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Lifting of a generalised almost *r*-contact structure in a tangent bundle

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Abstract

Different structures defined on a differentiable manifold M can be lifted to the same type of structures on its tangent bundle. Many researcher analysed herein obtained results in this vista. In this paper our aim is to study Lie derivatives in reference to the vertical and complete lifts of generalized almost r-contact structure in the tangent bundle. We investigate some theorems on induced Nijenhuis tensor in tangent bundle.

Keywords: Tangent bundle, complete lift, vertical lift, horizontal lift, Lie derivative

Mathematics Subject Classification: 58A30, 53D15.

Introduction

The tangent bundles of a differentiable manifold has imperative effects on the differential geometry because it helps in study several innovative problems of modern differential geometry. Many researchers [3, 6, 7, 8, 20, 21, 33, 37, 39] have made significant contributions on study of tangent bundles over differentiable manifold. Ocak [4] have studied horizontal and diagonal lifts of tensor of type (1, 1) on cross-section in cotangent bundle. M. Saxena and others [12, 30, 31, 32, 35] defines some new structure and studied its several properties. Numerous geometers studied lifts of various structures and connections in the tangent bundle including [2, 13, 15, 16, 18, 19, 28].

The main purpose of the present paper is to study of Lie derivative of generalized almost r-contact structure, induced Nijenhuis tensor and cross-section in tangent bundle.

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In Sect.2 we define horizontal, vertical and complete lifts, tangent bundle, generalized almost r-contact structure and most important Hsu-structure. We defined Hsu-structure in tangent bundle *T*(*M*) in over generalized almost r-contact manifold in Sect. 3. In Sect. 4, we study some formulas on Lie derivative and about almost analytic vector fields in tangent bundle. In Sect. 5, we proved certain theorems on induced Nijenhuis tensor of an element J of $\Im_1^1(T(M))$. Finally, in the last section we study the behavior of the lifts of Hsu-structure on the cross-section in *T*(*M*) determined by given vector field in *M*.

Preliminaries

Let *M* be an *n*-dimensonal differentiable manifold and *TM* be the tangent bundle over *M*. Let a function *f*, a vector field $\aleph$, a 1-form η, (1, 1) tensor field φ and an affine connection ∇ in *M* and $f^V, \aleph^V, \eta^V, \phi^V, \nabla^V$ and $f^C, \aleph^C, \eta^C, \phi^C, \nabla^C$ are vertical and complete lifts of $f, \aleph, \eta, \phi$ and ∇, respectively in *TM*. Then by [1, 14, 22, 34] we have

$$(f\aleph)^V = f^V \lambda_1^V, (f\aleph)^C = f^C \aleph^V + f^V \aleph^C, \tag{1}$$

$$\aleph^V f^V = 0, \aleph^V f^C = \aleph^C f^V = (\aleph f)^V, \aleph^C f^C = (\aleph f)^C, \tag{2}$$

$$\eta^V(f^V) = 0, \eta^V(\aleph^C) = \eta^C(\aleph^V) = \eta(\aleph)^V, \eta^C(\aleph^C) = \eta(\aleph)^C, \tag{3}$$

$$\phi^V \aleph^C = (\phi\aleph)^V, \phi^C \aleph^C = (\phi\aleph)^C, \tag{4}$$

$$[\aleph, \Re]^V = [\aleph^C, \Re^V] = [\aleph^V, \Re^C], [\aleph, \Re]^C = [\aleph^C, \Re^C], \tag{5}$$

$$\nabla^C_{\aleph^C} \Re^C = (\nabla_\aleph \Re)^C, \quad \nabla^C_{\aleph^C} \Re^V = (\nabla_\aleph \Re)^V. \tag{6}$$

$$\nabla^C_{\aleph^V} \Re^C = (\nabla_\aleph \Re)^V, \quad \nabla^C_{\aleph^V} \Re^V = 0. \tag{7}$$

Notations: Let $\Im_0^0(M), \Im_0^1(M), \Im_1^0(M), \Im_1^1(M)$ be the set of functions, vector fields, 1-forms and tensor fields of type (1, 1) in *M*, respectively. Similarly, let $\Im_0^0(TM), \Im_0^1(TM), \Im_1^0(TM), \Im_1^1(TM)$ be the set of functions, vector fields, 1-forms and a tensor fields of type (1, 1) in *TM*, respectively.

