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Fractional hybrid systems involving φ-Caputo derivative

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This paper considers a coupled hybrid thermostat system driven by the ψ-Caputo fractional derivative in a Banach algebra. We employ a version of Darbo's fixed-point theorem combined with the measure of noncompactness (MNC) technique to establish certain existence results. Additionally, an example is provided to illustrate our findings.

Key words and phrases: Hybrid differential equation, φ-Caputo derivative, measure of noncompactness, Banach algebra

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1. Introduction

It's commonly recognized that fractional calculus is beneficial in solving numerous real-world problems spanning various scientific and engineering fields, as evidenced by [1, 16, 20, 25, 28, 31, 39]. The main advantage of using a non-integer order derivative instead of an integer order derivative is that the first one is non-local in nature, while the second is local in nature and doesn't have any memory terms in the system. So, recently, Numerous studies have appeared focused on fractional differential equations (FDEs), see [2, 3, 6, 23, 24, 32–35, 37, 38, 40] and the references cited therein.

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In the literature, different types of integral and differential operators have been introduced by Kilbas et al. [20], including Caputo, Riemann–Liouville, Erdelyi–Kober, Riesz, and Hadamard operators. In the same regard, Almeida [5] generalized Caputo's fractional derivative (FD) to ψ-Caputo FD. This operator appears in various concrete models. For instance, several anomalous diffusions, including ultra-slow processes [22], financial crisis [27], random walks [18], Heston model [7] and Verhulst model [8]. Moreover, the increase in global population via the prism of the ψ-Caputo boundary value problem has been investigated in [36]. Therefore, considerable attention has been devoted to the quantitative and qualitative properties of solutions to various types of differential problems governed by ψ-Caputo FD [4, 9, 17, 29, 40, 41].

Hybrid differential equations have attracted a considerable amount of interest and investigation by several researchers. This category of differential systems includes the perturbations of primitive differential equations in different manners. For example, the authors in [15] discussed the following coupled system

$$\begin{cases} \mathcal{D}_{0^+}^{\vartheta}\left(\dfrac{\mathfrak{z}_1(\zeta)}{h(\zeta,\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta))}\right)=g(\zeta,\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta)), & \zeta\in[0,1],\\ \mathcal{D}_{0^+}^{\vartheta}\left(\dfrac{\mathfrak{z}_2(\zeta)}{h(\zeta,\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta))}\right)=g(\zeta,\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta)), & \zeta\in[0,1],\\ \mathfrak{z}_1(0)=\mathfrak{z}_1(0)=0 & \end{cases}$$

where $\vartheta \in (0,1)$ and D_{0+}^{ϑ} is the standard Riemann-Liouville FD.

Baleanu *et al.* [11] examined the existence of solutions for the following hybrid thermostat differential model:

$$\begin{cases} {}^{c}\mathcal{D}_{0^+}^{\vartheta}\left(\dfrac{\mathfrak{z}(\zeta)}{h(\zeta,\mathfrak{z}(\zeta))}\right)+g(\zeta,\mathfrak{z}(\zeta))=0, \quad \zeta\in[0,1],\\ D\left(\dfrac{\mathfrak{z}(\zeta)}{h(\zeta,\mathfrak{z}(\zeta))}\right)\Big|_{\zeta=a}=0,\\ \xi\,{}^{c}\mathcal{D}_{0^+}^{\vartheta-1}\left(\dfrac{\mathfrak{z}(\zeta)}{h(\zeta,\mathfrak{z}(\zeta))}\right)\Big|_{\zeta=1}+\left(\dfrac{\mathfrak{z}(\zeta)}{h(\zeta,\mathfrak{z}(\zeta))}\right)\Big|_{\zeta=\eta}=0 \end{cases}$$

where $1<\vartheta\leq 2$, $\eta\in[0,1]$, $D=\dfrac{d}{d\zeta}$, $\xi\in\mathbb{R}_+$ is a parameter, ${}^{c}\mathcal{D}_{0^+}^{\vartheta}$ and ${}^{c}\mathcal{D}_{0^+}^{\vartheta-1}$ denotes the Caputo FD of order ϑ and $\vartheta-1$, respectively, $h\in C\left([0,1]\times\mathbb{R},\mathbb{R}\setminus\{0\}\right)$ and $g\in C([0,1]\times\mathbb{R},\mathbb{R})$.

Motivated by the above papers and using the main idea of [12, 19, 26], this paper considers the following coupled hybrid system for the thermostat model involving ψ-Caputo FD:

$$\begin{cases} {}^{c}\mathcal{D}_{a^+}^{\vartheta_1;\psi}\left(\dfrac{\mathfrak{z}_1(\zeta)}{h(\zeta,\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta))}\right)+g(\zeta,\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta))=0, & \zeta\in\mathcal{J}:=[a,b],\\ {}^{c}\mathcal{D}_{a^+}^{\vartheta_2;\psi}\left(\dfrac{\mathfrak{z}_2(\zeta)}{h(\zeta,\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta))}\right)+g(\zeta,\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta))=0, & \zeta\in\mathcal{J}:=[a,b], \end{cases} \tag{1}$$

and

$$\begin{cases} D\left(\dfrac{\mathfrak{z}_1(\zeta)}{h(\zeta,\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta))}\right)|_{\zeta=a} = D\left(\dfrac{\mathfrak{z}_2(\zeta)}{h(\zeta,\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta))}\right)|_{\zeta=a} = 0, \\ \xi_1\, {}^c\mathcal{D}_{a^+}^{\vartheta_1-1;\psi}\left(\dfrac{\mathfrak{z}_1(\zeta)}{h(\zeta,\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta))}\right)|_{\zeta=b} + \left(\dfrac{\mathfrak{z}_1(\zeta)}{h(\zeta,\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta))}\right)|_{\zeta=\eta_1} = 0, \\ \xi_2\, {}^c\mathcal{D}_{a^+}^{\vartheta_2-1;\psi}\left(\dfrac{\mathfrak{z}_2(\zeta)}{h(\zeta,\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta))}\right)|_{\zeta=b} + \left(\dfrac{\mathfrak{z}_2(\zeta)}{h(\zeta,\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta))}\right)|_{\zeta=\eta_2} = 0, \end{cases} \tag{2}$$

where for i = 1, 2, $1<\vartheta_i \le 2$, $\eta_i \in \mathcal{J} := [a,b]$, $(0 \le a < b < \infty)$, $D = \dfrac{d}{d\zeta}$, ξ_i is a positive real parameter, ${}^c\mathcal{D}_{a^+}^{\vartheta_i;\psi}$ and ${}^c\mathcal{D}_{a^+}^{\vartheta_i-1;\psi}$ denotes the Caputo FD with respect to ψ of order ϑ_i and $\vartheta_i - 1$, respectively. Here $h \in C(\mathcal{J}\times\mathbb{R},\mathbb{R}\setminus\{0\})$, $g \in C(\mathcal{J}\times\mathbb{R},\mathbb{R})$ is a given function fulfilling some hypothesis that will be indicated later and $h(\zeta,0,0) \neq 0$ for all $\zeta \in \mathcal{J}, \psi$ is increasing and positive monotone function and $\psi'(t) > 0$.

Numerous physical systems undoubtedly merit an in-depth examination of (1)-(2), for instance, binary mixture convection [30], geophysical morphodynamics [21] and so on. The above-mentioned motivational models present notable benefits, but the complexity of their associated mathematical models often escalates, making it challenging to establish the existence of results. Hence, the investigation of hybrid coupled systems involving ψ-Caputo type TFD in a Banach algebra has become important.

The contributions of our paper can be outlined as follows:

- We consider a fractional hybrid system in a general configuration, enabling improvements over several earlier related papers [10, 11, 15, 19].
- By utilizing Darbo's fixed-point theorem and the MNC technique in a Banach algebra, we propose certain sufficient conditions to ensure the existence of solutions.

The paper is structured as follows: In Section 2, we collect the basic background necessary for subsequent discussions. Section 3 utilizes a version of Darbo's fixed point theorem to establish new existence criterion. Finally, our findings are illustrated through an example.

