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Second bounded variation in the sense of shiba with variable exponent

Ebner Pineda[1], Lorena Lopez[2], Luz Rodriguez[3]

[1]*Departamento de Matemáticas, Escuela Superior Politécnica del Litoral. ESPOL, FCNM, Campus Gustavo Galindo Km. 30.5 Vía Perimetral, P.O. Box 09-01-5863, Guayaquil, ECUADOR;* [2]*Departamento de Matemáticas, Universidad Centroccidental Lisandro Alvarado. Decanato de Ciencias Económicas y Empresariales, Barquisimeto, VENEZUELA;* [3]*Departamento de Matemáticas, Escuela Superior Politécnica del Litoral. ESPOL, FCNM, Campus Gustavo Galindo Km. 30.5 Vía Perimetral, P.O. Box 09-01-5863, Guayaquil, ECUADOR*

Abstract

In this paper we present a new notion of second bounded variation with variable exponent, studying the structure of these functions spaces, showing its basic properties and some inclusion results among them.

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1. Introduction

The interest generated by the classical notion of function of bounded variation, [6], has led to important generalizations of the concept. As in the classical case, these generalizations have found many applications in the study of certain differential and integral equations (see [2]). Consequently, the study of certain notions of generalized bounded variation takes an important direction in the field of mathematical analysis [1]. Two well-known generalizations are the functions of bounded p-variation and the functions of bounded Φ-variation, due to N. Wiener and L. C. Young respectively. In 1924 Wiener, [12], showed that the Fourier series of functions in one variable of finite p-variation converges almost everywhere. In 1937, L. C. Young [13], developed an integration theory with respect

Email addresses: epineda@espol.edu.ec (Ebner Pineda); lorena.lopez@ucla.edu.ve (Lorena Lopez); luzeurod@espol.edu.ec (Luz Rodríguez)

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to functions of finite φ-variation and showed that the Fourier series of such functions converges everywhere.

In 1972, Waterman [11] introduced the class of bounded variation functions ΛBV. In 1980, M. Shiba [9] generalizes this class and introduces the class $\Lambda_p BV$ $(1 \leq p < \infty)$. In [5], the authors introduce the notion of functions of second bounded variation $\Lambda_p^2 BV$, following the line traced by De la Vallée Poussin [4], in 1908, and M. Shiba [9], in 1980.

In more recent years, there has been a growing interest in the study of various mathematical problems with variable exponents. In [3], Castillo, Merentes and Rafeiro studied a new space of functions of bounded variation, they introduced the notion of bounded variation in the sense of Wiener with variable exponent. In [7], Mejías, Merentes and Sánchez showed important properties of the space of bounded variation in the sense of Wiener with variable exponent, they characterized this function space and conducted a study on the composition operator (Nemytskij).

Inspired by works [5] and [8], in this paper, we present a new notion of second bounded variation with variable exponent as a combination of the Wiener, Waterman and De La Vallee variations, studying the structure of these functions spaces, showing its basic properties and some inclusion results among them.

2. Preliminaries

Let us start with the definition that appears in Vyas [10], that is, the class $\Lambda_q BV$, given by the set of all functions $g : [c, d] \to \mathbb{R}$ of Λ_q−bounded variation on $[c, d]$.

Definition 2.1. *Given an interval $J = [c, d] \subset \mathbb{R}$ and a non-decreasing sequence of positive numbers $\Lambda = \{\lambda_j\}(j = 0, 1, 2, \cdots)$ such that $\sum(1/\lambda_j)$ diverges and $q \geq 1$. A function $g : J \to \mathbb{R}$ is said to be of Λ_q− bounded variation on J ($g \in \Lambda_q BV(J)$) if*

$$V_{\Lambda_q}(g) = V_\Lambda(g,q,J) = \sup_\xi V_\Lambda(\xi,g,q,J) < \infty,$$

where

$$V_\Lambda(\xi,g,q,J) := \left(\sum_{j=0}^{n} \frac{|g(t_j) - g(t_{j-1})|^q}{\lambda_j} \right)^{1/q},$$

and the supremum is taking over all partitions $\xi : c = t_0 < t_1 < \cdots < t_n = d$ of the interval J.

In [5], the authors introduce the notion of functions of second bounded variation $\Lambda_q^2 BV$, where it is shown that this class of functions is a normed vector space. In [8], it is proved in addition that this is a Banach space. The following definitions were considered in these papers.

Definition 2.2. Let $\Lambda = \{\lambda_j\}_{j=0}^\infty$ *a sequence of positive real numbers.* Λ is a $\mathcal{W}$-sequence if it is non-decreasing and $\sum(1/\lambda_j)$ diverges.

Definition 2.3. *Let $J = [c, d] \subset \mathbb{R}$, $\mathcal{P}_3(J)$ denotes the set of partitions $\xi = \{t_j\}_{j=0}^n$ of the interval J, with at least three points.*

Definition 2.4. [5] Let $1 \leq q < \infty$ and $\Lambda = \{\lambda_j\}_{j=0}^\infty$ *be a $\mathcal{W}$-sequence. The $(\Lambda, 2, q)$-th variation of g on $J = [c, d]$ is defined as*

$$V_{\Lambda,2,q}(g;J) = V_{\Lambda,2,q}(g) = \sup_\xi \left(\sum_{j=0}^{n-2} \frac{|D(g;t_{j+2},t_{j+1}) - D(g;t_{j+1},t_j)|^q}{\lambda_j} \right)^{1/q},$$

where $D(g;v,u) = \dfrac{g(v)-g(u)}{v-u}$ and the supremun is taken over all the partitions $\xi = \{t_j\}_{j=0}^n \in \mathcal{P}_3(J)$.

The sum in the definition 2.4, is called an approximate sum to $V_{\Lambda,2,q}(g)$. When $V_{\Lambda,2,q}(g) < \infty$, g has $(\Lambda, 2, q)$−th bounded variation on $J = [c, d]$.

Thus, $g \in \Lambda_q^2 BV(J)$.

Remark 2.5. *$\Lambda_q^2 BV$ together with the norm $\|g\|_{\Lambda,2,q} := \|g\|_\infty + V_{\Lambda,2,q}(g; J)$ is a Banach space. See* [8].

Some of the results tested in [5], were as follows.

Lemma 2.6. *If $J = [c, d]$, $1 \le q < \infty$ and $g \in \Lambda_q^2 BV(J)$, then $D(g; \cdot, \cdot)$ is bounded on $J \times J - \Delta$, where $\Delta = \{(x, x) : x \in J\}$.*

Lemma 2.7. *If $J = [c, d]$ and $g \in \Lambda_q^2 BV(J)$,* where $1 \le q < \infty$, *then g is Lipschitz on J.*

Remark 2.8. *It can be easily shown from the lemma above that if $g \in \Lambda_q^2 BV(J)$* then g is bounded.

