i \in \mathbb{N}$ and for every $\xi > 0$

$$\begin{aligned}
\mathbb{G}_i(\wp^i(\xi)) &= \Im_{\mathfrak{y}_i\mathfrak{y}_{i+1}}(\wp^i(\xi)) \\
&\geq \Im_{\mathfrak{y}_{i-1}\mathfrak{y}_i}(\wp^{i-1}(\xi)) \\
&\ \vdots \\
&\geq \Im_{\mathfrak{y}_0\mathfrak{y}_1}(\xi).
\end{aligned}$$

Given that the range of $\Im$ is contained within $\mathcal{D}^+$, we can infer from Lemma 2.10 that

$$\lim_{i\to+\infty} \Im_{\mathfrak{y}_i\mathfrak{y}_{i+1}}(\xi) = 1 \text{ for all } \xi > 0. \tag{5}$$

Likewise, we demonstrate that

$$\lim_{i\to+\infty} \Im_{\mathfrak{y}_i\mathfrak{y}_{i+2}}(\xi) = 1 \text{ for all } \xi > 0. \tag{6}$$

Subsequently, we establish by induction that for every $\xi > 0$ and for all $n \geq 1$

$$\Im_{\mathfrak{y}_n\mathfrak{y}_m}(\xi) \geq \Upsilon^{m-n}\left(\mathcal{C}_n(\xi)\right) \text{ for all } m > n, \tag{7}$$

where

$$\mathcal{C}_n(\xi) = \Upsilon\left(\Im_{\mathfrak{y}_n\mathfrak{y}_{n+1}}\left(\frac{\xi - \wp(\xi)}{2\upsilon\left(\frac{\xi - \wp(\xi)}{2}\right)}\right), \Im_{\mathfrak{y}_n\mathfrak{y}_{n+2}}\left(\frac{\xi - \wp(\xi)}{2}\right)\right).$$

Let's confirm that (7) holds for $m \in \{n + 1, n + 2\}$. Indeed, leveraging the monotonicity of $\Im$ and considering $\wp(\xi) < \xi$, we observe the following

For $m = n + 1$

$$\mathfrak{I}_{\mathfrak{y}_n\mathfrak{y}_{n+1}}(\xi) \geq \mathfrak{I}_{\mathfrak{y}_n\mathfrak{y}_{n+1}}\left(\frac{\xi-\wp(\xi)}{2}\right) \geq \mathfrak{I}_{\mathfrak{y}_n\mathfrak{y}_{n+1}}\left(\frac{\frac{\xi-\wp(\xi)}{2}}{\upsilon\left(\frac{\xi-\wp(\xi)}{2}\right)}\right) = \Upsilon\left(\mathfrak{I}_{\mathfrak{y}_n\mathfrak{y}_{n+1}}\left(\frac{\frac{\xi-\wp(\xi)}{2}}{\upsilon\left(\frac{\xi-\wp(\xi)}{2}\right)}\right),1\right)$$
$$\geq \Upsilon\left(\mathfrak{I}_{\mathfrak{y}_n\mathfrak{y}_{n+1}}\left(\frac{\frac{\xi-\wp(\xi)}{2}}{\upsilon\left(\frac{\xi-\wp(\xi)}{2}\right)}\right),\mathfrak{I}_{\mathfrak{y}_n\mathfrak{y}_{n+2}}\left(\frac{\xi-\wp(\xi)}{2}\right)\right)$$
$$= \Upsilon(\mathcal{C}_n(\xi),1) \geq \Upsilon^1(\mathcal{C}_n(\xi)).$$