Hsu-structure

Let us define *n*-dimensional manifold *M* of class C^∞. If (1, 1) type Tensor field *F* and class of the tensor field is of C^∞ then

$$F^2 = a^r I \tag{8}$$

where *r* be any integer, *I* is the unit tensor field and *a* is any real or complex number. Under said condition manifold *M* is endoweded with Hsu-structure [9, 36, 38].

Generalized almost r*-contact structure*

Consider differentiable manifold *M* (*dim* = *n*) and a (1, 1) tensor field *F*. Over *M* a vector field ξ_p and a 1-form η_p, $p \in [1, r]$ [5, 10, 11, 40].

(a) $F^2 = a^r I + \epsilon \sum_{p=1}^{r} \xi_p \otimes \eta_p$

(b) $F\xi_p = 0$

(c) $\eta_p \circ F = 0$

(d) $\eta_p(\xi_p) = -\frac{a^r}{\epsilon} \delta_q^p, p, q = 1, 2, \cdots r,$ (9)

where a, $\epsilon \neq 0$ are complex numbers. Differentiable manifold M is a generalized almost r-contact manifold and structure defined as $(F, \eta_p, \xi_p, a, \epsilon)$-structure.

The (1, 1) tensor field F is said to be a Hsu-structure if [17, 26, 29]

$$F^2 = a^r I. \tag{10}$$

Induced Structure on the Tangent Bundle

Considering complete lifts of equation (9) we get [23, 24, 25]:

$$\begin{aligned}
&\text{(a)} \quad F^C \xi_p^V = 0, F^C \xi_p^C = 0\\
&\text{(b)} \quad (F^C)^2 = a^r I + \epsilon \sum_{p=1}^{r} \left\{ \xi_p^V \otimes \eta_p^C + \xi_p^C \otimes \eta_p^V \right\}\\
&\text{(c)} \quad \eta_p^C(\xi_p^C) = \eta_p^V(\xi_p^V) = 0, \eta_p^C(\xi_p^V) = \eta_p^V(\xi_p^C) = -\frac{a^r}{\epsilon}\delta_{pq}\\
&\text{(d)} \quad \eta_p^V \circ F^C = 0, \eta_p^C \circ F^V = 0 \eta_p^C \circ F^C = 0, \eta_p^V \circ F^V = 0
\end{aligned} \tag{11}$$

Now consider the element J of $\Im_1^1(T(M))$ by

$$J = F^C + \frac{\epsilon}{a^{r/2}} \sum_{p=1}^{r} \left(\xi_p^V \otimes \eta_p^V + \xi_p^C \otimes \eta_p^C \right) \tag{12}$$

reference to equation (11), it is evident that

$$J^2 \aleph^C = a^r \aleph^C, J^2 \aleph^V = a^r \aleph^V$$

so, J is Hsu structure defined over tangent bundle $T(M)$.

Theorem 1. *If M be a differentiable manifold endowed with* $(F, \eta_p, \xi_p, a, \epsilon)$*-structure. Then J, given by (12) becomes a Hsu-structure on the tangent bundle T(M).*

Considering equation (12),

$$\begin{aligned}
&\text{(a)} \quad J\aleph^V = (F\aleph)^V + \frac{\epsilon}{a^{r/2}} \sum_{p=1}^{r} \left\{ (\eta_p(\aleph))^V \xi_p^C \right\}\\
&\text{(b)} \quad J\aleph^C = (F\aleph)^C + \frac{\epsilon}{a^{r/2}} \sum_{p=1}^{r} \left\{ (\eta_p(\aleph))^V \xi_p^V + (\eta_p(\aleph))^C \xi_p^C \right\}
\end{aligned} \tag{13}$$

$\forall \aleph \in \Im_0^1(M)$.

In particular $\eta_p(\aleph) = 0$, we have

$$\begin{aligned}
&\text{(a)} \quad J\xi_p^V = -\frac{a^{r/2}}{\epsilon}\xi_p^V, J\xi_p^C = -\frac{a^{r/2}}{\epsilon}\xi_p^V\\
&\text{(b)} \quad J\aleph^V = (F\aleph)^V, J\aleph^C = (F\aleph)^C
\end{aligned} \tag{14}$$

$\aleph$ being an arbitrary vector field in M such that $\eta_p(\aleph) = 0$.

Lie Derivatives with Respect to Complete and Vertical Lifts

Now we define lie derivative which represents as $\pounds_\aleph F$ and lie derivatiove is operated on a (1, 1) tensor field F with respect to a vector field $\aleph$ is given as [7].

$$(\pounds_\aleph F) = [\aleph, F\Re] - F[\aleph, \Re]$$

also [,] is the Lie bracket [14, 17].