Remark 2.9. *In general, if g is continuous on $J = [c, d]$ then*

$$\|g\|_\infty = \sup\{|g(x)| : x \in J\}.$$

3. Main Results

Next, we will present generalizations of several results shown in the articles [5] and [8]. We will start with the introduction of the notion of bounded second variation with variable exponent, then we will show their structure and some important properties, as well as a result of inclusion between these spaces.

In general, given a closed interval $J = [c, d] \subset \mathbb{R}$, we denote by $P(J)$ the set of all functions $q(\cdot) : J \to [1, +\infty)$. The elements of $P(J)$ are called Exponent Functions.

In what follows, J will be denote a closed interval $[c, d]$ in $\mathbb{R}$,

$$q_- = \inf\{q(x) : x \in J\} \quad \text{and} \quad q_+ = \sup\{q(x) : x \in J\}.$$

Definition 3.1. *A labeled partition of J, denoted by $\xi*$, is a partition $\xi = \{t_j\}_{j=0}^n \in \mathcal{P}_3(J)$ together with a finite sequence $\{x_j\}_{j=0}^{n-1}$ such that $t_j \le x_j \le t_{j+1}$ for all $j \in \{0, 1, \ldots, n-1\}$.*

Definition 3.2. *$\mathcal{P}_3^*(J)$ is the set of labeled partitions ξ^* of J.*

Definition 3.3. *Let Λ a $\mathcal{W}$-sequence, $q(\cdot) \in P(J)$ and $g : J \to \mathbb{R}$, the functional $V_{\Lambda,2}^{q(\cdot)}(g)$ given by*

$$V_{\Lambda,2}^{q(\cdot)}(g) = V_{\Lambda,2}^{q(\cdot)}(g;J) := \sup_{\xi^*} \sum_{j=0}^{n-2} \frac{|D(g;t_{j+2},t_{j+1}) - D(g;t_{j+1},t_j)|^{q(x_j)}}{\lambda_j}$$

is called the second variation with variable exponent of g on J, where $D(g;v,u) = \dfrac{g(v)-g(u)}{v-u}$ and the supremum is taken on all $\xi^* \in \mathcal{P}_3^*(J)$.

Definition 3.4. *The set of functions of second bounded variation with variable exponent, denote by $\Lambda_{q(\cdot)}^2 BV(J)$, is defined as*

$$\Lambda_{q(\cdot)}^2 BV(J) = \left\{ g : J \to \mathbb{R} \,/\, \exists \delta > 0 \text{ where } V_{\Lambda,2}^{q(\cdot)}\left(\frac{g}{\delta}\right) < \infty \right\}.$$

The following remark will be very useful in all this work.

Remark 3.5. *Since $q_- \le q(x) \le q_+$, $\forall x \in J$ then from the definition of $V_{\Lambda,2}^{q(\cdot)}(\cdot)$ and with $q_+ < \infty$, we have that*

(*i*) *If $0 < \gamma \le 1$ then*

$$\gamma^{q_+} V_{\Lambda,2}^{q(\cdot)}(g) \le V_{\Lambda,2}^{q(\cdot)}(\gamma g) \le \gamma^{q_-} V_{\Lambda,2}^{q(\cdot)}(g).$$

(*ii*) *If* $\gamma > 1$ *then*

$$\gamma^{q_-} V_{\Lambda,2}^{q(\cdot)}(g) \le V_{\Lambda,2}^{q(\cdot)}(\gamma g) \le \gamma^{q_+} V_{\Lambda,2}^{q(\cdot)}(g).$$

Now we will present some properties of the variation $V_{\Lambda,2}^{q(\cdot)}(g)$ necessary to prove that $\Lambda_{q(\cdot)}^2 BV(J)$ is a vector space.

Theorem 3.6. *Properties of* $V_{\Lambda,2}^{q(\cdot)}$.

(*i*) *If* $\delta_1 > \delta_2 > 0$ then $V_{\Lambda,2}^{q(\cdot)}\left(\frac{g}{\delta_1}\right) \le V_{\Lambda,2}^{q(\cdot)}\left(\frac{g}{\delta_2}\right)$

(*ii*) $V_{\Lambda,2}^{q(\cdot)}$ *is convex.*

(*iii*) $V_{\Lambda,2}^{q(\cdot)}(|c|g) = V_{\Lambda,2}^{q(\cdot)}(cg)$

(*iv*) *If* $V_{\Lambda,2}^{q(\cdot)}\left(\frac{g}{\gamma}\right) < \infty$ *then the function* $h : [\gamma, +\infty) \to \mathbb{R}$ *given by* $h(\delta) = V_{\Lambda,2}^{q(\cdot)}\left(\frac{g}{\delta}\right), \delta \ge \gamma$ *satisfies*

  (a) $h(\delta) \to 0$, $\delta \to \infty$.
  (b) *If* $q_+ < \infty$ *then h is continuous on* $[\gamma, +\infty)$.

Proof. Let $\xi^* \in \mathcal{P}_3^*(J)$, $q(\cdot) \in P(J)$ and Λ a $\mathcal{W}$-sequence.

(*i*) Let $\delta_1, \delta_2 > 0$ such that $\delta_1 > \delta_2$. Then,

$$\begin{aligned} &\left| D\left(\frac{g}{\delta_1}; t_{i+2}, t_{i+1}\right) - D\left(\frac{g}{\delta_1}; t_{i+1}, t_i\right) \right|^{q(x_i)} \\ &\le \left(\frac{1}{\delta_2} | D(g; t_{i+2}, t_{i+1}) - D(g; t_{i+1}, t_i) | \right)^{q(x_i)} \\ &= \left| D\left(\frac{g}{\delta_2}; t_{i+2}, t_{i+1}\right) - D\left(\frac{g}{\delta_2}; t_{i+1}, t_i\right) \right|^{q(x_i)}. \end{aligned}$$

Dividing by λ_i, adding and taking supreme over all partitions $\xi^* \in \mathcal{P}_3^*(J)$, we obtain

$$V_{\Lambda,2}^{q(\cdot)}\left(\frac{g}{\delta_1}\right) \le V_{\Lambda,2}^{q(\cdot)}\left(\frac{g}{\delta_2}\right).$$

(*ii*) Let $\gamma, \delta \ge 0$ such that $\gamma + \delta = 1$. Using the fact that the function $f(x) = x^a$, $a \ge 1$ is increasing and convex, we have

$$\begin{aligned} &| D(\gamma g + \delta h; t_{i+2}, t_{i+1}) - D(\gamma g + \delta h; t_{i+1}, t_i) |^{q(x_i)} \\ &\le (\gamma | D(g; t_{i+2}, t_{i+1}) - D(g; t_{i+1}, t_i) | + \delta | D(h; t_{i+2}, t_{i+1}) - D(h; t_{i+1}, t_i) |)^{q(x_i)} \\ &\le \gamma | D(g; t_{i+2}, t_{i+1}) - D(g; t_{i+1}, t_i) |^{q(x_i)} + \delta | D(h; t_{i+2}, t_{i+1}) - D(h; t_{i+1}, t_i) |^{q(x_i)}, \end{aligned}$$

which implies that

$$V_{\Lambda,2}^{q(\cdot)}(\gamma g + \delta h) \le \gamma V_{\Lambda,2}^{q(\cdot)}(g) + \delta V_{\Lambda,2}^{q(\cdot)}(h).$$

(iii) Let $c \in \mathbb{R}$, using the definition of $V_{\Lambda,2}^{q(\cdot)}(g)$ for cg and $|c|g$ the result is obtained immediately.