2. Preliminaries

Let $\mathcal{J} := [a,b]$, be a finite interval. Consider $C(\mathcal{J})$ the space of continuous functions $f : \mathcal{J} \to \mathbb{R}$ with the supremum (uniform) norm:

$$\| f \| = \sup_{\zeta\in\mathcal{J}} | f(\zeta) |,$$

$L^1(\mathcal{J})$ the space of Lebesgue integrable real-valued functions on $\mathcal{J}$ equipped with the norm

$$\| f \|_{L^1} = \int_{\mathcal{J}} | f(\zeta) | d\zeta.$$

We define

$$\mathbb{T}_+^1(\mathcal{J},\mathbb{R}) = \{\psi : \ \psi \in C^1(\mathcal{J}) \ \text{ and } \ \psi'(\zeta) > 0 \ \text{ for all } \ \zeta \in \mathcal{J}\}.$$

For $\psi \in \mathbb{T}_+^1(\mathcal{J},\mathbb{R})$ and $\zeta, s \in \mathcal{J}$, $(\zeta > s)$, we pose

$$\psi(\zeta,s) = \psi(\zeta) - \psi(s) \ \text{ and } \ \psi(\zeta,s)^{\vartheta} = \left(\psi(\zeta) - \psi(s)\right)^{\vartheta}.$$

Definition 1. [5, 20] *The ψ-Riemann-Liouville fractional integral of a function $f \in L^1(\mathcal{J})$ of order ϑ>0 is given by*

$$\mathcal{I}_{a^+}^{\vartheta,\psi} f(\zeta) = \frac{1}{\Gamma(\vartheta)} \int_a^{\zeta} \psi(\zeta,s)^{\vartheta-1} \psi'(s) f(s) ds, \quad \zeta > a,$$

with $\psi \in \mathbb{T}_+^1(\mathcal{J},\mathbb{R})$ and $\Gamma(\cdot)$ the gamma function.

Definition 2. [5, 20] *Let* $0 < \vartheta \le 1$ *and* $\psi \in \mathbb{T}_{+}^{1}(\mathcal{J},\mathbb{R})$. *The* ψ*-Caputo FD at order* ϑ *of a function* $f \in C(\mathcal{J})$ *of order* ϑ *is defined as*

$$\left({}^{C}\mathcal{D}_{a^+}^{\vartheta;\psi} f\right)(\zeta) = \mathcal{I}_{a^+}^{1-\vartheta;\psi}\left(\frac{1}{\psi'(\zeta)}\frac{d}{dt}\right) f(\zeta).$$

Lemma 1. [5, 20] *Let* $n-1 < \vartheta < n$, $n \in \mathbb{N}^*$, $\gamma > 0$, *then*

(1) $\mathcal{I}_{a^+}^{\vartheta;\psi}\ \psi(\zeta,a)^{\gamma-1} = \dfrac{\Gamma(\gamma)}{\Gamma(\vartheta+\gamma)}\psi(\zeta,a)^{\vartheta+\gamma-1}.$

(2) ${}^{C}\mathcal{D}_{a^+}^{\vartheta;\psi}\psi(\zeta,a)^{k} = 0$ *for* $k < n$.

Lemma 2. [5, 20] *Let* $n-1 < \vartheta < n$. *Then, the following equality holds*

(1) if $f \in L^1(\mathcal{J})$ and $\gamma > 0$, then $\mathcal{I}_{a^+}^{\vartheta;\psi}\mathcal{I}_{a^+}^{\gamma;\psi} f(\zeta) = \mathcal{I}_{a^+}^{\vartheta+\gamma;\psi} f(\zeta)$,

(2) if $f \in C(\mathcal{J})$, then ${}^{C}\mathcal{D}_{a^+}^{\vartheta;\psi}\mathcal{I}_{a^+}^{\vartheta;\psi}\ f(\zeta) = f(\zeta)$,

(3) if $f \in C^n(\mathcal{J})$, then $\mathcal{J}_{a^+}^{\vartheta;\psi}{}^{C}\mathcal{D}_{a^+}^{\vartheta;\psi}\ f(\zeta) = f(\zeta) - \displaystyle\sum_{j=0}^{n-1}\frac{\psi(\zeta,a)^j}{j!}\left(\frac{1}{\psi'(\zeta)}\frac{d}{dt}\right)^{j} f(a)$.

Now, let us assume that $(\mathbb{X},\|\cdot\|)$ is a real Banach space and the zero element 0. If $\mathbb{V} \subset \mathbb{X}$ is non-empty, then $Conv\mathbb{V}$ and $\overline{\mathbb{V}}$ denote the convex hull and the closure of $\mathbb{V}$, respectively. If $\mathbb{V} \subset \mathbb{X}$ is a bounded, $diam\mathbb{V}$ represents the diameter of $\mathbb{V}$ and

$$\|\mathbb{V}\| = \sup\{\|v\| \ :\ v \in \mathbb{V}\}.$$

We denote by $\mathfrak{M}_{\mathbb{X}}$ is the family of all bounded subsets of $\mathbb{X}$ and by $\mathfrak{N}_{\mathbb{X}}$ its subfamily comprising of the relatively compact subsets. Moreover, uv denotes the product of elements $u,v \in \mathbb{X}$. Also, $\mathbb{U}\mathbb{V}$ is the product of subsets $\mathbb{U},\mathbb{V}$ of $\mathbb{X}$ i.e., $\mathbb{U}\mathbb{V} = \{uv\ :\ u \in \mathbb{U}, v \in \mathbb{V}\}$.

Definition 3. [13] *We say that* $\Lambda : \mathfrak{M}_{\mathbb{X}} \to [0,\infty)$ *is a MNC in* $\mathbb{X}$ *if all the assumptions below are satisfied:*

(1) $Ker\Lambda = \{\mathbb{V} \in \mathfrak{M}_{\mathbb{X}},\ \Lambda(\mathbb{V}) = 0\}$ is non-empty and $Ker\Lambda \in \mathfrak{N}_{\mathbb{X}}$,

(2) $\mathbb{U} \subset \mathbb{V} \Rightarrow \Lambda(\mathbb{U}) \le \Lambda(\mathbb{V})$,

(3) $\Lambda(\mathbb{V}) = \Lambda(\overline{\mathbb{V}}) = \Lambda(conv(\mathbb{V}))$,

(4) $\Lambda(\lambda\mathbb{V} + (1-\lambda)\mathbb{U}) \le \lambda\Lambda(\mathbb{V}) + (1-\lambda)\Lambda(\mathbb{U})$, *for* $\lambda \in \mathcal{J}$

(5) If $(\mathbb{V}_n)$ is a sequence of closed subsets of $\mathfrak{M}_{\mathbb{X}}$ where $\mathbb{V}_{n+1} \subset \mathbb{V}_n$; $n = 1,2,\cdots$, and $\lim_{n\to\infty}\Lambda(\mathbb{V}_n) = 0$ *then* $\mathbb{V}_\infty = \bigcap_{n=1}^{\infty}\mathbb{V}_n \neq \emptyset$

Observe that $\mathbb{V}_\infty$ is in $\mathrm{Ker}\Lambda$ and $\Lambda(\mathbb{V}_\infty) \le \Lambda(\mathbb{V}_n)$ *for* $n = 1,2,\cdots$.

Definition 4. [14] *We say that the MNC* Λ *in the Banach algebra* $\mathbb{X}$ *satisfies condition* (m) *if:*

$$\Lambda(\mathbb{U}\mathbb{V}) \le \|\mathbb{U}\|\,\Lambda(\mathbb{V}) + \|\mathbb{V}\|\,\Lambda(\mathbb{U}),$$

for any $\mathbb{U},\mathbb{V} \in \mathfrak{M}_{\mathbb{X}}$.

Definition 5. [13] *Let us fix* $\mathbb{Y} \in \mathfrak{M}_{C(\mathcal{J})}$ *and* $\varsigma > 0$. *For* $y \in \mathbb{Y}$, *we denote by* $\varpi(y,\varsigma)$ *the modulus of continuity of* y; *that is,*

$$\varpi(y,\varsigma) = \sup\{|y(\zeta) - y(s)|\ :\ \zeta,s \in \mathcal{J},\ |\zeta - s| \le \varsigma\}, \tag{3}$$

Moreover, put

$$\varpi(\mathbb{Y},\varsigma) = \sup\{\varpi(y,\varsigma)\ :\ y \in \mathbb{Y}\}, \tag{4}$$

and

$$\varpi_0(\mathbb{Y}) = \lim_{\varsigma\to 0}\varpi(\mathbb{Y},\varsigma). \tag{5}$$

Lemma 3. [15] *The MNC $\varpi_0(\mathbb{Y})$ on $C(\mathcal{J})$ satisfies condition (m).*

We need to introduce the class $\mathbb{E}$ of functions:

$$\mathbb{E} = \left\{\kappa : (a,\infty)\to(0,\infty),\ \kappa \text{ is nondecreasing and } \lim_{n\to\infty}\kappa^n(\zeta) = 0 \text{ for any } \zeta > a.\right\}$$

where κ^n denotes the n-iteration of κ.