Theorem 2. $\forall \aleph, \Re \in \Im_0^1(M)$, then

(a) $\pounds_{\aleph^V} f^V = 0,$

(b) $\pounds_{\aleph^C} f^C = (\pounds_{\aleph} f)^C$

(c) $\pounds_{\aleph^C} f^V = (\pounds_{\aleph} f)^V,$

(d) $\pounds_{\aleph^V} f^C = (\pounds_{\aleph} f)^V,$ (16)

Theorem 3. $\forall \aleph, Y \in \Im_0^1(M)$, then

(a) $\pounds_{\aleph^C} \Re^V = (\pounds_{\aleph} \Re)^V,$

(b) $\pounds_{\aleph^C} \Re^C = (\pounds_{\aleph} \Re)^C.$

(c) $\pounds_{\aleph^V} \Re^V = 0,$

(d) $\pounds_{\aleph^V} \Re^C = (\pounds_{\aleph} \Re)^V,$ (17)

Theorem 4. *For the vector field* $\aleph$, $J \in \Im_1^1(T(M))$ $\pounds_{\aleph}$ *the Lie derivation with respect to vector field defined by (12) and* $\eta_p(\Re) = 0$, *we have the following postulates*

(a) $(\pounds_{\aleph^V} J)\Re^V = 0$

(b) $(\pounds_{\aleph^V} J)\Re^C = \left((\pounds_{\aleph} F)\Re\right)^V + \frac{\epsilon}{a^{r/2}} \sum_{p=1}^{r} ((\pounds_{\aleph} \eta_p)\Re)^V \xi_p^C$

(c) $(\pounds_{\aleph^C} J)\Re^C = \left((\pounds_{\aleph} F)\Re\right)^V + \frac{\epsilon}{a^{r/2}} \sum_{p=1}^{r} ((\pounds_{\aleph} \eta_p)\Re)^V \xi_p^C$

(d) $(\pounds_{\aleph^C} J)\Re^C = \left((\pounds_{\aleph} F)\Re\right)^C + \frac{\epsilon}{a^{r/2}} \sum_{p=1}^{r} \eta_p^V (\pounds_{\aleph} \Re)^V \xi_p^V - \frac{\epsilon}{a^{r/2}} \sum_{p=1}^{r} \left(\eta_p (\pounds_{\aleph} \Re)\right)^C \xi_p^C.$ (18)

Proof. The proof can be obtained from equations (11), (13), (15) and [7].

Corollary 5. *If we put* $\Re = \xi_p$ *i.e.* $\eta_p^C(\xi_p^C) = \eta_p^V(\xi_p^V) = 0, \eta_p^C(\xi_p^V) = \eta_p^V(\xi_p^C) = -\frac{a^r}{\epsilon}\delta_{pq}$ *has condition (12), then we get different results*

(a) $(\pounds_{\aleph^V} J)\xi_p^V = -a^{r/2} \sum_{p=1}^{r} (\pounds_{\aleph} \xi_p)^V$

(b) $(\pounds_{\aleph^V} J)\xi^C = a^{r/2} \sum_{p=1}^{r} ((\pounds_{\aleph} F)\xi_p)^V$

(c) $(\pounds_{\aleph^C} J)\xi_p^V = ((\pounds_{\aleph} F)\xi_p)^V + \frac{\epsilon}{a^{r/2}} \sum_{p=1}^{r} \left((\pounds_{\aleph} \eta_p)\xi_p\right)^V \xi_p^C - a^{r/2} \sum_{p=1}^{r} \left((\pounds_{\aleph^C} \xi_p^C\right)$

(d) $(\pounds_{\aleph^C} J)\xi_p^C = ((\pounds_{\aleph} F)\xi_p)^C - a^{r/2} \sum_{p=1}^{r} \left(\pounds_{\aleph} \xi_p\right)^V + \frac{\epsilon}{a^{r/2}} \sum_{p=1}^{r} \left((\pounds_{\aleph} \eta_p)\xi_p\right)^V \xi_p^V - \frac{\epsilon}{a^{r/2}} \sum_{p=1}^{r} \left((\pounds_{\aleph} \eta_p)\xi_p\right)^C \xi_p^C.$ (19)

Theorem 6. *Let M be a differentiable manifold endowed with* $(F, \eta_p, \xi_p, a, \epsilon)$*-structure. Then the vertical lift* $\aleph^V$ *of a vector field* $\aleph$ *given in M is almost analytic with respect to the Hsu-structure J defined by (13) in T(M) iff the conditions*

$$\pounds_{\aleph} F = 0, \quad \pounds_{\aleph} \eta_p = 0, \quad \pounds_{\aleph} \xi_p = 0$$

are satisfied in M.