For $m = n + 2$

$$\mathfrak{I}_{\mathfrak{y}_n\mathfrak{y}_{n+2}}(\xi) \geq \mathfrak{I}_{\mathfrak{y}_n\mathfrak{y}_{n+2}}\left(\frac{\xi-\wp(\xi)}{2}\right) = \Upsilon\left(\mathfrak{I}_{\mathfrak{y}_n\mathfrak{y}_{n+2}}\left(\frac{\xi-\wp(\xi)}{2}\right),1\right) \geq \Upsilon\left(\mathfrak{I}_{\mathfrak{y}_n\mathfrak{y}_{n+2}}\left(\frac{\xi-\wp(\xi)}{2}\right),\mathfrak{I}_{\mathfrak{y}_n\mathfrak{y}_{n+1}}\left(\frac{\frac{\xi-\wp(\xi)}{2}}{\upsilon\left(\frac{\xi-\wp(\xi)}{2}\right)}\right)\right)$$
$$\geq \Upsilon\left(\mathfrak{I}_{\mathfrak{y}_n\mathfrak{y}_{n+1}}\left(\frac{\frac{\xi-\wp(\xi)}{2}}{\upsilon\left(\frac{\xi-\wp(\xi)}{2}\right)}\right),\mathfrak{I}_{\mathfrak{y}_n\mathfrak{y}_{n+2}}\left(\frac{\xi-\wp(\xi)}{2}\right)\right)$$
$$= \mathcal{C}_n(\xi)$$
$$= \Upsilon(\mathcal{C}_n(\xi),1) \geq \Upsilon^2(\mathcal{C}_n(\xi)).$$

Consequently, (7) remains valid for $m \in \{n + 1, n + 2\}$. Let's assume that (7) holds for $m > n + 2$.

$$\mathfrak{I}_{\mathfrak{y}_n\mathfrak{y}_{m+1}}(\xi) = \mathfrak{I}_{\mathfrak{y}_n\mathfrak{y}_{m+1}}\left(\frac{\xi-\wp(\xi)}{2} + \upsilon\left(\frac{\xi-\wp(\xi)}{2}\right)\frac{\frac{\xi-\wp(\xi)}{2}}{\upsilon\left(\frac{\xi-\wp(\xi)}{2}\right)} + \wp(\xi)\right)$$
$$\geq \mathfrak{I}_{\mathfrak{y}_n\mathfrak{y}_{m+1}}\left(\frac{\xi-\wp(\xi)}{2} + \upsilon\left(\frac{\frac{\xi-\wp(\xi)}{2}}{\upsilon\left(\frac{\xi-\wp(\xi)}{2}\right)}\right)\frac{\frac{\xi-\wp(\xi)}{2}}{\upsilon\left(\frac{\xi-\wp(\xi)}{2}\right)} + \wp(\xi)\right)$$
$$\geq \Upsilon\left(\mathfrak{I}_{\mathfrak{y}_n\mathfrak{y}_{n+2}}\left(\frac{\xi-\wp(\xi)}{2}\right),\Upsilon\left(\mathfrak{I}_{\mathfrak{y}_{n+2}\mathfrak{y}_{n+1}}\left(\frac{\frac{\xi-\wp(t)}{2}}{\upsilon\left(\frac{\xi-\wp(\xi)}{2}\right)}\right),\mathfrak{I}_{\mathfrak{y}_{n+1}\mathfrak{y}_{m+1}}(\wp(\xi))\right)\right)$$
$$\geq \Upsilon\left(\mathfrak{I}_{\mathfrak{y}_n\mathfrak{y}_{n+2}}\left(\frac{\xi-\wp(\xi)}{2}\right),\Upsilon\left(\mathfrak{I}_{\mathfrak{y}_{n+2}\mathfrak{y}_{n+1}}\left(\wp\left(\frac{\frac{\xi-\wp(\xi)}{2}}{\upsilon\left(\frac{\xi-\wp(\xi)}{2}\right)}\right)\right),\mathfrak{I}_{\mathfrak{y}_{n+1}\mathfrak{y}_{m+1}}(\wp(\xi))\right)\right)$$
$$\geq \Upsilon\left(\mathfrak{I}_{\mathfrak{y}_n\mathfrak{y}_{n+2}}\left(\frac{\xi-\wp(\xi)}{2}\right),\Upsilon\left(\mathfrak{I}_{\mathfrak{y}_{n+1}\mathfrak{y}_n}\left(\frac{\frac{\xi-\wp(\xi)}{2}}{\upsilon\left(\frac{\xi-\wp(\xi)}{2}\right)}\right),\mathfrak{I}_{\mathfrak{y}_n\mathfrak{y}_m}(\xi)\right)\right)$$
$$\geq \Upsilon\left(\mathfrak{I}_{\mathfrak{y}_n\mathfrak{y}_{n+2}}\left(\frac{\xi-\wp(\xi)}{2}\right),\Upsilon\left(\mathfrak{I}_{\mathfrak{y}_n\mathfrak{y}_{n+1}}\left(\frac{\frac{\xi-\wp(\xi)}{2}}{\upsilon\left(\frac{\xi-\wp(\xi)}{2}\right)}\right),\mathfrak{I}_{\mathfrak{y}_n\mathfrak{y}_m}(\xi)\right)\right)$$
$$\geq \Upsilon\left(\Upsilon\left(\mathfrak{I}_{\mathfrak{y}_n\mathfrak{y}_{n+1}}\left(\frac{\frac{\xi-\wp(\xi)}{2}}{\upsilon\left(\frac{\xi-\wp(\xi)}{2}\right)}\right),\mathfrak{I}_{\mathfrak{y}_n\mathfrak{y}_{n+2}}\left(\frac{\xi-\wp(\xi)}{2}\right)\right),\mathfrak{I}_{\mathfrak{y}_n\mathfrak{y}_m}(\xi)\right).$$