(*iv*) (a) Let $\delta \geq \gamma > 0$, by the remark 3.5, $h(\delta) \leq \left(\frac{\gamma}{\delta}\right)^{q_-} h(\gamma)$. Thus, $h(\delta) \to \delta \to \infty$.

(b) Let $\delta, \delta_0 \geq \gamma > 0$ then $h(\delta), h(\delta_0) \leq h(\gamma) < \infty$.

Let us first consider the case where $\delta > \delta_0$. As $\frac{\delta_0}{\delta} < 1$, by the remark 3.5

$$\begin{aligned} &\left(\frac{\delta_0}{\delta}\right)^{q_+} V_{\Lambda,2}^{q(\cdot)}\left(\frac{g}{\delta_0}\right) \leq V_{\Lambda,2}^{q(\cdot)}\left(\frac{g}{\delta}\right) \leq \left(\frac{\delta_0}{\delta}\right)^{q_-} V_{\Lambda,2}^{q(\cdot)}\left(\frac{g}{\delta_0}\right) \\ &\Rightarrow \left(\frac{\delta_0}{\delta}\right)^{q_+} h(\delta_0) \leq h(\delta) \leq \left(\frac{\delta_0}{\delta}\right)^{q_-} h(\delta_0). \end{aligned} \tag{1}$$

From inequality (1), we get

$$\left[\left(\frac{\delta_0}{\delta}\right)^{q_+} - 1\right] h(\delta_0) \leq h(\delta) - h(\delta_0) \leq \left[\left(\frac{\delta_0}{\delta}\right)^{q_-} - 1\right] h(\delta_0),$$

and taking limit when $\delta \to \delta_0^+$ we have to

$$\lim_{\delta \to \delta_0^+} h(\delta) = h(\delta_0). \tag{2}$$

Now, Let us consider the case $\delta < \delta_0$. As $\frac{\delta_0}{\delta} > 1$, again by the remark 3.5,

$$\begin{aligned} &\left(\frac{\delta_0}{\delta}\right)^{q_-} V_{\Lambda,2}^{q(\cdot)}\left(\frac{g}{\delta_0}\right) \leq V_{\Lambda,2}^{q(\cdot)}\left(\frac{g}{\delta}\right) \leq \left(\frac{\delta_0}{\delta}\right)^{q_+} V_{\Lambda,2}^{q(\cdot)}\left(\frac{g}{\delta_0}\right) \\ &\Rightarrow \left(\frac{\delta_0}{\delta}\right)^{q_-} h(\delta_0) \leq h(\delta) \leq \left(\frac{\delta_0}{\delta}\right)^{q_+} h(\delta_0). \end{aligned} \tag{3}$$

From inequality (3), we get

$$\left[\left(\frac{\delta_0}{\delta}\right)^{q_-} - 1\right] h(\delta_0) \leq h(\delta) - h(\delta_0) \leq \left[\left(\frac{\delta_0}{\delta}\right)^{q_+} - 1\right] h(\delta_0),$$

and taking limit when $\delta \to \delta_0^-$ we have to

$$\lim_{\delta \to \delta_0^-} h(\delta) = h(\delta_0). \tag{4}$$

From (2) and (4) we conclude that h is continuous in δ_0, $\forall \delta_0 \geq \gamma > 0$. □

Now we prove that the set of functions of second bounded variation with variable exponent is a vector space.

Theorem 3.7. $\Lambda^2_{q(\cdot)}BV\,(J)$ is a vector space.

Proof. By definition is clear that $0 \in \Lambda^2_{q(\cdot)}BV(J)$.

Let Λ a W-sequence and $g, h \in \Lambda^2_{q(\cdot)}BV\,(J)$. Thus, there exist δ_1, $\delta_2 > 0$ such that

$$V^{q(\cdot)}_{\Lambda,2}\left(\frac{g}{\delta_1}\right) < \infty \ \ and \ \ V^{q(\cdot)}_{\Lambda,2}\left(\frac{h}{\delta_2}\right) < \infty.$$

Let $\hat{\delta} = \max\{\delta_1, \delta_2\} > 0$, by theorem 3.6 we have

$$V^{q(\cdot)}_{\Lambda,2}\left(\frac{g}{\hat{\delta}}\right) \le V^{q(\cdot)}_{\Lambda,2}\left(\frac{g}{\delta_1}\right) < \infty$$

and

$$V^{q(\cdot)}_{\Lambda,2}\left(\frac{h}{\hat{\delta}}\right) \le V^{q(\cdot)}_{\Lambda,2}\left(\frac{h}{\delta_2}\right) < \infty.$$

Let $\gamma \in \mathbb{R}$ and $\mu = (|\gamma| + 1)\,\hat{\delta} > 0$ and $\xi^* \in \mathcal{P}^*_3(J)$. Then

$$\begin{aligned}&|D(\gamma g + h; t_{i+2}, t_{i+1}) - D(\gamma g + h; t_{i+1}, t_i)|\\ &\quad \le |\gamma|\,|D(g; t_{i+2}, t_{i+1}) - D(g; t_{i+1}, t_i)| + |D(h; t_{i+2}, t_{i+1}) - D(h; t_{i+1}, t_i)|.\end{aligned}$$

Using again the convexity and monotony of $f(x) = x^a$, $a \ge 1$ and since $\frac{|\gamma|}{|\gamma|+1} + \frac{1}{|\gamma|+1} = 1$, we have

$$\begin{aligned}&\left|D\left(\frac{\gamma g+h}{\mu};t_{i+2},t_{i+1}\right) - D\left(\frac{\gamma g+h}{\mu};t_{i+1},t_i\right)\right|^{q(x_i)}\\
&\le \left(\frac{|\gamma|}{\mu}|D(g;t_{i+2},t_{i+1}) - D(g;t_{i+1},t_i)| + \frac{1}{\mu}|D(h;t_{i+2},t_{i+1}) - D(h;t_{i+1},t_i)|\right)^{q(x_i)}\\
&= \left(\frac{|\gamma|}{|\gamma|+1}\left|D\left(\frac{g}{\hat{\delta}};t_{i+2},t_{i+1}\right) - D\left(\frac{g}{\hat{\delta}};t_{i+1},t_i\right)\right| + \frac{1}{|\gamma|+1}\left|D\left(\frac{h}{\hat{\delta}};t_{i+2},t_{i+1}\right) - D\left(\frac{h}{\hat{\delta}};t_{i+1},t_i\right)\right|\right)q(x_i)\\
&\le \frac{|\gamma|}{|\gamma|+1}\left|D\left(\frac{g}{\hat{\delta}};t_{i+2},t_{i+1}\right) - D\left(\frac{g}{\hat{\delta}};t_{i+1},t_i\right)\right|^{q(x_i)} + \frac{1}{|\gamma|+1}\left|D\left(\frac{h}{\hat{\delta}};t_{i+2},t_{i+1}\right) - D\left(\frac{h}{\hat{\delta}};t_{i+1},t_i\right)\right|^{q(x_i)}.\end{aligned}$$