Remark 1. [15] *Notice that if $\kappa \in \mathbb{E}$, then $\kappa(\zeta) < \zeta$, for any $\zeta > a$. Moreover, if $\kappa \in \mathbb{E}$, then κ is continuous at $\zeta_a = a$.*

Theorem 1. [15, Theorem 13] *Let Λ be an MNC in the Banach space $\mathbb{X}$ and $\mathbb{V}\subset\mathbb{X}$ be a nonempty, closed, bounded, and convex. $\mathbb{X}$. Assume that $\mathcal{S}:\mathbb{V}\times\mathbb{V}\to\mathbb{V}$ is a continuous mapping satisfying*

$$\Lambda(\mathcal{S}(V_1\times V_2)) \le \kappa(\max\{\Lambda(V_1),\Lambda(V_2)\}), \tag{6}$$

for any nonempty subsets V_1 and V_2 of $\mathbb{V}$, *where $\kappa\in\mathbb{E}$. Then $\mathcal{S}$ possesses a fixed point in $\mathbb{V}$.*

Lemma 4. *Let $\rho > 1,\ \zeta_1,\zeta_2 \in \mathcal{J}$ with $\zeta_2 > \zeta_1$ then*

$$\psi(\zeta_2,a)^\rho - \psi(\zeta_1,a)^\rho \le \rho\psi'(b)\psi(b,a)^{\rho-1}\,|t_2 - t_1|.$$

Proof. Let $F:\mathbb{R}_+\to\mathbb{R}_+$ be the function defined by $F(\zeta)=\psi(\zeta,a)^\rho$. Then, Lagrange's Mean Value Theorem implies that there exists $\varepsilon\in[\zeta_1,\zeta_2]$ such that

$$\frac{F(\zeta_2)-F(\zeta_1)}{\zeta_2-\zeta_1} \le F'(\varepsilon).$$

Then, for $\varepsilon \le \zeta_2 \le b,$ we get

$$\psi(\zeta_2,a)^\rho - \psi(\zeta_1,a)^\rho \le \rho\psi'(\varepsilon)\psi(\varepsilon,a)^{\rho-1}\,|\zeta_2-\zeta_1| \le \rho\psi'(b)\psi(b,a)^{\rho-1}\,|\zeta_2-\zeta_1|.$$

3. Main Results

Before stating our result, we need to present the auxiliary lemma.

Lemma 5. *Let $1 < \vartheta_1 \le 2$. Suppose that $h\in C(\mathcal{J}\times\mathbb{R}\times\mathbb{R},\mathbb{R}\setminus\{0\}),\ \bar{g}\in C(\mathcal{J})$ such that for all $\zeta\in\mathcal{J},\ \bar{g}(\zeta)\neq 0$. Then the solution of the following problem*

$$ {}^{c}\mathcal{D}_{a^+}^{\vartheta_1;\psi}\left(\frac{\mathfrak{z}_1(\zeta)}{h(\zeta,\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta))}\right) + \bar{g}(\zeta) = 0, \quad \zeta\in\mathcal{J} := [a,b], \tag{7}$$

and

$$\begin{cases} D\left(\dfrac{\mathfrak{z}_1(\zeta)}{h(\zeta,\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta))}\right)\Big|_{\zeta=a} = 0, \\ \xi_1\,{}^{c}\mathcal{D}_{a^+}^{\vartheta_1-1;\psi}\left(\dfrac{\mathfrak{z}_1(\zeta)}{h(\zeta,\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta))}\right)\Big|_{\zeta=b} + \left(\dfrac{\mathfrak{z}_1(\zeta)}{h(\zeta,\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta))}\right)\Big|_{\zeta=\eta_1} = 0, \end{cases} \tag{8}$$

satisfies the following integral equation

$$\mathfrak{z}_1(\zeta) = h(\zeta,\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta))\left[-\int_a^\zeta \frac{\psi'(s)\psi(\zeta,s)^{\vartheta_1-1}}{\Gamma(\vartheta_1)}\bar{g}(s)ds + \int_a^\eta \frac{\psi'(s)\psi(\eta,s)^{\vartheta_1-1}}{\Gamma(\vartheta_1)}\bar{g}(s)ds + \xi\int_a^b \psi'(s)\bar{g}(s)ds\right]. \tag{9}$$

Proof. Applying the operator $\mathcal{I}_{a^+}^{\vartheta_1,\psi}$ to both sides of (7), by Lemma 2, one gets

$$\frac{\mathfrak{z}_1(\zeta)}{h(\zeta,\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta))}-\frac{\mathfrak{z}_1(a)}{h(a,\mathfrak{z}_1(a),\mathfrak{z}_2(a))}-\frac{1}{\psi'(\zeta)}\frac{d}{d\zeta}\left(\frac{\mathfrak{z}_1(\zeta)}{h(\zeta,\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta))}\right)\Big|_{\zeta=a}\psi(\zeta,a)=-\mathcal{I}_{a^+}^{\vartheta_1,\psi}\overline{g}(\zeta).$$

By using the condition $D\left(\frac{\mathfrak{z}_1(\zeta)}{h(\zeta,\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta))}\right)\Big|_{\zeta=a}=0,$ we get

$$\frac{\mathfrak{z}_1(\zeta)}{h(\zeta,\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta))}=-\mathcal{I}_{a^+}^{\vartheta_1,\psi}\overline{g}(\zeta)+c_a,$$

where $c_a=\frac{\mathfrak{z}_1(a)}{h(a,\mathfrak{z}_1(a),\mathfrak{z}_2(a))}$. Thus, the solution of (7) is

$$\mathfrak{z}_1(\zeta)=h(\zeta,\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta))\left(-\mathcal{I}_{a^+}^{\vartheta_1,\psi}\overline{g}(\zeta)+c_0\right),\quad c_a\in\mathbb{R}. \tag{10}$$

Then, by using Lemmas 1, we have

$${}^{c}\mathcal{D}_{a^+}^{\vartheta_1-1;\psi}\left(\frac{\mathfrak{z}_1(\zeta)}{h(\zeta,\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta))}\right)\Big|_{\zeta=b}=-\mathcal{I}_{a^+}^{1,\psi}\overline{g}(b)=-\int_a^b\psi'(s)\overline{g}(s)ds,$$

and

$$\left(\frac{\mathfrak{z}_1(\zeta)}{h(\zeta,\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta))}\right)\Big|_{\zeta=\eta_1}=-\mathcal{I}_{a^+}^{\vartheta_1,\psi}\overline{g}(\eta_1)+c_a=-\int_a^{\eta_1}\frac{\psi'(s)\psi(\eta_1,s)^{\vartheta_1-1}}{\Gamma(\vartheta_1)}\overline{g}(s)ds+c_a,$$

which, together with the boundary condition

$$\xi_1\,{}^{c}\mathcal{D}_{a^+}^{\vartheta_1-1;\psi}\left(\frac{\mathfrak{z}_1(\zeta)}{h(\zeta,\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta))}\right)\Big|_{\zeta=b}+\left(\frac{\mathfrak{z}_1(\zeta)}{h(\zeta,\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta))}\right)\Big|_{\zeta=\eta_1}=0,$$

implies that

$$c_a=\mathcal{I}_{a^+}^{\vartheta_1,\psi}\overline{g}(\eta_1)+\xi_1\mathcal{I}_{a^+}^{1,\psi}\overline{g}(b)=\int_a^{\eta_1}\frac{\psi'(s)\psi(\eta_1,s)^{\vartheta_1-1}}{\Gamma(\vartheta_1)}\overline{g}(s)ds+\xi_1\int_a^b\psi'(s)\overline{g}(s)ds.$$

Substituting the value of c_a in (10) we get (9).

Let us introduce the following assertions:

(H1) $g\in C(\mathcal{J}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $h\in C(\mathcal{J}\times\mathbb{R}\times\mathbb{R},\mathbb{R}\setminus\{0\})$ where $g(\zeta,0,0)\neq 0$ for all $\zeta\in\mathcal{J}$;

(H2) The functions h and g verifies

$$|h(\zeta,u_1,u_2)-h(\zeta,v_1,v_2)|\leq\kappa_1(\max\{|u_1-v_1|,|u_2-v_2|\}),$$
$$|g(\zeta,u_1,u_2)-g(\zeta,v_1,v_2)|\leq\kappa_2(\max\{|u_1-v_1|,|u_2-v_2|\}),$$

for any $\zeta\in\mathcal{J}$, and $v_1,v_2,u_1,u_2\in\mathbb{R}$, where $\kappa_1,\kappa_2\in\mathbb{E}$, and κ_1 are continuous.