Considering the induction hypothesis, we deduce

$$\mathfrak{I}_{\mathfrak{y}_n\mathfrak{y}_{m+1}}(\xi) \geq \Upsilon(\mathcal{C}_n(\xi),\Upsilon^{m-n}(\mathcal{C}_n(\xi))) = \Upsilon^{m-n+1}(\mathcal{C}_n(\xi)).$$

This suggests that (7) holds for all $m > n$. Let $\vartheta > 0$ and $\delta \in (0,1)$. Given that Υ is a t-norm of H-type, let $\lambda \in (0,1)$ be such that

$$\text{If } \xi > 1-\lambda \text{ then } \Upsilon^n(\xi) > 1-\delta \text{ for all } n \geq 1. \tag{8}$$

Utilizing the continuity of Υ at (1,1), along with (5) and (6), we deduce

$$\lim_{n\to+\infty} \Upsilon\left(\mathfrak{I}_{\mathfrak{y}_n\mathfrak{y}_{n+1}}\left(\frac{\vartheta-\wp(\vartheta)}{2\upsilon\left(\frac{\vartheta-\wp(\vartheta)}{2}\right)}\right), \mathfrak{I}_{\mathfrak{y}_n\mathfrak{y}_{n+2}}\left(\frac{\vartheta-\wp(\vartheta)}{2}\right)\right) = \Upsilon(1,1) = 1.$$

Thus, there exists $\mathfrak{H}_0 \in \mathbb{N}$ for which

$$\mathcal{C}_n(\vartheta) > 1-\lambda \text{ for all } n \geq \mathfrak{H}_0. \tag{9}$$

This inference from (7) and (8) suggests that

$$\mathfrak{I}_{\mathfrak{y}_n\mathfrak{y}_m}(\vartheta) > 1-\delta \text{ for all } m > n \geq \mathfrak{H}_0.$$

Therefore, $\{\mathfrak{y}_n\}$ forms a Cauchy sequence in $\mathcal{W}$.

Now, we are prepared to present and establish the main fixed-point theorem of this paper.

Theorem 3.3. *In a probabilistic controlled generalized metric space (pcgms)* $(\mathcal{W}, \mathfrak{I}, \Upsilon, \upsilon)$ *with a t-norm* Υ *of H-type, where* $Ran(\mathfrak{I}) \subset \mathcal{D}^+$ *and* υ *is an increasing mapping, let* $\mathfrak{f}: \mathcal{W} \to \mathcal{W}$ *be a* $\wp$*-probabilistic contraction with* $\wp \in \Psi$*; then*

(1) $\mathfrak{f}$ *possesses a unique fixed point* $\mathfrak{y}$.
(2) $\lim_{n\to+\infty} \mathfrak{f}^n x = \mathfrak{y}$, *for all* $x \in \mathcal{W}$.