Thus,

$$V^{q(\cdot)}_{\Lambda,2}\left(\frac{\gamma g+h}{\mu}\right) \le \frac{|\gamma|}{|\gamma|+1}V^{q(\cdot)}_{\Lambda,2}\left(\frac{g}{\hat{\delta}}\right) + \frac{|\gamma|}{|\gamma|+1}V^{q(\cdot)}_{\Lambda,2}\left(\frac{g}{\hat{\delta}}\right) < \infty$$

Therefore,

$$\gamma\, g + h \in \Lambda^2_{q(\cdot)}BV\,(J),$$

and we conclude that $\Lambda^2_{q(\cdot)}BV\,(J)$ is a vector space. □

The following result is crucial to define the norm of the space.

Lemma 3.8. If $g \in \Lambda^2_{q(\cdot)}BV(J)$ then $D(g;\ \cdot,\ \cdot)$ is bounded on $J \times J - \Delta$, where $\Delta = \{(x, x) : x \in J\}$.

Proof. Let $g \in \Lambda^2\,\Lambda^2_{q(\cdot)}BV\,(J)$ then there exists $\delta > 0$ such that $V^{q(\cdot)}_{\Lambda,2}\left(\frac{g}{\delta}\right) < \infty$. Let us $h = \frac{g}{\delta}$.

First, we note that if $c \leq t_0 < t_1 < t_2 < t_3 \leq d$, then, by the definition of $V_{\Lambda,2}^{q(\cdot)}(h)$, it follows that

$$\begin{aligned}
&|D(h;t_3,t_2) - D(h;t_1,t_0)| \\
&\quad = |D(h;t_3,t_2) - D(h;t_2,t_1) + D(h;t_2,t_1) - D(h;t_1,t_0)| \\
&\quad \leq |D(h;t_3,t_2) - D(h;t_2,t_1)| + |D(h;t_2,t_1) - D(h;t_1,t_0)|.
\end{aligned}$$

It is clear that, $a < 1 + a^p$, $\forall a \geq 0$ and $p \geq 1$. Thus, we have in the previous inequality

(*i*) For $c = t_0 < t_1 < t_2 < t_3 \leq d$, since $\lambda_0 \leq \lambda_1 \leq \lambda_2$, we have

$$\begin{aligned}
&|D(h;t_3,t_2) - D(h;t_1,t_0)| \\
&\leq 2 + |D(h;t_3,t_2) - D(h;t_2,t_1)|^{q(x_1)} + |D(h;t_2,t_1) - D(h;t_1,t_0)|^{q(x_0)} \\
&\leq 2 + \lambda_1 \frac{|D(h;t_3,t_2) - D(h;t_2,t_1)|^{q(x_1)}}{\lambda_1} + \lambda_0 \frac{|D(h;t_2,t_1) - D(h;t_1,t_0)|^{q(x_0)}}{\lambda_0} \\
&\leq 2 + \lambda_2 V_{\Lambda,2}^{q(\cdot)}(h).
\end{aligned} \tag{5}$$

(*ii*) For $c < t_0 < t_1 < t_2 < t_3 \leq d$, again since $\lambda_1 \leq \lambda_2$, we have

$$\begin{aligned}
&|D(h;t_3,t_2) - D(h;t_1,t_0)| \\
&\leq 2 + |D(h;t_3,t_2) - D(h;t_2,t_1)|^{q(x_2)} + |D(h;t_2,t_1) - D(h;t_1,t_0)|^{q(x_1)} \\
&\leq 2 + \lambda_2 \frac{|D(h;t_3,t_2) - D(h;t_2,t_1)|^{q(x_2)}}{\lambda_2} + \lambda_1 \frac{|D(h;t_2,t_1) - D(h;t_1,t_0)|^{q(x_1)}}{\lambda_1} \\
&\leq 2 + \lambda_2 V_{\Lambda,2}^{q(\cdot)}(h).
\end{aligned} \tag{6}$$

Now, taking an arbitrary point $z \in (c, d)$, $v_0, v_1 \in J$ and $B = |D(h;z,c)|$.

The proof of lemma depend on how v_0, v_1 are located with respect to c, d and z.

Case 1. Suppose that $c < v_0 < z < v_1 < d$. If v_2 is a point such that $v_1 < v_2 < d$ then $c < v_0 < z < v_1 < v_2 < d$, using (5) we get

$$\begin{aligned}
&|D(h;v_1,v_0)| \\
&= \left|D(h;v_1,v_0) - D(h;v_2,v_1) + D(h;v_2,v_1) - D(h;z,c) + D(h;z,c)\right| \\
&\leq \left|D(h;v_1,v_0) - D(h;v_2,v_1)\right| + \left|D(h;v_2,v_1) - D(h;z,c)\right| + \left|D(h;z,c)\right| \\
&\leq 1 + \lambda_1 \frac{|D(h;v_2,v_1) - D(h;v_1,v_0)|^{q(x_1)}}{\lambda_1} + 2 + \lambda_2 V_{\Lambda,2}^{q(\cdot)}(h) + B \\
&\leq 3 + \lambda_1 V_{\Lambda,2}^{q(\cdot)}(h) + \lambda_2 V_{\Lambda,2}^{q(\cdot)}(h) + B \\
&\leq 3 + 2\lambda_2 V_{\Lambda,2}^{q(\cdot)}(h) + B.
\end{aligned}$$

Case 2. Suppose that $c < v_0 < z < v_1 = d$. If v_2 is a point such that $z < v_2 < v_1 = d$ then $c < v_0 < z < v_2 < v_1 = d$, using (5) we get

$$\begin{aligned}
&\left|D(h;v_1,v_0)\right| \\
&= \left|D(h;v_1,v_0) - D(h;v_0,c) + D(h;v_0,c) - D(h;v_1,v_2) + D(h;v_1,v_2) - D(h;z,c) + D(h;z,c)\right| \\
&\leq \left|D(h;v_1,v_0) - D(h;v_0,c)\right| + \left|D(h;v_0,c) - D(h;v_1,v_2)\right| + \left|D(h;v_1,v_2) - D(h;z,c)\right| + \left|D(h;z,c)\right| \\
&\leq 1 + \lambda_0 \frac{\left|D(h;v_1,v_0) - D(h;v_0,c)\right|^{q(x_0)}}{\lambda_0} + 2 + \lambda_2 V_{\Lambda,2}^{q(\cdot)}(h) + 2 + \lambda_2 V_{\Lambda,2}^{q(\cdot)}(h) + B \\
&\leq 5 + \lambda_0 V_{\Lambda,2}^{q(\cdot)}(h) + 2\,\lambda_2 V_{\Lambda,2}^{q(\cdot)}(h) + B \\
&\leq 5 + 3\lambda_2 V_{\Lambda,2}^{q(\cdot)}(h) + B.
\end{aligned}$$