(H3) There exists a constant $K>0$ such that

$$\left(\kappa_1(K)+h^*\right)\left(\kappa_2(K)+g^*\right)(2\max_{1\leq i\leq 2}\{\psi(b,a)^{\vartheta_i}\}+\xi_{\max}\psi(b,a))\leq K, \tag{11}$$

and

$$\left(\kappa_2(K)+g^*\right)(2\max_{1\leq i\leq 2}\{\psi(b,a)^{\vartheta_i}\}+\xi_{\max}\psi(b,a))\leq 1,$$

where: $\xi_{\max}=\max_{1\leq i\leq 2}\{\xi_i\}$.

Notice that hypothesis (H1) gives us the existence of two constants g*, h* > 0 such that $|g(\zeta,0,0)|\leq g^*$ and $|h(\zeta,0,0)|\leq h^*$, for any $\zeta\in\mathcal{J}$.

Theorem 2. *Assume that (H1)–(H3) hold. Then the system* (1)-(2) *admits at least one solution defined in* $C(\mathcal{J})\times C(\mathcal{J})$.

Proof. We equip the Banach algebra $\mathbb{P}=C(\mathcal{J})\times C(\mathcal{J})$ with the norm

$$\|(\mathfrak{z}_1,\mathfrak{z}_2)\|_{\mathbb{P}}=\max\{\|\mathfrak{z}_1\|,\|\mathfrak{z}_2\|\},$$

for all $(\mathfrak{z}_1,\mathfrak{z}_2)\in\mathbb{P}$. Now, from Lemma 5, we introduce an operator $\mathcal{U}:\mathbb{P}\to\mathbb{P}$ as follows:

$$(\mathcal{U}(\mathfrak{z}_1,\mathfrak{z}_2))(\zeta)=((\mathcal{U}_1(\mathfrak{z}_1,\mathfrak{z}_2))(\zeta),(\mathcal{U}_2(\mathfrak{z}_1,\mathfrak{z}_2))(\zeta)), \tag{12}$$

where, $\mathcal{U}_i:\mathbb{P}\to C(\mathcal{J}),\ i=1,2$ are defined by:

$$\mathcal{U}_i(\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta))=h(\zeta,\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta))\left[-\int_a^\zeta\frac{\psi'(s)\psi(\zeta,s)^{\vartheta_i-1}}{\Gamma(\vartheta_i)}g(s,\mathfrak{z}_1(s),\mathfrak{z}_2(s))ds\right.$$
$$\left.+\int_a^{\eta_i}\frac{\psi'(s)\psi(\eta_i,s)^{\vartheta_i-1}}{\Gamma(\vartheta_i)}g(s,\mathfrak{z}_1(s),\mathfrak{z}_2(s))ds+\xi_i\int_a^b\psi'(s)g(s,\mathfrak{z}_1(s),\mathfrak{z}_2(s))ds\right]. \tag{13}$$

Clearly, the fixed points of the operator $\mathcal{U}$ coincide with the solutions of problem (1)-(2). We introduce the operators $\mathcal{H}=(\mathcal{H}_1,\mathcal{H}_2)$ and $\mathcal{L}=(\mathcal{L}_1,\mathcal{L}_2)$ as follows (for i = 1, 2):

$$\mathcal{H}_i(\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta))=h(\zeta,\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta)), \tag{14}$$

and

$$\mathcal{L}_i(\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta))=-\int_a^\zeta\frac{\psi'(s)\psi(\zeta,s)^{\vartheta_i-1}}{\Gamma(\vartheta_i)}g(s,\mathfrak{z}_1(s),\mathfrak{z}_2(s))ds$$
$$+\int_a^{\eta_i}\frac{\psi'(s)\psi(\eta_i,s)^{\vartheta_i-1}}{\Gamma(\vartheta_i)}g(s,\mathfrak{z}_1(s),\mathfrak{z}_2(s))ds+\xi_i\int_a^b\psi'(s)g(s,\mathfrak{z}_1(s),\mathfrak{z}_2(s))ds. \tag{15}$$

for all $(\mathfrak{z}_1,\mathfrak{z}_2)\in\mathbb{P}$ and any $\zeta\in\mathcal{J}$. Then,

$$\mathcal{U}_i(\mathfrak{z}_1,\mathfrak{z}_2)=\mathcal{H}_i(\mathfrak{z}_1,\mathfrak{z}_2)\cdot\mathcal{L}_i(\mathfrak{z}_1,\mathfrak{z}_2). \tag{16}$$

Clearly, the fixed point of operator $\mathcal{U}$ coincides with the solution of the problem (1)-(2).

For K > 0, we define the ball

$$\mathbb{B}_K=\{(\mathfrak{z}_1,\mathfrak{z}_2)\in\mathbb{P}:\ \|(\mathfrak{z}_1,\mathfrak{z}_2)\|_{\mathbb{P}}\leq K\}.$$

It's clear that $\mathbb{B}_K$ is closed, bounded and convex in $\mathbb{P}$.

Step 1. The operator $\mathcal{U}$ maps $\mathbb{P}$ into $C(\mathcal{J})$.

To demonstrate that $\mathcal{U}_i(\mathfrak{z}_1,\mathfrak{z}_2)\in C(\mathcal{J})$, it suffices to show that $\mathcal{H}_i(\mathfrak{z}_1,\mathfrak{z}_2),\mathcal{L}_i(\mathfrak{z}_1,\mathfrak{z}_2)\in C(\mathcal{J})$, for all $(\mathfrak{z}_1,\mathfrak{z}_2)\in\mathbb{P}$. Since the product of continuous functions is continuous.

Obviously, hypotheses (H1), ensure that if $(\mathfrak{z}_1,\mathfrak{z}_2)\in\mathbb{P}$, then $\mathcal{H}_i(\mathfrak{z}_1,\mathfrak{z}_2)\in C(\mathcal{J})$.

Next, we will demonstrate that if $(\mathfrak{z}_1,\mathfrak{z}_2)\in\mathbb{P}$ then $\mathcal{L}_i(\mathfrak{z}_1,\mathfrak{z}_2)\in C(\mathcal{J})$. To do so, let $\zeta\in\mathcal{J}$ be fixed and consider a sequence $\{\zeta_n\}$ in $\mathcal{J}$ such that $\zeta_n\to\zeta$ as $n\to\infty$. Without loss of generality, we may suppose $\zeta n>\zeta$. Thus, we obtain:

$$
\begin{aligned}
&\left|\mathcal{L}_i(\mathfrak{z}_1,\mathfrak{z}_2)(\zeta_n)-\mathcal{L}_i(\mathfrak{z}_1,\mathfrak{z}_2)(\zeta)\right| \\
&\quad\le \left|\int_a^{\zeta_n}\frac{\psi'(s)\psi(\zeta_n,s)^{\vartheta_i-1}}{\Gamma(\vartheta_i)}g(s,\mathfrak{z}_1(s),\mathfrak{z}_2(s))ds-\int_a^{\zeta}\frac{\psi'(s)\psi(\zeta,s)^{\vartheta_i-1}}{\Gamma(\vartheta_i)}g(s,\mathfrak{z}_1(s),\mathfrak{z}_2(s))ds\right| \\
&\quad\le \left|\int_a^{\zeta_n}\frac{\psi'(s)\psi(\zeta_n,s)^{\vartheta_i-1}}{\Gamma(\vartheta_i)}g(s,\mathfrak{z}_1(s),\mathfrak{z}_2(s))ds-\int_a^{\zeta_n}\frac{\psi'(s)\psi(\zeta,s)^{\vartheta_i-1}}{\Gamma(\vartheta_i)}g(s,\mathfrak{z}_1(s),\mathfrak{z}_2(s))ds\right| \\
&\quad+ \left|\int_a^{\zeta_n}\frac{\psi'(s)\psi(\zeta,s)^{\vartheta_i-1}}{\Gamma(\vartheta_i)}g(s,\mathfrak{z}_1(s),\mathfrak{z}_2(s))ds-\int_a^{\zeta}\frac{\psi'(s)\psi(\zeta,s)^{\vartheta_i-1}}{\Gamma(\vartheta_i)}g(s,\mathfrak{z}_1(s),\mathfrak{z}_2(s))ds\right| \\
&\quad\le \frac{1}{\Gamma(\vartheta_i)}\int_a^{\zeta_n}\left|\psi(\zeta_n,s)^{\vartheta_i-1}-\psi(\zeta,s)^{\vartheta_i-1}\right|\psi'(s)\left|g(s,\mathfrak{z}_1(s),\mathfrak{z}_2(s))\right|ds \\
&\quad+\frac{1}{\Gamma(\vartheta_i)}\int_\zeta^{\zeta_n}\psi'(s)\psi(\zeta,s)^{\vartheta_i-1}\left|g(s,\mathfrak{z}_1(s),\mathfrak{z}_2(s))\right|ds.
\end{aligned}
$$