Proof. Let $\mathfrak{y}_0 \in \mathcal{W}$ be given, and let's define the sequence $\{\mathfrak{y}_n\}$ as follows

$$\mathfrak{y}_n = \mathfrak{f}^n \mathfrak{y}_0 \text{ for } n \in \mathbb{N}.$$

If there is some $i \in \mathbb{N}$ where $\mathfrak{y}_{i+1} = \mathfrak{y}_i$, then $\mathfrak{f}(\mathfrak{y}_i) = \mathfrak{y}_i$, and the proof is complete.

Now, let's assume that $\mathfrak{y}_n \neq \mathfrak{y}_{n+1}$ for all $n \in \mathbb{N}$. Then, according to Lemma 3.1, we deduce that $\mathfrak{y}_i \neq \mathfrak{y}_j$ for all $i, j \in \mathbb{N}$ where $i \neq j$. In this context, referring to (1), we observe

$$\mathfrak{I}_{\mathfrak{y}_{n+1}\mathfrak{y}_{m+1}}(\wp(\xi)) \geq \mathfrak{I}_{\mathfrak{y}_n\mathfrak{y}_m}(\xi) \text{ for all } n,m \in \mathbb{N} \text{ and all } \xi > 0.$$

By Lemma 3.2, we deduce that $\{\mathfrak{y}_n\}$ forms a Cauchy sequence in $\mathcal{W}$. As $\mathcal{W}$ is complete, there exists $\mathfrak{y} \in \mathcal{W}$ for which

$$\lim_{n\to+\infty} \mathfrak{y}_n = \mathfrak{y}. \tag{10}$$

Next, we aim to demonstrate that $\mathfrak{f}(\mathfrak{y}) = \mathfrak{y}$. Consider $\vartheta > 0$, $\delta \in (0,1)$, and $n \in \mathbb{N}$. Given that $\wp(\vartheta) < \vartheta$, and leveraging the monotonicity of $\mathfrak{I}$ alongside (1), we deduce

$$\mathfrak{I}_{\mathfrak{y}_{n+1}\mathfrak{f}\mathfrak{y}}(\vartheta) \geq \mathfrak{I}_{\mathfrak{y}_{n+1}\mathfrak{f}\mathfrak{y}}(\wp(\vartheta)) = \mathfrak{I}_{\mathfrak{f}\mathfrak{y}_n\mathfrak{f}\mathfrak{y}}(\wp(\vartheta)) \geq \mathfrak{I}_{\mathfrak{y}_n\mathfrak{y}}(\vartheta).$$

As $\{\mathfrak{y}_n\}$ converges to $\mathfrak{y}$, there exists $\mathfrak{H}_0 \in \mathbb{N}$ for which

$$\mathfrak{I}_{\mathfrak{y}_n\mathfrak{y}}(\vartheta) > 1-\delta \text{ for all } n \geq \mathfrak{H}_0.$$

Thus, we have

$$\mathfrak{I}_{\mathfrak{y}_{n+1}\mathfrak{f}\mathfrak{y}}(\vartheta) > 1-\delta \text{ for all } n \geq \mathfrak{H}_0.$$

Consequently,

$$\lim_{n\to+\infty} \mathfrak{y}_{n+1} = \mathfrak{f}\mathfrak{y}. \tag{11}$$

Applying the quadrilateral inequality, we derive

$$\mathfrak{I}_{\mathfrak{y}\mathfrak{f}\mathfrak{y}}(\varepsilon) \geq \mathfrak{I}_{\mathfrak{y}\mathfrak{f}\mathfrak{y}}\left(\frac{\varepsilon}{3} + \frac{\varepsilon}{3\upsilon\left(\frac{\varepsilon}{3}\right)}\upsilon\left(\frac{\varepsilon}{3}\right) + \frac{\varepsilon}{3}\right) \geq \mathfrak{I}_{\mathfrak{y}\mathfrak{f}\mathfrak{y}}\left(\frac{\varepsilon}{3} + \frac{\varepsilon}{3\upsilon\left(\frac{\varepsilon}{3}\right)}\upsilon\left(\frac{\frac{\varepsilon}{3}}{\upsilon\left(\frac{\varepsilon}{3}\right)}\right) + \frac{\varepsilon}{3}\right)$$