Case 3. Suppose that $c < v_0 < v_1 \leq z < d$. If v_2 is a point such that $z < v_2 < d$ we have that $c < v_0 < v_1 \leq z < v_2 < d$, using again (5) we get

$$\begin{aligned}
&|D(h;v_1,v_0)| \\
&= |D(h;v_1,v_0) - D(h;v_2,v_1) + D(h;v_2,v_1) - D(h;d,v_2) + D(h;d,v_2) - D(h;z,c) + D(h;z,c)| \\
&\leq |D(h;v_1,v_0) - D(h;v_2,v_1)| + |D(h;v_2,v_1) - D(h;d,v_2)| + |D(h;d,v_2) - D(h;z,c)| + |D(h;z,c)| \\
&\leq 1 + \lambda_1 \frac{|D(h;v_2,v_1) - D(h;v_1,v_0)|^{q(x_1)}}{\lambda_1} + 1 + \lambda_2 \frac{|D(h;d,v_2) - D(h;v_2,v_1)|^{q(x_2)}}{\lambda_2} + 2 + \lambda_2 V_{\Lambda,2}^{q(\cdot)}(h) + B \\
&\leq 2 + \lambda_2 V_{\Lambda,2}^{q(\cdot)}(h) + 2 + \lambda_2 V_{\Lambda,2}^{q(\cdot)}(h) + B \\
&= 4 + 2\lambda_2 V_{\Lambda,2}^{q(\cdot)}(h) + B.
\end{aligned}$$

The remaining three cases,

- $c = v_0 < z < v_1 < v_2 < d$
- $c = v_0 < z < v_2 < v_1 = d$
- $c < z \leq v_0 < v_1 < v_2 < d,$

can be shown in analogous way, using (5) and (6).

We conclude that $D(h; \cdot, \cdot)$ is bounded.

Hence, there exists $C > 0$ such that

$$\left| D\left(\frac{g}{\delta}; u,v\right) \right| \leq C, \quad \forall u,v \in J, u \neq v.$$

This implies that

$$|D(g;u,v)| \leq C\,\delta, \quad \forall u,v \in J, u \neq v.$$

Therefore, $D(g; \cdot, \cdot)$ is bounded. □

Corollary 3.9. If $g \in \Lambda^2_{q(\cdot)}BV(J)$ then g is Lipschitz on J . In consequence, g es continuous and bounded on J .

Proof. Let $g \in \Lambda^2_{q(\cdot)}BV(J)$. Then, by lemma 3.8, there exists $C > 0$ such that

$$|D(g;u,v)| \leq C, \quad \forall u,v \in J, u \neq v.$$

Thus,

$$\left| \frac{g(v) - g(u)}{v - u} \right| \leq C \Rightarrow |g(v) - g(u)| \leq C\,|v - u|, \ \forall u,v \in J,$$

therefore, g is Lipschitz on J. □

We define the functional $\|\cdot\|_{\Lambda^2_{q(\cdot)}} : \Lambda^2_{q(\cdot)}BV(J) \to \mathbb{R}$ *by*

$$\|g\|_{\Lambda^2_{q(\cdot)}} := g_\infty + \mu_{\Lambda^2_{q(\cdot)}}(g), \text{ where } \mu_{\Lambda^2_{q(\cdot)}}(g) = \inf\left\{\delta > 0 : V_{\Lambda,2}^{q(\cdot)}\left(\frac{g}{\delta}\right) \leq 1\right\}.$$

Moreover, $\mu_{\Lambda^2_{q(\cdot)}}(g) < \infty$ by theorem 3.6 and $\|g\|_\infty < \infty$ by lemma 3.8, for $g \in \Lambda^2_{q(\cdot)}BV(J)$.

Now, we prove that $(\Lambda^2_{q(\cdot)}BV(J), \|\cdot\|_{\Lambda^2_{q(\cdot)}})$ is a normed vector space.

Theorem 3.10. *The functional* $\|\cdot\|_{\Lambda^2_{q(\cdot)}}$ *is a norm on* $\Lambda^2_{q(\cdot)}BV(J)$

Proof. Let g, $h \in \Lambda^2_{q(\cdot)}BV(J)$, $\beta \in \mathbb{R}$ and Λ a W-sequence.

(i) It is clear that

$$\|g\|_{\Lambda^2_{q(\cdot)}} \geq 0, \forall g \in \Lambda^2_{q(\cdot)}BV(J).$$

Moreover, $\|g\|_{\Lambda^2_{q(\cdot)}} = 0 \Leftrightarrow g = 0$.

If $g = 0$, $V^{q(\cdot)}_{\Lambda,2}\left(\frac{g}{\delta}\right) = 0 \leq 1, \forall \delta > 0$, so that $\mu_{\Lambda^2_{q(\cdot)}}(g) = 0$ and therefore $\|g\|_{\Lambda^2_{q(\cdot)}} = 0$.

Now suppose that $\|g\|_{\Lambda^2_{q(\cdot)}} = 0$. Then $\|g\|_\infty + \mu_{\Lambda^2_{q(\cdot)}}(g) = 0$, which implies that $\|g\|_\infty = 0$, therefore $g = 0$.

(ii) Let us suppose $\beta = 0$, thus $\|\beta g\|_{\Lambda^2_{q(\cdot)}} = |\beta|\,\|g\|_{\Lambda^2_{q(\cdot)}} = 0$.
Now, if $\beta \neq 0$, we obtain

$$\begin{aligned}
\|\beta g\|_{\Lambda^2_{q(\cdot)}} &= \|\beta g\|_\infty + \mu_{\Lambda^2_{q(\cdot)}}(\beta g) \\
&= \|\beta g\|_\infty + \inf\left\{\delta > 0 : V^{q(\cdot)}_{\Lambda,2}\left(\frac{\beta g}{\delta}\right) \leq 1\right\} \\
&= |\beta|\|g\|_\infty + |\beta| \inf\left\{\frac{\delta}{|\beta|} > 0 : V^{q(\cdot)}_{\Lambda,2}\left(\frac{g}{\frac{\delta}{|\beta|}}\right) \leq 1\right\} \\
&= |\beta|\left(\|g\|_\infty + \inf\left\{\hat{\delta} > 0 : V^{q(\cdot)}_{\Lambda,2}\left(\frac{g}{\hat{\delta}}\right) \leq 1\right\}\right) \\
&= |\beta|\|g\|_{\Lambda^2_{q(\cdot)}}.
\end{aligned}$$