Proof. In view of (18a), (18b), (19a) and (19b), then

$$\pounds_{\aleph^C} J = 0$$

is equivalent to

$$\pounds_{\aleph} F = 0, \quad \pounds_{\aleph} \eta_p = 0, \quad \pounds_{\aleph} \xi_p = 0.$$

Theorem 7. *Let M be a differentiable manifold endowed with $(F,\eta_p,\xi_p,a,\epsilon)$-structure. Then the complete lift $\aleph^C$ of a vector field $\aleph$ given in M is almost analytic with respect to the Hsu-structure J defined by (12) in T(M) iff the conditions*

$$\pounds_{\aleph} F = 0, \quad \pounds_{\aleph} \eta_p = -\frac{\epsilon}{a^{r/2}} \eta_p, \quad \pounds_{\aleph} \xi_p = \frac{\epsilon}{a^{r/2}} \xi_p$$

are satisfied in M.

Proof. In view of (18c), (18d), (19c) and (19d), then

$$\pounds_{\aleph^C} J = 0$$

is equivalent to

$$\pounds_{\aleph} F = 0, \quad \pounds_{\aleph} \eta_p = -\frac{\epsilon}{a^{r/2}} \eta_p, \quad \pounds_{\aleph} \xi_p = \frac{\epsilon}{a^{r/2}} \xi_p.$$

Nijenhuis Tensor

Let us define a tensor field S of type (1, 2) by

$$S(\aleph,\Re) = N(\aleph,\Re) + (\aleph(\eta_p(\Re)) - \Re(\eta_p(\aleph)) - \eta_p([\aleph,\Re])\xi_p \tag{20}$$

$\forall \aleph,\Re \in \Im_0^1(M)$, N being the Nijenhuis tensor of F.

Theorem 8. *If* $\eta_p(\aleph) = 0, \eta_p(\Re) = 0,$ *then we have*

(a) $S(\aleph,\Re) = [F\aleph, F\Re] + a^r[\aleph,\Re] - F[F\aleph,\Re] - F[\aleph,F\Re]$

(b) $(S(\aleph,\xi_p))^V = a^r[\aleph,\eta_p]^V - (F[F\aleph,\eta_p])^V$

(c) $(S(\aleph,\xi_p))^C = a^r[\aleph,\eta_p]^C - (F[F\aleph\,\aleph,\eta_p])^C$ (21)

Proof. Since $\eta_p(\aleph) = 0, \eta_p(\Re) = 0,$ from (20) we have

$$\begin{aligned}(23a): \ S(\aleph,\Re) &= [F\aleph, F\Re] + F^2[\aleph,\Re] - F[F\aleph,\Re] - F[\aleph,F\Re]\\ S(\aleph,\Re) &= [F\aleph, F\Re] + a^r[\aleph,\Re] - F[F\aleph,\Re] - F[\aleph,F\Re].\end{aligned}$$

$$\begin{aligned}(23b): \ (S(\aleph,\xi_p))^V &= (N(\aleph,\xi_p))^V + (\aleph(\eta_p(\xi_p)) - \Re(\eta_p(\aleph)) - \eta_p([\aleph,\xi_p])\xi_p)^V\\ &= (N(\aleph,\xi_p))^V + (\aleph(\eta_p(\xi_p)))^V - (\Re(\eta_p(\aleph)))^V - (\eta_p([\aleph,\xi_p])\xi_p)^V\\ &= (N(\aleph,\xi_p))^V,\\ &= [F\aleph, F\xi_p]^V + a^r[\aleph,\xi_p]^V - (F[F\aleph,\xi_p])^V - (F[\aleph,F\xi_p])^V\\ &= a^r[\aleph,\xi_p]^V - (F[F\aleph,\xi_p])^V \quad by\ (11).\end{aligned}$$

$$\begin{aligned}(23c): \ (S(\aleph,\xi_p))^C &= (N(\aleph,\xi_p))^C + (\aleph(\eta_p(\xi_p)) - \Re(\eta_p(\aleph)) - \eta_p([\aleph,\xi_p])\xi_p)^C\\ &= (N(\aleph,\xi_p))^C + (\aleph(\eta_p(\xi_p)))^C - (\Re(\eta_p(\aleph)))^C - (\eta_p([\aleph,\xi_p])\xi_p)^C\\ &= (N(\aleph,\xi_p))^C,\\ &= [F\aleph, F\xi_p]^C + a^r[\aleph,\xi_p]^C - (F[F\aleph,\xi_p])^C - (F[\aleph,F\xi_p])^C\\ &= a^r[\aleph,\xi_p]^C - (F[F\aleph,\xi_p])^C \quad by\ (11).\end{aligned}$$