$$\geq \Upsilon\left(\mathfrak{I}_{\mathfrak{y}\mathfrak{y}_n}(\frac{\varepsilon}{3}), Y\left(\mathfrak{I}_{\mathfrak{y}_n\mathfrak{y}_{n+1}}(\frac{\varepsilon}{3\upsilon(\frac{\varepsilon}{3})}), \mathfrak{I}_{\mathfrak{y}_{n+1}\mathfrak{f}\mathfrak{y}}(\frac{\varepsilon}{3})\right)\right), \tag{12}$$

for all $\varepsilon > 0$, $n \geq 1$.

Taking the limit as $n \to \infty$ in (12) and utilizing (10), (11), along with the fact that $\{\mathfrak{y}_n\}$ is a Cauchy sequence, we conclude that

$$\mathfrak{I}_{\mathfrak{f}\mathfrak{y}\mathfrak{y}}(\varepsilon) \geq 1 \text{ for all } \varepsilon > 0.$$

This remains valid unless $\mathfrak{I}_{f\mathfrak{y}\mathfrak{y}} = \mathbb{H}$, which implies $\mathfrak{f}\mathfrak{y} = \mathfrak{y}$. Consequently, $\mathfrak{f}$ possesses a fixed point.

Finally, let's establish the uniqueness of the fixed point. To establish this, let's assume that $\mathfrak{f}$ possesses another fixed point $\mathfrak{x} \in \mathcal{W}$.

Consider $\xi > 0$, then from (1), we can derive

$$\mathfrak{I}_{\mathfrak{y}\mathfrak{x}}(\wp(\xi)) = \mathfrak{I}_{\mathfrak{f}\mathfrak{y}\mathfrak{f}\mathfrak{x}}(\wp(\xi)) \geq \mathfrak{I}_{\mathfrak{y}\mathfrak{x}}(\xi).$$

Thus, based on Lemma 2.9, we can infer that $\mathfrak{y} = \mathfrak{x}$.

To wrap up this study, we will showcase the applicability of Theorem 3.1 through two consecutive examples.

Example 3.4. *Let* $\mathcal{W} = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}\}$ *and* $\mathfrak{I} : \mathcal{W} \times \mathcal{W} \to \Delta^+$ *defined by*

$$\mathfrak{I}_{\mathfrak{p}\mathfrak{q}}(\xi) = \begin{cases} \mathbb{H}(\xi) & \textit{if } \mathfrak{p} = \mathfrak{q}, \\ \mathbb{H}(\xi - 4) & \text{if} (\mathfrak{p}, \mathfrak{q}) \in \{(\mathfrak{c}, \mathfrak{e}), (\mathfrak{e}, \mathfrak{c})\} \ \mathfrak{p} \neq \mathfrak{q}, \\ \mathbb{H}(\xi - 1) & \textit{otherwise}. \end{cases}$$

Now, let's establish the mapping $\upsilon : \mathbb{R}^+ \to [1, +\infty)$ *using the formula* $\upsilon(t) = 4t+1$.

It's evident that $(\mathcal{W}, \mathfrak{I}, \Upsilon_M, \upsilon)$ *constitutes a probabilistic controlled generalized metric space. Furthermore,* $(\mathcal{W}, \mathfrak{I}, \Upsilon_M, \upsilon)$ *is complete because any Cauchy sequence in* $\mathcal{W}$ *must eventually become constant.*

Let's consider the function $\mathfrak{f}$: $\mathcal{W} \to \mathcal{W}$ *defined as* $\mathfrak{f}(\chi) = \mathfrak{e}$ for $\chi \in \mathcal{W}$.