In view of hypothesis (H1), g is bounded on the compact set $\mathcal{J}\times[-\|\mathfrak{z}_1\|,\|\mathfrak{z}_1\|]\times[-\|\mathfrak{z}_2\|,\|\mathfrak{z}_2\|]$ Denote by

$$N=\sup\left\{|g(s,\mathfrak{z}_1,\mathfrak{z}_2)|:\ s\in\mathcal{J},\ \mathfrak{z}_1\in[-\|\mathfrak{z}_1\|,\|\mathfrak{z}_1\|],\ \mathfrak{z}_2\in[-\|\mathfrak{z}_2\|,\|\mathfrak{z}_2\|]\right\}.$$

From the last estimate, we obtain

$$\left|\mathcal{L}_i(\mathfrak{z}_1,\mathfrak{z}_2)(\zeta_n)-\mathcal{L}_i(\mathfrak{z}_1,\mathfrak{z}_2)(\zeta)\right|\le\frac{N}{\Gamma(\vartheta_i)}\int_a^{\zeta_n}\left|\psi(\zeta_n,s)^{\vartheta_i-1}-\psi(\zeta,s)^{\vartheta_i-1}\right|\psi'(s)ds+\frac{N}{\Gamma(\vartheta_i)}\int_\zeta^{\zeta_n}\psi'(s)\psi(\zeta,s)^{\vartheta_i-1}ds.$$

Taking into account that $1<\vartheta\le 2$ and $\zeta_n>\zeta$, we infer that

$$
\begin{aligned}
\left|\mathcal{L}_i(\mathfrak{z}_1,\mathfrak{z}_2)(\zeta_n)-\mathcal{L}_i(\mathfrak{z}_1,\mathfrak{z}_2)(\zeta)\right|&\le\frac{N}{\Gamma(\vartheta_i)}\int_a^{\zeta_n}\left(\psi(\zeta_n,s)^{\vartheta-1}-\psi(\zeta,s)^{\vartheta-1}\right)\psi'(s)ds \\
&+\frac{N}{\Gamma(\vartheta_i)}\int_\zeta^{\zeta_n}\psi'(s)\psi(\zeta,s)^{\vartheta-1}ds\le\frac{N}{\Gamma(\vartheta_i)}\left(\frac{\psi(\zeta_n,a)^{\vartheta_i}}{\vartheta_i}-\frac{\psi(\zeta,a)^{\vartheta_i}}{\vartheta_i}\right).
\end{aligned}
$$

From the above inequality, we conclude that $\mathcal{L}_i(\mathfrak{z}_1,\mathfrak{z}_2)(\zeta_n)\to\mathcal{L}_i(\mathfrak{z}_1,\mathfrak{z}_2)(\zeta)$ as $n\to\infty$. Therefore, $\mathcal{L}_i(\mathfrak{z}_1,\mathfrak{z}_2)\in C(\mathcal{J})$.

Step 2. Estimate $\|\mathcal{U}(\mathfrak{z}_1,\mathfrak{z}_2)\|$ for $(\mathfrak{z}_1,\mathfrak{z}_2)\in\mathbb{P}$.

Let us fix $(\mathfrak{z}_1,\mathfrak{z}_2)\in\mathbb{P}$, for all $\zeta\in\mathcal{J}$, we have

$$\left|\mathcal{U}_i(\mathfrak{z}_1,\mathfrak{z}_2)(\zeta)\right|=\left|\mathcal{H}_i(\mathfrak{z}_1,\mathfrak{z}_2)(\zeta)\right|\left|\mathcal{L}_i(\mathfrak{z}_1,\mathfrak{z}_2)(\zeta)\right|,\quad i=1,2. \tag{17}$$

On one hand, by using hypothesis (H1) and (H2), we get

$$
\begin{aligned}
\left|\mathcal{H}_i(\mathfrak{z}_1,\mathfrak{z}_2)(\zeta)\right|&=\left|h(\zeta,\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta))\right|\le\left|h(\zeta,\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta))-h(\zeta,0,0)\right| \\
&+|h(\zeta,0,0)|\le\kappa_1\left(\max\left\{|\mathfrak{z}_1(\zeta)|,|\mathfrak{z}_2(\zeta)|\right\}\right) \\
&+h^*\le\kappa_1\left(\max\{\|\mathfrak{z}_1\|,\|\mathfrak{z}_2\|\}\right)+h^*\le\kappa_1\left(\|(\mathfrak{z}_1,\mathfrak{z}_2)\|_{\mathbb{P}}\right)+h^*,\quad i=1,2,
\end{aligned}
\tag{18}
$$

Similarly, we obtain

$$\left|g(s,\mathfrak{z}_1(s),\mathfrak{z}_2(s))\right|\le\left|g(s,\mathfrak{z}_1(s),\mathfrak{z}_2(s))-g(s,0,0)\right|+|g(s,0,0)|\le\kappa_2\left(\|(\mathfrak{z}_1,\mathfrak{z}_2)\|_{\mathbb{P}}\right)+g^*,\quad i=1,2.$$

On the other hand,

$$\begin{aligned}\left|\mathcal{L}_i(\mathfrak{z}_1,\mathfrak{z}_2)(\zeta)\right| &\le \left(\kappa_{i,2}(\|(\mathfrak{z}_1,\mathfrak{z}_2)\|_{\mathbb{P}})+g^*\right)\left(\int_a^\zeta \frac{\psi'(s)\psi(\zeta,s)^{\vartheta_i-1}}{\Gamma(\vartheta_i)}ds+\int_a^{\eta_i}\frac{\psi'(s)\psi(\eta_i,s)^{\vartheta_i-1}}{\Gamma(\vartheta_i)}ds+\xi_i\int_a^b\psi'(s)ds\right)\\ &\le \left(\kappa_2(\|(\mathfrak{z}_1,\mathfrak{z}_2)\|_{\mathbb{P}})+g^*\right)\left(\frac{\psi(\zeta,a)^{\vartheta_i}}{\Gamma(\vartheta_i+1)}+\frac{\psi(\eta_i,a)^{\vartheta_i}}{\Gamma(\vartheta_i+1)}+\xi_i\psi(b,a)\right)\\ &\le \left(\kappa_2(\|(\mathfrak{z}_1,\mathfrak{z}_2)\|_{\mathbb{P}})+g^*\right)\left(\frac{2\psi(b,a)^{\vartheta_i}}{\Gamma(\vartheta_i+1)}+\xi_i\psi(b,a)\right).\end{aligned} \tag{19}$$

Therefore, from (17), (18) and (19) and by $\Gamma(\vartheta_i+1)\ge 1$ for $1<\vartheta_i\le 2$, we have

$$\begin{aligned}\|\mathcal{U}_i(\mathfrak{z}_1,\mathfrak{z}_2)\| &\le \left(\kappa_1(\|(\mathfrak{z}_1,\mathfrak{z}_2)\|_{\mathbb{P}})+h^*\right)\left(\kappa_2(\|(\mathfrak{z}_1,\mathfrak{z}_2)\|_{\mathbb{P}})+g^*\right)\left(\frac{2\psi(b,a)^{\vartheta_i}}{\Gamma(\vartheta_i+1)}+\xi_i\psi(b,a)\right)\\ &\le \left(\kappa_1(\|(\mathfrak{z}_1,\mathfrak{z}_2)\|_{\mathbb{P}})+h^*\right)\left(\kappa_2(\|(\mathfrak{z}_1,\mathfrak{z}_2)\|_{\mathbb{P}})+g^*\right)\left(2\max_{1\le i\le 2}\left\{\psi(b,a)^{\vartheta_i}\right\}+\xi_{\max}\psi(b,a)\right).\end{aligned}$$

Hence, by (H3), we get

$$\|\mathcal{U}(\mathfrak{z}_1,\mathfrak{z}_2)\|=\max\left\{\|\mathcal{U}_1(\mathfrak{z}_1,\mathfrak{z}_2)\|,\|\mathcal{U}_2(\mathfrak{z}_1,\mathfrak{z}_2)\|\right\}\le K.$$

We deduce that the operator $\mathcal{U}$ maps $\mathbb{B}_K$ into itself, Moreover, from the last estimates, it follows that

$$\|\mathcal{H}(\mathbb{B}_K\times\mathbb{B}_K)\|\le\left(\kappa_1(K)+h^*\right),\|\mathcal{L}\left(\mathbb{B}_K\times\mathbb{B}_K\right)\|\le\left(\kappa_2(K)+g^*\right)(2\max_{1\le i\le 2}\left\{\psi(b,a)^{\vartheta_i}\right\}+\xi_{\max}\psi(b,a)). \tag{20}$$

Step 3. The continuity of the operators $\mathcal{H}$ and $\mathcal{L}$ on $\mathbb{B}_K$.