(iii) Let us consider δ_1, $\delta_2 > 0$ such that

$$\delta_1 > \mu_{\Lambda^2_{q(\cdot)}}(g) \quad \text{and} \quad \delta_2 > \mu_{\Lambda^2_{q(\cdot)}}(h).$$

By the characterization of infimum, there exist $\hat{\delta}_1$, $\hat{\delta}_2$ such that

$$\mu_{\Lambda^2_{q(\cdot)}}(g) < \hat{\delta}_1 < \delta_1 \quad \text{with} \quad V^{q(\cdot)}_{\Lambda,2}\left(\frac{g}{\hat{\delta}_1}\right) \leq 1$$

and

$$\mu_{\Lambda^2_{q(\cdot)}}(h) < \hat{\delta}_2 < \delta_2 \quad \text{with} \quad V^{q(\cdot)}_{\Lambda,2}\left(\frac{h}{\hat{\delta}_2}\right) \leq 1,$$

then, by theorem 3.6, we obtain

$$V^{q(\cdot)}_{\Lambda,2}\left(\frac{g}{\delta_1}\right) \leq 1 \quad \text{and} \quad V^{q(\cdot)}_{\Lambda,2}\left(\frac{h}{\delta_2}\right) \leq 1.$$

By the convexity of $V^{q(\cdot)}_{\Lambda,2}(g)$ and taking $\hat{\delta} = \delta_1 + \delta_2$, we obtain

$$\begin{aligned} V^{q(\cdot)}_{\Lambda,2}\left(\frac{g+h}{\hat{\delta}}\right) &= V^{q(\cdot)}_{\Lambda,2}\left(\frac{\delta_1}{\hat{\delta}}\frac{g}{\delta_1} + \frac{\delta_2}{\hat{\delta}}\frac{h}{\delta_2}\right) \\ &\leq \frac{\delta_1}{\delta_1+\delta_2} V^{q(\cdot)}_{\Lambda,2}\left(\frac{g}{\delta_1}\right) + \frac{\delta_2}{\delta_1+\delta_2} V^{q(\cdot)}_{\Lambda,2}\left(\frac{h}{\delta_2}\right) \\ &\leq 1, \end{aligned}$$

thus, $\hat{\delta} \in \left\{\delta > 0 : V^{q(\cdot)}_{\Lambda,2}\left(\frac{g+h}{\delta}\right) \leq 1\right\}$ and in consequence

$$\begin{aligned} \|g+h\|_{\Lambda^2_{q(\cdot)}} &= \|g+h\|_\infty + \mu_{\Lambda^2_{q(\cdot)}}(g+h) \\ &\leq \|g\|_\infty + \|h\|_\infty + \hat{\delta} \\ &= \|g\|_\infty + \|h\|_\infty + \delta_1 + \delta_2 \end{aligned}$$

in particular for $\delta_1 = \mu_{\Lambda^2_{q(\cdot)}}(g) + \frac{1}{2n}$ and $\delta_2 = \mu_{\Lambda^2_{q(\cdot)}}(h) + \frac{1}{2n}$ with $n \in \mathbb{N}$, we obtain

$$\begin{aligned} \|g+h\|_{\Lambda^2_{q(\cdot)}} &\leq \|g\|_\infty + \|h\|_\infty + \mu_{\Lambda^2_{q(\cdot)}}(g) + \frac{1}{2n} + \mu_{\Lambda^2_{q(\cdot)}}(h) + \frac{1}{2n} \\ &= \|g\|_\infty + \mu_{\Lambda^2_{q(\cdot)}}(g) + \|h\|_\infty + \mu_{\Lambda^2_{q(\cdot)}}(h) + \frac{1}{n}, \end{aligned}$$

which implies that

$$\begin{aligned} \|g+h\|_{\Lambda^2_{q(\cdot)}} &\leq \|g\|_\infty + \mu_{\Lambda^2_{q(\cdot)}}(g) + \|h\|_\infty + \mu_{\Lambda^2_{q(\cdot)}}(h) \\ &= \|g\|_{\Lambda^2_{q(\cdot)}} + \|h\|_{\Lambda^2_{q(\cdot)}}. \end{aligned}$$

Therefore $\|\cdot\|_{\Lambda^2_{q(\cdot)}}$ is a norm on $\Lambda^2_{q(\cdot)}BV(J)$. □

To show that $\Lambda^2_{q(\cdot)}BV(J)$ is a Banach space we will use the following characterization of completeness.

Theorem 3.11. *A normed space X is Banach if and only if every absolutely convergent series in X is convergent.*

Theorem 3.12. *Let $q(\cdot) \in P(J)$ and* Λ a *W*-sequence, then $(\Lambda^2_{q(\cdot)}BV(J); \|\cdot\|_{\Lambda^2_{q(\cdot)}})$ is a Banach space.

Proof. Let Λ a W -sequence and $\{f_n\}_{n\geq 1}$ a sequence in $\Lambda^2_{q(\cdot)}BV(J)$ such that $\sum_{n=1}^{\infty}\|f_n\|_{\Lambda^2_{q(\cdot)}} < \infty$. Let us take $M = \sum_{n=1}^{\infty}\|f_n\|_{\Lambda^2_{q(\cdot)}}$.

Thus, $\sum_{n=1}^{\infty}\|fn\|_\infty \leq M < \infty$ and since $L^\infty(J)$ is a Banach space, there exists $f \in L^\infty(J)$ such that $\sum_{n=1}^{\infty} f_n = f$ in $L^\infty(J)$.

Let us define $h_n = \sum_{k=1}^{n} f_k, \forall n \in \mathbb{N}$ then $h_n \to f$ in $L^\infty(J)$, this is $\|h_n - f\|_\infty \to 0$, $n \to \infty$.

Now, by corollary 3.9, f_n is continuous $\forall n \in \mathbb{N}$, then h_n is continuous in J and so $\|h_n - f\|_\infty \to 0$, $n \to \infty$.

Therefore, h_n converges uniformly to f in J, in consequence f is continuous in J.