For any $\mathfrak{p}$, $\mathfrak{q} \in \mathcal{W}$, *and* $\xi > 0$, *it's evident that* $\mathfrak{I}_{\mathfrak{f}\mathfrak{p}\mathfrak{f}\mathfrak{q}}(\wp(\xi)) = 1$, *which implies* $\mathfrak{I}_{\mathfrak{f}\mathfrak{p}\mathfrak{f}\mathfrak{q}}(\wp(\xi)) \geq \mathfrak{I}_{\mathfrak{p}\mathfrak{q}}(\xi)$. *Thus,* $\mathfrak{f}$ *constitutes a* $\wp$*-probabilistic contraction for any* $\wp \in \Psi$. *All prerequisites outlined in Theorem 3.1 are met, and* $\mathfrak{e}$ *stands as a fixed point of* $\mathfrak{f}$. *However, it's important to note that we cannot apply the results from* [6] *in this instance, as* $(\mathcal{W}, \mathfrak{I}, \Upsilon_M)$ *does not qualify as a probabilistic generalized metric space since it fails to satisfy the Quadrilateral inequality:*

$$\mathfrak{I}_{\mathfrak{c}\mathfrak{e}}(4) = 0 < \Upsilon_M^2\left(\mathfrak{I}_{\mathfrak{c}\mathfrak{b}}\left(\frac{4}{3}\right), \mathfrak{I}_{\mathfrak{b}\mathfrak{d}}\left(\frac{4}{3}\right), \mathfrak{I}_{\mathfrak{d}\mathfrak{e}}\left(\frac{4}{3}\right)\right) = 1.$$

Example 3.5. *Let* $\mathbb{K} = \{0,3\}$ *and* $\mathbb{V} = \left\{\frac{1}{n} : n \in \mathbb{N}^*\right\}$, with $\mathcal{W} = \mathbb{K} \cup \mathbb{V}$.

Define $\mathfrak{I} : \mathcal{W} \times \mathcal{W} \to \Delta^+$ *as follow*

$$\mathfrak{I}_{\mathfrak{k}\mathfrak{l}}(\xi)=\begin{cases}\mathbb{H}(\xi) & \textit{if } \mathfrak{k}=\mathfrak{l},\\ \mathbb{H}(\xi-3) & \textit{if } \mathfrak{k} \textit{ and } \mathfrak{l} \textit{ are in } \mathbb{K}\ \mathfrak{k}\neq\mathfrak{l},\\ \mathbb{H}(\xi-1) & \textit{if } \mathfrak{k} \textit{ and } \mathfrak{l} \textit{ are in } \mathbb{V}\ \mathfrak{k}\neq\mathfrak{l},\\ \mathbb{H}(\xi-\mathfrak{k}) & \textit{if } \mathfrak{l}\in\mathbb{K} \textit{ and } \mathfrak{k}\in\mathbb{V}.\end{cases}$$

Consider the function $\upsilon:\mathbb{R}^{+}\to[1,+\infty)$ *given by* $\upsilon(t)=4$.

It can be easily demonstrated that $(\mathcal{W},\mathfrak{I},\Upsilon_M,\upsilon)$ *constitutes a pcgms space. Indeed, it's apparent that conditions* (i) *through* (iii) *are satisfied by* $\mathfrak{I}$. *To verify condition* (iv), *let* $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ *denote distinct points in* $\mathcal{W}$, *and* $\mathfrak{s}$, $\mathfrak{r}$, $\mathfrak{t}>0$, *we need to show that*

$$\mathfrak{I}_{\mathfrak{a}\mathfrak{d}}(\mathfrak{s}+\mathfrak{r}\upsilon(\mathfrak{r})+\mathfrak{t})\geq\min\left(\mathfrak{I}_{\mathfrak{a}\mathfrak{b}}(\mathfrak{s}),\mathfrak{I}_{\mathfrak{b}\mathfrak{c}}(\mathfrak{r}),\mathfrak{I}_{\mathfrak{c}\mathfrak{d}}(\mathfrak{t})\right).$$

If

$$\min\left(\mathfrak{I}_{\mathfrak{a}\mathfrak{b}}(\mathfrak{s}),\mathfrak{I}_{\mathfrak{b}\mathfrak{c}}(\mathfrak{r}),\mathfrak{I}_{\mathfrak{c}\mathfrak{d}}(\mathfrak{t})\right)=0.$$