Theorem 9. *Let us define tensor fields S_1, S_2 and S_3*

$$\begin{aligned} &\text{(a)} \quad S_1(\aleph,\Re) = \frac{\epsilon}{a^{r/2}}\eta_p([F\aleph,\Re]+[\aleph,F\Re]) \\ &\text{(b)} \quad S_2(\aleph) = a^{r/2}([\xi_p,F\aleph]-F[\xi_p,\aleph]) \\ &\text{(c)} \quad S_3(\aleph) = -\eta_p[\xi_p,\aleph] \end{aligned} \tag{22}$$

$\forall \aleph,\Re \in \Im_0^1(M)$. *Let $\tilde{H}$ be the Nijenhuis tensor of J defined by (12). Then* $\forall \aleph,\Re \in \Im_0^1(M)$ *such that* $\eta_p(\aleph)=\eta_p(\Re)=0$, *we have*

$$\begin{aligned} &\text{(a)} \quad \tilde{H}(\aleph^V,\Re^V)=0, \\ &\text{(b)} \quad \tilde{H}(\aleph^V,\Re^C)=(S(\aleph,\Re))^V-(S_1(\aleph,\Re))^V\xi_p^C, \\ &\text{(c)} \quad \tilde{H}(\aleph^C,\Re^C)=(S(\aleph,\Re))^C-(S_1(\aleph,\Re))^V\xi_p^V-(S_1(\aleph,\Re))^C\xi_p^C, \\ &\text{(d)} \quad \tilde{H}(\aleph^V,\xi_p^V)=(S_2(\aleph))^V-(S_3(\aleph))^V\xi_p^C, \\ &\text{(e)} \quad \tilde{H}(\aleph^V,\xi_p^C)=(S(\aleph,\xi_p))^V-(S_1(\aleph,\xi_p))^V\xi_p^C, \\ &\text{(f)} \quad \tilde{H}(\aleph^C,\xi_p^V)=(S_2(\aleph))^C+(S(\aleph,\xi_p))^V+(S_3(\aleph))^V\xi_p^C-(S_1(\aleph,\xi_p))^V\xi_p^C-(S_3(\aleph))^C\xi_p^V, \\ &\text{(g)} \quad \tilde{H}(\xi_p^V,\xi_p^C)=0. \end{aligned} \tag{23}$$

Proof. The proof can be obtained from equations (11), (12), (13), (14) and [7, 27].

Theorem 10. *Let $\tilde{H}$ be the Nijenhuis tensor of J defined by (12), the condition $\tilde{H}=0$ is equivalent to the condition*

$$S=0, \qquad S_1=0, \qquad S_2=0, \qquad S_3=0$$

and hence to the condition $S=0$.

Proof. As the consequence of (23), we equate the components of both sides of (23), we get

$$\begin{aligned} S(\aleph,\Re)=0,\ \ S_1(\aleph,\Re)=0,\ \ S_2(\aleph)=0,\ \ S_3(\aleph)=0 \\ S(\aleph,\xi_p)=0,\ \ S_1(\aleph,\xi_p)=0. \end{aligned}$$

since S and S_1 are skew symmetric then

$$S(\xi_p,\xi_p)=0,\ \ S_1(\xi_p,\xi_p)=0.$$

substitute $\aleph=\xi_p$ in equations (22b) and (22c), we get

$$\begin{aligned} S_2(\xi_p)=a^{r/2}([\xi_p,F\xi_p]-F[\xi_p,\xi_p]=0), \\ S_3(\xi_p)=-\eta_p[\xi_p,\xi_p]=0. \\ as\ F\xi_p=0,[\xi_p,\xi_p]=0 \end{aligned}$$

Therefore the conditions of (23) are equivalent to conditions

$$\begin{aligned} S=0,\ \ S_1=0,\ \ S_2=0,\ \ S_3=0 \\ \Rightarrow S=0 \end{aligned}$$

because

$$S=0 \Rightarrow S_1=0,\ \ S_2=0,\ \ S_3=0.$$

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