On the other hand,

$$\|h_n\|_{\Lambda^2_{q(\cdot)}} = \left\|\sum_{k=1}^{n} f_k\right\|_{\Lambda^2_{q(\cdot)}} \le \sum_{k=1}^{n}\|f_k\|_{\Lambda^2_{q(\cdot)}} \le M, \quad \forall n \in \mathbb{N},$$

this implies that

$$\mu_{\Lambda^2_{q(\cdot)}}(h_n) \le \|h_n\|_{\Lambda^2_{q(\cdot)}} \le M < M+1, \ \ \forall n \in \mathbb{N}.$$

Thus, $V^{q(\cdot)}_{\Lambda,2}\left(\dfrac{h_n}{M+1}\right) \le 1, \forall n \in \mathbb{N}.$

Let $\xi^* \in \mathcal{P}^*(J)$, then

$$\begin{aligned}
&\sum_{i=0}^{k-2}\frac{\left|D\left(\frac{f}{M+1};t_{i+2},t_{i+1}\right)-D\left(\frac{f}{M+1};t_{i+1},t_i\right)\right|^{q(x_i)}}{\lambda_i}\\
&=\lim_{m\to+\infty}\sum_{i=0}^{k-2}\frac{\left|D\left(\frac{h_m}{M+1};t_{i+2},t_{i+1}\right)-D\left(\frac{h_m}{M+1};t_{i+1},t_i\right)\right|^{q(x_i)}}{\lambda_i}\\
&\le \limsup_{m\to+\infty} V^{q(\cdot)}_{\Lambda,2}\left(\frac{h_m}{M+1}\right) \le 1,
\end{aligned}$$

hence, $V^{q(\cdot)}_{\Lambda,2}\left(\dfrac{f}{M+1}\right) \le 1$, in consequence $f \in \Lambda^2_{q(\cdot)}BV(J)$ and $\mu_{\Lambda^2_{q(\cdot)}}(f) \le M+1$.

Let us see that $h_n \to f$ in $\Lambda^2_{q(\cdot)}BV(J)$.

Using again that $\sum_{n=1}^{\infty}\|f_n\|_{\Lambda^2_{q(\cdot)}} < \infty$ and $h_n \to f$ in $L^\infty(J)$, we have that given $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that for $n \ge N$,

$$\sum_{k=n+1}^{\infty}\|f_k\|_{\Lambda^2_{q(\cdot)}} < \frac{\varepsilon}{2} \quad \text{and} \quad \|h_n - f\|_\infty < \frac{\varepsilon}{2}.$$

Fixing $n \ge N$ and given $m > n$, we obtain

$$\|h_m - h_n\|_{\Lambda^2_{q(\cdot)}} = \left\|\sum_{k=n+1}^{m} f_k\right\|_{\Lambda^2_{q(\cdot)}} \le \sum_{k=n+1}^{m}\|f_k\|_{\Lambda^2_{q(\cdot)}} < \frac{\varepsilon}{2},$$

hence

$$\mu_{\Lambda^2_{q(\cdot)}}(h_n - h_m) \le \|h_m - h_n\|_{\Lambda^2_{q(\cdot)}} < \frac{\varepsilon}{2} \quad \forall m > n.$$

This implies that $V^{q(\cdot)}_{\Lambda,2}\left(\dfrac{h_n - h_m}{\varepsilon/2}\right) \le 1, \forall m > n.$

Thus,

$$\sum_{i=0}^{k-2}\frac{\left|D\left(\frac{h_n-f}{\varepsilon/2};t_{i+2},t_{i+1}\right)-D\left(\frac{h_n-f}{\varepsilon/2};t_{i+1},t_i\right)\right|^{q(x_i)}}{\lambda_i}$$
$$=\lim_{m\to+\infty}\sum_{i=0}^{k-2}\frac{\left|D\left(\frac{h_n-h_m}{\varepsilon/2};t_{i+2},t_{i+1}\right)-D\left(\frac{h_n-h_m}{\varepsilon/2};t_{i+1},t_i\right)\right|^{q(x_i)}}{\lambda_i}$$
$$\leq \limsup_{m\to+\infty} V_{\Lambda,2}^{q(\cdot)}\left(\frac{h_n-h_m}{\varepsilon/2}\right)\leq 1,$$

so

$$V_{\Lambda,2}^{q(\cdot)}\left(\frac{h_n-f}{\varepsilon/2}\right)\leq 1 \Rightarrow \mu_{\Lambda^2_{q(\cdot)}}(h_n-f)\leq\frac{\varepsilon}{2}.$$

Finally, given $\varepsilon>0$, there exists $N=N(\varepsilon)\in\mathbb{N}$ such that for $n\geq N$ we obtain

$$\|h_n-f\|_{\Lambda^2_{q(\cdot)}}=\|h_n-f\|_\infty+\mu_{\Lambda^2_{q(\cdot)}}(h_n-f)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Thus, the sequence $\{h_n\}_{n\geq 1}$ converge to f in $\Lambda^2_{q(\cdot)}BV(J)$, that is to say, $\sum_{n=1}^{\infty}f_n=f$ in $\Lambda^2_{q(\cdot)}BV(J)$.

By theorem 3.11, we conclude that $\Lambda^2_{q(\cdot)}BV(J)$ is a Banach space. □

Lemma 3.13. If $q_+<\infty$ then

$$g\in\Lambda^2_{q(\cdot)}BV(J)\Leftrightarrow V_{\Lambda,2}^{q(\cdot)}(g)<+\infty.$$

Proof. Let $g\in\Lambda^2_{q(\cdot)}BV(J)$

Then there exists $\delta>0$ such that $V_{\Lambda,2}^{q(\cdot)}\left(\frac{g}{\delta}\right)<+\infty$. By hypothesis $q_+<\infty$ and since $q_-\leq q(\cdot)\leq q_+$ we have

(*i*) If $\delta<1$,

$$V_{\Lambda,2}^{q(\cdot)}(g)=V_{\Lambda,2}^{q(\cdot)}\left(\delta\frac{g}{\delta}\right)\leq\delta^{q_-}V_{\Lambda,2}^{q(\cdot)}\left(\frac{g}{\delta}\right)<+\infty.$$

(*ii*) If $\delta\geq 1$,

$$V_{\Lambda,2}^{q(\cdot)}(g)=V_{\Lambda,2}^{q(\cdot)}\left(\delta\frac{g}{\delta}\right)\leq\delta^{q_+}V_{\Lambda,2}^{q(\cdot)}\left(\frac{g}{\delta}\right)<+\infty.$$

The reciprocal is immediate by definition. □

The following result relates the norm in space with the second variation.