Then

$$\mathfrak{I}_{\mathfrak{a}\mathfrak{d}}(\mathfrak{s}+\mathfrak{r}\upsilon(\mathfrak{r})+\mathfrak{t})\geq\min\left(\mathfrak{I}_{\mathfrak{a}\mathfrak{b}}(\mathfrak{s}),\mathfrak{I}_{\mathfrak{b}\mathfrak{c}}(\mathfrak{r}),\mathfrak{I}_{\mathfrak{c}\mathfrak{d}}(\mathfrak{t})\right).$$

If

$$\min\left(\mathfrak{I}_{\mathfrak{a}\mathfrak{b}}(\mathfrak{s}),\mathfrak{I}_{\mathfrak{b}\mathfrak{c}}(\mathfrak{r}),\mathfrak{I}_{\mathfrak{c}\mathfrak{d}}(\mathfrak{t})\right)=1.$$

We examine the following cases:

Case 1: If $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ *are all in* $\mathbb{V}$, *then we have* $\mathfrak{s}+\upsilon(\mathfrak{r})\mathfrak{r}+\mathfrak{t}>4$, *implying* $\mathfrak{I}_{\mathfrak{a}\mathfrak{d}}(\mathfrak{s}+\upsilon(\mathfrak{r})\mathfrak{r}+\mathfrak{t})=1$.

Case 2: If either $\{\mathfrak{a},\ \mathfrak{b}\}$ or $\{\mathfrak{b},\ \mathfrak{c}\}$ or $\{\mathfrak{c},\ \mathfrak{d}\}$ *is a subset of* $\mathbb{K}$, *then we obtain* $\mathfrak{s}+\upsilon(\mathfrak{r})\mathfrak{r}+\mathfrak{t}>3$, *implying* $\mathfrak{I}_{\mathfrak{a}\mathfrak{d}}(\mathfrak{s}+\upsilon(\mathfrak{r})\mathfrak{r}+\mathfrak{t})=1$.

Case 3: If $\mathfrak{b}$, $\mathfrak{c}$ *are both in* $\mathbb{V}$, *we have* $\mathfrak{r}>1$, *thus* $\mathfrak{s}+\upsilon(\mathfrak{r})\mathfrak{r}+\mathfrak{t}>3$, *implying* $\mathfrak{I}_{\mathfrak{a}\mathfrak{d}}(\mathfrak{s}+\upsilon(\mathfrak{r})\mathfrak{r}+\mathfrak{t})=1$.

Case 4: If $\mathfrak{b}\in\mathbb{V}$ *and* $\mathfrak{c}\in\mathbb{K}$, *then* $\mathfrak{d}\in\mathbb{V}$ *and* $\mathfrak{t}>\mathfrak{d}$ *if* $\mathfrak{a}\in\mathbb{K}$.

$$\mathfrak{s}+\upsilon(\mathfrak{r})\mathfrak{r}+\mathfrak{t}>\mathfrak{d} \textit{ and } \mathfrak{I}_{\mathfrak{a}\mathfrak{d}}(\mathfrak{s}+\upsilon(\mathfrak{r})\mathfrak{r}+\mathfrak{t})=1.$$

If $\mathfrak{a}\in\mathbb{V}$, *we have* $\mathfrak{s}>1$, *hence* $\mathfrak{s}+\upsilon(\mathfrak{r})\mathfrak{r}+\mathfrak{t}>1$ *and* $\mathfrak{I}_{\mathfrak{a}\mathfrak{d}}(\mathfrak{s}+\upsilon(\mathfrak{r})\mathfrak{r}+\mathfrak{t})=1$.

Case 5: If $\mathfrak{b}\in\mathbb{K}$ *and* $\mathfrak{c}\in\mathbb{V}$, *then* $\mathfrak{a}\in\mathbb{V}$ *and* $\mathfrak{s}>\mathfrak{a}$ if $\mathfrak{d}\in\mathbb{K}$.

$$\mathfrak{s}+\upsilon(\mathfrak{r})\mathfrak{r}+\mathfrak{t}>\mathfrak{a} \textit{ and } \mathfrak{I}_{\mathfrak{a}\mathfrak{d}}(\mathfrak{s}+\upsilon(\mathfrak{r})\mathfrak{r}+\mathfrak{t})=1.$$

If $\mathfrak{d}\in\mathbb{V}$, *we have* $\mathfrak{t}>1$, *thus* $\mathfrak{s}+\upsilon(\mathfrak{r})\mathfrak{r}+\mathfrak{t}>1$ *and* $\mathfrak{I}_{\mathfrak{a}\mathfrak{d}}(\mathfrak{s}+\upsilon(\mathfrak{r})\mathfrak{r}+\mathfrak{t})=1$.