Firstly, we demonstrate that the operator $\mathcal{H}$ is continuous on $\mathbb{B}_K$. To do so, fix $\varsigma > 0$ and take $(\mathfrak{z}_1,\mathfrak{z}_2),(y_1,y_2)\in\mathbb{B}_K$ such that

$$\|(\mathfrak{z}_1,\mathfrak{z}_2)-(y_1,y_2)\|=\|(\mathfrak{z}_1-y_1,\mathfrak{z}_2-y_2)\|=\max\left\{\|\mathfrak{z}_1-y_1\|,\|\mathfrak{z}_2-y_2\|\right\}\le\varsigma.$$

Then, for $\zeta\in\mathcal{J}$, one has

$$\begin{aligned}\left|\mathcal{H}_i(\mathfrak{z}_1,\mathfrak{z}_2)(\zeta)-\mathcal{H}_i(y_1,y_2)(\zeta)\right| &= \left|h(\zeta,\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta))-h(\zeta,y_1(\zeta),y_2(\zeta))\right|\\ &\le \kappa_1\left(\max\left\{|\mathfrak{z}_1(\zeta)-y_1(\zeta)|,|\mathfrak{z}_2(\zeta)-y_2(\zeta)|\right\}\right)\\ &\le \kappa_1\left(\max\left\{\|\mathfrak{z}_1-y_1\|,\|\mathfrak{z}_2-y_2\|\right\}\right)\le\kappa_1(\varsigma)<\varsigma.\end{aligned}$$

where we have used Remark 1. Then

$$\|\mathcal{H}(\mathfrak{z}_1,\mathfrak{z}_2)-\mathcal{H}(y_1,y_2)\|<\varsigma.$$

The above inequality entails the continuity of the operator $\mathcal{H}$ on $\mathbb{B}_K$.
Now, the operator $\mathcal{L}$ is continuous on $\mathbb{B}_K$. In fact,

$$\begin{aligned}\left|\mathcal{L}_i(\mathfrak{z}_1,\mathfrak{z}_2)(\zeta)-\mathcal{L}_i(y_1,y_2)(\zeta)\right| &\le \frac{1}{\Gamma(\vartheta_i)}\int_a^\zeta\psi'(s)\psi(\zeta,s)^{\vartheta_i-1}\left|g(s,\mathfrak{z}_1(s),\mathfrak{z}_2(s))ds-g(s,y_1(s),y_2(s))\right|ds\\ &+\frac{1}{\Gamma(\vartheta_i)}\int_a^{\eta_i}\psi'(s)\psi(\eta_i,s)^{\vartheta_i-1}\left|g(s,\mathfrak{z}_1(s),\mathfrak{z}_2(s))ds-g(s,y_1(s),y_2(s))\right|ds\\ &+\xi_i\int_a^b\psi'(s)\left|g(s,\mathfrak{z}_1(s),\mathfrak{z}_2(s))-g(s,y_1(s),y_2(s))\right|ds.\end{aligned}$$

By (H2), we get

$$\left|\mathcal{L}_i(\mathfrak{z}_1,\mathfrak{z}_2)(\zeta)-\mathcal{L}_i(y_1,y_2)(\zeta)\right|\le \kappa_2\left(\max\{\|\mathfrak{z}_1-y_1\|,\|\mathfrak{z}_2-y_2\|\}\right)$$
$$\left[\int_a^\zeta \frac{\psi'(s)\psi(\zeta,s)^{\vartheta_i-1}}{\Gamma(\vartheta_i)}ds+\int_a^{\eta_i}\frac{\psi'(s)\psi(\eta_i,s)^{\vartheta_i-1}}{\Gamma(\vartheta_i)}ds+\xi_i\int_a^b\psi'(s)ds\right]$$
$$\le \kappa_2\left(\max\{\|\mathfrak{z}_1-y_1\|,\|\mathfrak{z}_2-y_2\|\}\right)\left(\frac{2\psi(b,a)^{\vartheta_i}}{\Gamma(\vartheta_i+1)}+\xi_i\psi(b,a)\right)$$
$$\le \left(\frac{2\psi(b,a)^{\vartheta_i}}{\Gamma(\vartheta_i+1)}+\xi_i\psi(b,a)\right)\kappa_2(\varsigma)\le\left(\frac{2\psi(b,a)^{\vartheta_i}}{\Gamma(\vartheta_i+1)}+\xi_i\psi(b,a)\right)\varsigma.$$

Therefore,

$$\|\mathcal{L}_i(\mathfrak{z}_1,\mathfrak{z}_2)-\mathcal{L}_i(y_1,y_2)\|\le\left(2\max_{1\le i\le 2}\left\{\psi(b,a)^{\vartheta_i}\right\}+\xi_{\max}\psi(b,a)\right)\varsigma.$$

Thus, the preceding inequality demonstrates that the operator $\mathcal{L}$ is a continuous operator on $\mathbb{B}_K$. Consequently, we conclude that $\mathcal{U}$ is a continuous operator on $\mathbb{B}_K$.

Step 4. For all subsets V_1 and V_2 of $\mathbb{B}_K$. We must solely examine the condition (6).

To accomplish this, we fix $\varsigma>0$, $\zeta_1,\zeta_2\in\mathcal{J}$ with $|\zeta_1-\zeta_2|\le\varsigma$ and $(\mathfrak{z}_1,\mathfrak{z}_2)\in V\times V_2$. Then, based on hypothesis (H2), we obtain

$$\left|\mathcal{H}_i(\mathfrak{z}_1,\mathfrak{z}_2)(\zeta_2)-\mathcal{H}_i(\mathfrak{z}_1,\mathfrak{z}_2)(\zeta_1)\right|=\left|h(\zeta_2,\mathfrak{z}_1(\zeta_2),\mathfrak{z}_2(\zeta_2))-h(\zeta_1,\mathfrak{z}_1(\zeta_1),\mathfrak{z}_2(\zeta_1))\right|$$
$$\le\left|h(\zeta_2,\mathfrak{z}_1(\zeta_2),\mathfrak{z}_2(\zeta_2))-h(\zeta_2,\mathfrak{z}_1(\zeta_1),\mathfrak{z}_2(\zeta_1))\right|$$
$$+\left|h(\zeta_2,\mathfrak{z}_1(\zeta_1),\mathfrak{z}_2(\zeta_1))-h(\zeta_1,\mathfrak{z}_1(\zeta_1),\mathfrak{z}_2(\zeta_1))\right|$$
$$\le\kappa_1\left(\max\{|\mathfrak{z}_1(\zeta_2)-\mathfrak{z}_1(\zeta_1)|,|\mathfrak{z}_2(\zeta_2)-\mathfrak{z}_2(\zeta_1)|\}\right)+\varpi(h,\varsigma)$$
$$\le\kappa_1\left(\max\{\varpi(\mathfrak{z}_1,\varsigma),\varpi(\mathfrak{z}_2,\varsigma)\}\right)+\varpi(h,\varsigma),\quad i=1,2,$$

where $\varpi(h,\varsigma)$ denotes the quantity

$$\varpi(h,\varsigma)=\sup\left\{\left|h(\zeta_2,\mathfrak{z}_1,\mathfrak{z}_2)-h(\zeta_1,\mathfrak{z}_1,\mathfrak{z}_2)\right|:\zeta_1,\zeta_2\in\mathcal{J},|\zeta_2-\zeta_1|\le\varsigma,\mathfrak{z}_1,\mathfrak{z}_2\in[-K,K]\right\}.$$

Thus,

$$\varpi(\mathcal{H}_i(V_1\times V_2),\varsigma)\le\kappa_1\left(\max\{\varpi(V_1,\varsigma),\varpi(V_2,\varsigma)\}\right)+\varpi(h,\varsigma),\quad i=1,2.$$

Note that the function $h(\zeta,\mathfrak{z}_1,\mathfrak{z}_2)$ is uniformly continuous on the set $\mathcal{J}\times[-K,K]\times[-K,K]$. Therefore, we infer that $\varpi(h,\varsigma)\to 0$ as $\varsigma\to 0$. Thus, The above inequality implies

$$\varpi_0(\mathcal{H}_i(V_1\times V_2))\le\lim_{\varsigma\to 0}\kappa_1\left(\max\{\varpi(V_1,\varsigma),\varpi(V_2,\varsigma)\}\right),\quad i=1,2.$$

By (H2) and since κ_1 is continuous, we get

$$\varpi_0\left(\mathcal{H}_i(V_1\times V_2)\right)\le\kappa_1\left(\max\{\varpi_0(V_1),\varpi_0(V_2)\}\right),\quad i=1,2. \tag{21}$$

Next, we estimate $\varpi_0(\mathcal{L}_i(V_1\times V_2))$, $i=1,2$.