Lemma 3.14. Let $g\in\Lambda^2_{q(\cdot)}BV(J)$. If $\|g\|_{\Lambda^2_{q(\cdot)}}\leq 1$ then

$$V_{\Lambda,2}^{q(\cdot)}(g)\leq\|g\|^{q_-}_{\Lambda^2_{q(\cdot)}}\leq\|g\|_{\Lambda^2_{q(\cdot)}}.$$

Proof. Recall that if $g \in \Lambda^2_{q(\cdot)}BV(J)$

$$\|g\|_{\Lambda^2_{q(\cdot)}} := \|g\|_\infty + \mu_{\Lambda^2_{q(\cdot)}}(g) \ \text{ where } \ \mu_{\Lambda^2_{q(\cdot)}}(g) = \inf\left\{\delta > 0 : V^{q(\cdot)}_{\Lambda,2}\left(\frac{g}{\delta}\right) \leq 1\right\}.$$

Suppose $\|g\|_{\Lambda^2_{q(\cdot)}} \leq 1$, then

(*i*) If $\|g\|_\infty = 0$, we have that $g \equiv 0$ and therefore

$$V^{q(\cdot)}_{\Lambda,2}(g) = 0 \leq \|g\|^{q_-}_{\Lambda^2_{q(\cdot)}} \leq \|g\|_{\Lambda^2_{q(\cdot)}}.$$

(*ii*) If $\|g\|_\infty > 0$, we have that $\|g\|_\infty + \mu_{\Lambda^2_{q(\cdot)}}(g) > \mu_{\Lambda^2_{q(\cdot)}}(g)$, thus,

$$\mu_{\Lambda^2_{q(\cdot)}}(g) < \|g\|_{\Lambda^2_{q(\cdot)}} \leq 1.$$

Therefore, there exists $\delta > 0$ such that $\mu_{\Lambda^2_{q(\cdot)}}(g) \leq \delta < \|g\|_{\Lambda^2_{q(\cdot)}}$ and $V^{q(\cdot)}_{\Lambda,2}\left(\frac{g}{\delta}\right) \leq 1$.

Using the theorem 3.6, when $\delta < \|g\|_{\Lambda^2_{q(\cdot)}}$ we have

$$V^{q(\cdot)}_{\Lambda,2}\left(\frac{g}{\|g\|_{\Lambda^2_{q(\cdot)}}}\right) \leq V^{q(\cdot)}_{\Lambda,2}\left(\frac{g}{\delta}\right) \leq 1.$$

Hence, by remark 3.5

$$\begin{aligned} V^{q(\cdot)}_{\Lambda,2}(g) &= V^{q(\cdot)}_{\Lambda,2}\left(\left(\|g\|\right]_{\Lambda^2_{q(\cdot)}} \frac{g}{\|g\|_{\Lambda^2_{q(\cdot)}}}\right) \\ &\leq \left(\|g\|_{\Lambda^2_{q(\cdot)}}\right)^{q_-} V^{q(\cdot)}_{\Lambda,2}\left(\frac{g}{\|g\|_{\Lambda^2_{q(\cdot)}}}\right) \\ &\leq \left(\|g\|_{\Lambda^2_{q(\cdot)}}\right)^{q_-} \leq \|g\|_{\Lambda^2_{q(\cdot)}} \end{aligned}$$

□

Finally, the following result shows us the inclusion between the spaces of second variation with variable exponent when $r(\cdot) \leq q(\cdot)$.

Theorem 3.15. If $r(\cdot) \leq q(\cdot)$ and $q_+ < \infty$ then

$$\Lambda^2_{r(\cdot)}BV(J) \subset \Lambda^2_{q(\cdot)}BV(J).$$

Proof. Let ξ^* be a labeled partition of J and $g \in \Lambda^2_{r(\cdot)}BV(J)$. By lemma 3.13, $V^{r(\cdot)}_{\Lambda,2}(g) < \infty$ since $r_+ \leq q_+ < \infty$.

Let us take

$$A = \{i \in \{0, 1, 2, \cdots, k-2\} : |D(g; t_{i+2}, t_{i+1}) - D(g; t_{i+1}, t_i)| \leq 1\}$$

then

$$\begin{aligned}
&\sum_{i=0}^{k-2}\frac{|D(g;t_{i+2},t_{i+1})-D(g;t_{i+1},t_i)|^{q(x_i)}}{\lambda_i}\\
&\quad=\sum_{i\in A}\frac{|D(g;t_{i+2},t_{i+1})-D(g;t_{i+1},t_i)|^{q(x_i)}}{\lambda_i}\\
&\quad+\sum_{i\notin A}\frac{|D(g;t_{i+2},t_{i+1})-D(g;t_{i+1},t_i)|^{q(x_i)}}{\lambda_i}\\
&\quad\le\sum_{i\in A}\frac{|D(g;t_{i+2},t_{i+1})-D(g;t_{i+1},t_i)|^{r(x_i)}}{\lambda_i}\\
&\quad+\sum_{i\notin A}\frac{|D(g;t_{i+2},t_{i+1})-D(g;t_{i+1},t_i)|^{q(x_i)}}{\lambda_i}\\
&\quad\le\sum_{i=0}^{k-2}\frac{|D(g;t_{i+2},t_{i+1})-D(g;t_{i+1},t_i)|^{r(x_i)}}{\lambda_i}\\
&+\sum_{i\notin A}\frac{|D(g;t_{i+2},t_{i+1})-D(g;t_{i+1},t_i)|^{q(x_i)}}{\lambda_i}\\
&\le V_{\Lambda,2}^{r(\cdot)}(g)+\sum_{i\notin A}\frac{|D(g;t_{i+2},t_{i+1})-D(g;t_{i+1},t_i)|^{q(x_i)}}{\lambda_i}.
\end{aligned}\tag{7}$$

On the other hand, if $i\notin A$, $|D(g;\,t_{i+2},\,t_{i+1})-D(g;\,t_{i+1},\,t_i)|>1$ and this implies that

$$\begin{aligned}
\sum_{i\notin A}\frac{1}{\lambda_i}&<\sum_{i\notin A}\frac{|D(g;t_{i+2},t_{i+1})-D(g;t_{i+1},t_i)|^{r(x_i)}}{\lambda_i}\\
&\le V_{\Lambda,2}^{r(\cdot)}(g)<\infty.
\end{aligned}$$

Using the lemma 3.8, there exists $M>1$ such that $|D(g;\,u,\,v)|\le M$; $\forall u,\,v\in J$ with $u\neq v$. Therefore,

$$\begin{aligned}
\sum_{i\notin A}\frac{\left|D(g;t_{i+2},t_{i+1})-D(g;t_{i+1},t_i)\right|^{q(x_i)}}{\lambda_i}&\le\sum_{i\notin A}\frac{(2M)^{q+}}{\lambda_i}\\
&=(2M)^{q_+}\sum_{i\notin A}\frac{1}{\lambda_i}\\
&\le(2M)^{q_+}V_{\Lambda,2}^{r(\cdot)}(g).
\end{aligned}$$

Going back to the equation (7) and using the previous inequality

$$\sum_{i=0}^{k-2}\frac{|D(g;t_{i+2},t_{i+1})-D(g;t_{i+1},t_i)|^{q(x_i)}}{\lambda_i}\le V_{\Lambda,2}^{r(\cdot)}(g)+(2M)^{q_+}V_{\Lambda,2}^{r(\cdot)}(g).$$

Hence,

$$V_{\Lambda,2}^{q(\cdot)}(g)\le[1+(2M)^{q_+}]V_{\Lambda,2}^{r(\cdot)}(g)<\infty,$$

therefore, $g\in\Lambda^2_{q(\cdot)}$. □

Based on the results of this article, in future work we will study the composition operator in these spaces.

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