Consequently, from all the aforementioned cases, we conclude that $\mathfrak{I}_{\mathfrak{a}\mathfrak{d}}(\mathfrak{s}+\upsilon(\mathfrak{r})\mathfrak{r}+\mathfrak{t})=1$, *then*

$$\mathfrak{I}_{\mathfrak{a}\mathfrak{d}}(\mathfrak{s}+\mathfrak{r}\upsilon(\mathfrak{r})+\mathfrak{t})\geq\min\left(\mathfrak{I}_{\mathfrak{a}\mathfrak{b}}(\mathfrak{s}),\mathfrak{I}_{\mathfrak{b}\mathfrak{c}}(\mathfrak{r}),\mathfrak{I}_{\mathfrak{c}\mathfrak{d}}(\mathfrak{t})\right).$$

Thereby fulfilling condition (iv). *Hence,* $(\mathcal{W},\mathfrak{I},\Upsilon_M,\upsilon)$ *constitutes a pcgms space, with* Υ_M *being continuous, and there exists no* $\xi>0$ *where* $\mathbb{U}_0(\xi)\cap\mathbb{U}_3(\xi)=\emptyset$. *Consequently,* $(\mathcal{W},\mathfrak{I},\Upsilon_M,\upsilon)$ *equipped with the topology* $\mathfrak{I}$ *does not constitute a Hausdorff topological space. Also,* $(\mathcal{W},\mathfrak{I},\Upsilon_M,\upsilon)$ *is considered complete due to the property that every Cauchy sequence in* $\mathcal{W}$ *must eventually become constant.*

Now, let's examine the function $\mathfrak{f}:\mathcal{W}\to\mathcal{W}$ *given by* $\mathfrak{f}(\chi)=3$ for $\chi\in\mathcal{W}$. *For any* $\mathfrak{p}$, $\mathfrak{q}\in\mathcal{W}$ *and* $s>0$, $\mathfrak{I}_{\mathfrak{f}\mathfrak{q}\mathfrak{f}\mathfrak{p}}(\wp(s))=1$, *which implies* $\mathfrak{I}_{\mathfrak{f}\mathfrak{q}\mathfrak{f}\mathfrak{p}}(\wp(s))\geq\mathfrak{I}_{\mathfrak{p}\mathfrak{q}}(s)$. *Hence,* $\mathfrak{f}$ *is a* $\wp$*-probabilistic contraction for any* $\wp\in\Psi$. *All prerequisites outlined in Theorem 3.1 are met, and 3 stands as a fixed point of* $\mathfrak{f}$. *Nonetheless, we cannot apply the findings from* [6] *in this scenario, as* $(\mathcal{W},\ \mathfrak{I},\ \Upsilon_M)$ *does not qualify as a probabilistic generalized metric space due to its failure to adhere to the Quadrilateral inequality:*

$$\mathfrak{I}_{03}(3) = 0 < \Upsilon_M^2\left(\mathfrak{I}_{0\frac{1}{5}}\left(\frac{1}{3}\right), \mathfrak{I}_{\frac{1}{5}\frac{1}{3}}\left(\frac{31}{15}\right), \mathfrak{I}_{\frac{1}{3}3}\left(\frac{3}{5}\right)\right) = 1.$$

In conclusion, this paper contributes to the field of probabilistic (fuzzy) metric spaces by introducing the concept of probabilistic controlled generalized metric spaces and establishing its connection to classical probabilistic metric spaces. Moreover, it provides a significant theoretical result regarding the existence and uniqueness of fixed points for the $\wp$-probabilistic contraction mapping, even without imposing the Hausdorff condition. This research paves the way for further exploration and application in the realm of probabilistic (fuzzy) metric spaces.

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