Fix $\varsigma>0$, $\zeta_1,\zeta_2\in\mathcal{J}$ with $|\zeta_2-\zeta_1|\le\varsigma$ and $(\mathfrak{z}_1,\mathfrak{z}_2)\in V_1\times V_2$, we can suppose that $\zeta_1<\zeta_2$; Then, according to hypothesis (H2), we obtain

$$
\begin{aligned}
&\left|\mathcal{L}_i(\mathfrak{Z}_1,\mathfrak{Z}_2)(\zeta_2)-\mathcal{L}_i(\mathfrak{Z}_1,\mathfrak{Z}_2)(\zeta_1)\right|\\
&\quad\le\left|\int_a^{\zeta_2}\frac{\psi'(s)\psi(\zeta_2,s)^{\vartheta_i-1}}{\Gamma(\vartheta_i)}g(s,\mathfrak{Z}_1(s),\mathfrak{Z}_2(s))ds-\int_a^{\zeta_1}\frac{\psi'(s)\psi(\zeta_1,s)^{\vartheta_i-1}}{\Gamma(\vartheta_i)}g(s,\mathfrak{Z}_1(s),\mathfrak{Z}_2(s))ds\right|\\
&\quad\le\frac{1}{\Gamma(\vartheta_i)}\int_{\zeta_1}^{\zeta_2}\psi'(s)\psi(\zeta_2,s)^{\vartheta_i-1}\left|g(s,\mathfrak{Z}_1(s),\mathfrak{Z}_2(s))\right|ds\\
&\quad+\frac{1}{\Gamma(\vartheta_i)}\int_a^{\zeta_1}\left|\psi(\zeta_2,s)^{\vartheta_i-1}-\psi(\zeta_1,s)^{\vartheta_i-1}\right|\psi'(s)\left|g(s,\mathfrak{Z}_1(s),\mathfrak{Z}_2(s))\right|ds,\quad i=1,2.
\end{aligned}
$$

From $1<\vartheta\le 2$ and $\zeta_2>\zeta_1$, we can find

$$
\begin{aligned}
\left|\mathcal{L}_i(\mathfrak{Z}_1,\mathfrak{Z}_2)(\zeta_2)-\mathcal{L}_i(\mathfrak{Z}_1,\mathfrak{Z}_2)(\zeta_1)\right|&\le\frac{1}{\Gamma(\vartheta_i)}\int_{\zeta_1}^{\zeta_2}\psi'(s)\psi(\zeta_2,s)^{\vartheta_i-1}\left|g(s,\mathfrak{Z}_1(s),\mathfrak{Z}_2(s))\right|ds\\
&+\frac{1}{\Gamma(\vartheta_i)}\int_a^{\zeta_1}\left(\psi(\zeta_2,s)^{\vartheta_i-1}-\psi(\zeta_1,s)^{\vartheta_i-1}\right)\psi'(s)\left|g(s,\mathfrak{Z}_1(s),\mathfrak{Z}_2(s))\right|ds,\quad i=1,2.
\end{aligned}
$$

Since $g\in C(\mathcal{J}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ is bounded on the compact subsets of $\mathcal{J}\times\mathbb{R}\times\mathbb{R}$, particularly, on $\mathcal{J}\times[-K,K]\times[-K,K]$. Put

$$
L=\sup\left\{\left|g(\zeta,\mathfrak{Z}_1,\mathfrak{Z}_2)\right|:\ \zeta\in\mathcal{J},\ \ \mathfrak{Z}_1,\mathfrak{Z}_2\in[-K,K]\right\}.
$$

Then,

$$
\begin{aligned}
\left|\mathcal{L}_i(\mathfrak{Z}_1,\mathfrak{Z}_2)(\zeta_2)-\mathcal{L}_i(\mathfrak{Z}_1,\mathfrak{Z}_2)(\zeta_1)\right|&\le\frac{L}{\Gamma(\vartheta_i)}\int_{\zeta_1}^{\zeta_2}\psi'(s)\psi(\zeta_2,s)^{\vartheta_i-1}ds\\
&+\frac{L}{\Gamma(\vartheta_i)}\int_a^{\zeta_1}\left(\psi(\zeta_2,s)^{\vartheta_i-1}-\psi(\zeta_1,s)^{\vartheta_i-1}\right)\psi'(s)ds\\
&\le\frac{L}{\Gamma(\vartheta_i)}\left(\frac{\psi(\zeta_2,a)^{\vartheta_i}}{\vartheta_i}-\frac{\psi(\zeta_1,a)^{\vartheta_i}}{\vartheta_i}\right)\le\frac{L}{\Gamma(\vartheta_i)}\psi'(b)\psi(b,a)^{\vartheta_i-1}\,|\,\zeta_2-\zeta_1\,|\\
&\le\frac{L}{\Gamma(\vartheta_i)}\psi'(b)\psi(b,a)^{\vartheta_i-1}\varsigma,\quad i=1,2.
\end{aligned}
$$

Using Lemma 4, one has

$$
\varpi(\mathcal{L}_i(V_1\times V_2),\varsigma)\le\frac{L\psi'(b)\psi(b,a)^{\vartheta_i-1}}{\Gamma(\vartheta_i)}\varsigma,\quad i=1,2.
$$

From this, it follows that

$$
\varpi_0\left(\mathcal{L}_i(V_1\times V_2)\right)=0,\quad i=1,2. \tag{22}
$$

After that, by Lemma 3, and using (20), (21), and (22), with i = 1, 2, we have:

$$
\begin{aligned}
\varpi_0\left(\mathcal{U}_i(V_1\times V_2)\right)&=\varpi_0\left(\mathcal{H}_i(V_1\times V_2)\cdot\mathcal{L}_i(V_1\times V_2)\right)\\
&\le\|\mathcal{H}_i(V_1\times V_2)\|\varpi_0\left(\mathcal{L}_i(V_1\times V_2)\right)+\|\mathcal{L}_i(V_1\times V_2)\|\varpi_0\left(\mathcal{H}_i(V_1\times V_2)\right)\\
&\le\|\mathcal{H}_i(\mathbb{B}_K\times\mathbb{B}_K)\|\varpi_0\left(\mathcal{L}_i(V_1\times V_2)\right)+\|\mathcal{L}_i(\mathbb{B}_K\times\mathbb{B}_K)\|\varpi_0\left(\mathcal{H}_i(V_1\times V_2)\right)\\
&\le\left(\kappa_2(K)+g^*\right)\left(2\max_{1\le i\le 2}\left\{\psi(b,a)^{\vartheta_i}\right\}+\xi_{\max}\psi(b,a)\right)\kappa_1\left(\max\{\varpi_0(V_1),\varpi_0(V_2)\}\right).
\end{aligned}
$$

Since

$$
\left(\kappa_2(K)+g^*\right)\left(2\max_{1\le i\le 2}\left\{\psi(b,a)^{\vartheta_i}\right\}+\xi_{\max}\psi(b,a)\right)\le 1.
$$

From (H3), we deduce from the most recent estimate that

$$\varpi_0\left(\mathcal{U}_i(V_1\times V_2)\right)\le \kappa_1\left(\max\{\varpi_0(V_1),\varpi_0(V_2)\}\right),\quad i=1,2.$$

and

$$\varpi_0\left(\mathcal{U}(V_1\times V_2)\right)=\max\left\{\varpi_0\left(\mathcal{U}_1(V_1\times V_2)\right),\varpi_0\left(\mathcal{U}_2(V_1\times V_2)\right)\right\}\le \kappa_1\left(\max\{\varpi_0(V_1),\varpi_0(V_2)\}\right).$$

Finally, by employing Theorem 1, we deduce that $\mathcal{U}$ possesses at least one coupled fixed point in $\mathbb{B}_K$. Therefore, problem (1)-(2) admits at least one solution in $\mathbb{B}_K$.

4. An example

In this section, we give examples to illustrate the usefulness of our main results. Consider the following coupled system :

$$\begin{cases} {}^{c}\mathcal{D}_{0^+}^{\frac{3}{2};\zeta}\left(\dfrac{13\mathfrak{z}_1(\zeta)}{5+\arctan|\mathfrak{z}_1(\zeta)|+\mathfrak{z}_2(\zeta)}\right)+\dfrac{1+\mathfrak{z}_1(\zeta)}{91}+\dfrac{|\mathfrak{z}_2(\zeta)|}{91(1+|\mathfrak{z}_2(\zeta)|)}=0,\quad \zeta\in\mathcal{J}:=[0,2],\\ {}^{c}\mathcal{D}_{0^+}^{\frac{4}{3};\zeta}\left(\dfrac{13\mathfrak{z}_2(\zeta)}{5+\arctan|\mathfrak{z}_1(\zeta)|+\mathfrak{z}_2(\zeta)}\right)+\dfrac{1+\mathfrak{z}_1(\zeta)}{91}+\dfrac{|\mathfrak{z}_2(\zeta)|}{91(1+|\mathfrak{z}_2(\zeta)|)}=0,\quad \zeta\in\mathcal{J}:=[0,2] \end{cases}\tag{23}$$

and

$$\begin{cases} D\left(\dfrac{13\mathfrak{z}_1(\zeta)}{5+\arctan|\mathfrak{z}_1(\zeta)|+\mathfrak{z}_2(\zeta)}\right)\Bigg|_{\zeta=0}=D\left(\dfrac{13\mathfrak{z}_2(\zeta)}{5+\arctan|\mathfrak{z}_1(\zeta)|+\mathfrak{z}_2(\zeta)}\right)\Bigg|_{\zeta=0}=0,\\ \dfrac{1}{91}\,{}^{c}\mathcal{D}_{0^+}^{\frac{1}{2};\zeta}\left(\dfrac{13\mathfrak{z}_1(\zeta)}{5+\arctan|\mathfrak{z}_1(\zeta)|+\mathfrak{z}_2(\zeta)}\right)\Bigg|_{\zeta=2}+\left(\dfrac{13\mathfrak{z}_1(\zeta)}{5+\arctan|\mathfrak{z}_1(\zeta)|+\mathfrak{z}_2(\zeta)}\right)\Bigg|_{\zeta=\frac{3}{2}}=0,\\ \dfrac{1}{10}\,{}^{c}\mathcal{D}_{0^+}^{\frac{1}{3};\zeta}\left(\dfrac{13\mathfrak{z}_2(\zeta)}{5+\arctan|\mathfrak{z}_1(\zeta)|+\mathfrak{z}_2(\zeta)}\right)\Bigg|_{\zeta=2}+\left(\dfrac{13\mathfrak{z}_2(\zeta)}{5+\arctan|\mathfrak{z}_1(\zeta)|+\mathfrak{z}_2(\zeta)}\right)\Bigg|_{\zeta=\frac{4}{3}}=0. \end{cases}\tag{24}$$

Corresponding to the problem (1)-(2), we have that

$$\mathcal{J}=[a,b]=[0,2],\quad \psi(\zeta)=\zeta,\quad \xi_1=\frac{1}{91},\quad \xi_2=\frac{1}{10},\quad \eta_1=\vartheta_1=\frac{3}{2},\quad \eta_2=\vartheta_2=\frac{4}{3},$$

and

$$h(\zeta,\mathfrak{z}_1,\mathfrak{z}_2)=\frac{1}{13}(5+\arctan|\mathfrak{z}_1|+\mathfrak{z}_2),\quad g(\zeta,\mathfrak{z}_1,\mathfrak{z}_2)=\frac{1}{91}+\frac{\mathfrak{z}_1}{91}+\frac{|\mathfrak{z}_2|}{91(1+|\mathfrak{z}_2|)},$$

It is evident that hypotheses (H1) hold, and

$$h^*=\sup\left\{\left|h(\zeta,0,0)\right| : \zeta\in\mathcal{J}\right\}=\frac{5}{13},$$

$$g^*=\sup\left\{\left|g(\zeta,0,0)\right| : \zeta\in\mathcal{J}\right\}=\frac{1}{91}.$$

Moreover, for $\zeta\in\mathcal{J}$ and $u_1,u_2,v_1,v_2\in\mathbb{R}$, we have

$$\begin{aligned}\left|h(\zeta,u_1,u_2)-h(\zeta,v_1,v_2)\right| &\le \frac{1}{13}\left(|\arctan|u_1|-\arctan|v_1||+|u_2-v_2|\right)\\ &\le \frac{1}{13}\left(\arctan||u_1|-|v_1||+|u_2-v_2|\right)\\ &\le \frac{1}{13}\left(\arctan(|u_1-v_1|)+|u_2-v_2|\right)\\ &\le \frac{1}{13}\left(\sigma_1(|u_1-v_1|)+\sigma_2(|u_2-v_2|)\right),\end{aligned}$$

where $\sigma_1(u) = \arctan(u)$ and $\sigma_2(u) = u,$ then

$$\left|h(\zeta,u_1,u_2) - h(\zeta,v_1,v_2)\right| \le \frac{1}{13}\max_{1\le i\le 2}\{\sigma_i\}(|u_1 - v_1|) + \frac{1}{13}\max_{1\le i\le 2}\{\sigma_i\}(|u_2 - v_2|)$$
$$\le \frac{2}{13}\max_{1\le i\le 2}\{\sigma_i\}\left(\max\left\{|u_1 - v_1|,|u_2 - v_2|\right\}\right).$$

Therefore, $\kappa_1(\zeta) = \dfrac{2}{13}\max\left(\sigma_1(\zeta),\sigma_2(\zeta)\right)$ and $\kappa_1 \in \mathbb{E}.$.

On the other hand,

$$\left|g(\zeta,u_1,u_2) - g(\zeta,v_1,v_2)\right| \le \frac{1}{91}\left|\frac{|u_2|}{1+|u_2|} - \frac{|v_2|}{1+|v_2|}\right| + \frac{|u_1 - v_1|}{91} \le \frac{1}{91}\left|\frac{|u_2|-|v_2|}{(1+|u_2|)(1+|v_2|)}\right| + \frac{|u_1 - v_1|}{91}$$
$$\le \frac{1}{91}\frac{|u_2 - v_2|}{1+|u_2 - v_2|} + \frac{|u_1 - v_1|}{91} \le \frac{\sigma_3(|u_2 - v_2|)}{91} + \frac{\sigma_2(|u_1 - v_1|)}{91},$$

where $\sigma_3(u) = \dfrac{u}{1+u},$ then

$$\left|g(\zeta,u_1,u_2) - g(\zeta,v_1,v_2)\right| \le \frac{2}{91}\max_{2\le i\le 3}\{\sigma_i\}\left(\max\left\{|u_1 - v_1|,|u_2 - v_2|\right\}\right)$$

Thus, $\kappa_2(\zeta) = \dfrac{2}{91}\max\left(\sigma_2(\zeta),\sigma_3(\zeta)\right)$ and $\kappa_2 \in \mathbb{E}.$ Hence, hypothesis (H2) is fulfilled.

Lastly, note that hypothesis (H3) is equivalent to

$$\left(\frac{2}{13}\max(\arctan(K),K) + 5/13\right)\left(\frac{2}{91}\max\left(K,\frac{K}{1+K}\right) + 1/91\right)\left(2^{5/2} + 1/5\right) \le K,$$

and

$$\left(\frac{2}{91}\max\left(K,\frac{K}{1+K}\right) + 1/91\right)\left(2^{5/2} + 1/5\right) \le 1,$$

which is verified by K_0 = 1. Therefore, all hypotheses of Theorem 2 are met, and as a result, the problem (23)-(24) admits at least one solution $(\mathfrak{z}_1,\mathfrak{z}_2) \in \mathbb{P}$ such that $\max\left\{\|\mathfrak{z}_1\|,\|\mathfrak{z}_2\|\right\} \le 1.$

5. Concluding

This paper presents existing results for the coupled ψ-Caputo hybrid fractional thermostat system. Our analysis is based on the MNC technique along with Darbo's fixed-point theorem. Additionally, we investigate future research directions regarding the extension of this system within the ψ-Hilfer FD framework.

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