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Hyperbolic sine function in Hilbert's discrete inequality of three variables

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Through this manuscript, we deduce a new discrete form of Hilbert's inequality of three variables of Hyperbolic Sine Function, also, we will show that the constant in the main inequality is the best constant. Also, we will give the reverse form of the main inequality and the equivalent forms of it.

Key words and phrases: Hilbert's inequality of discrete form, Hölder inequality, Best constant, Hyperbolic Sine Function.

Mathematics Subject Classification (2020): 26D15.

1. Introduction

If $\{a_m\}$ and $\{b_m\}$ are real sequences, satisfying $\sum_{m=1}^{\infty} a_m \in (0,\infty)$ and $\sum_{n=1}^{\infty} b_n \in (0,\infty)$, for $p>1, \frac{1}{p}+\frac{1}{q}$, then we have the famous inequality

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}, \tag{1}$$

where the constant factor $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$ is the best possible, and the constant in (1) will be π incase $p = q = 2$. The inequality (1) is the discrete analogue of Hilbert's inequality of two variables. The historical and prehistorical origins of the Hardy inequality were studied in [1, 2].

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In the last two decades, the inequality (1) and the integral form of it, have many extensions and equivalent forms in many different ways, see [3–11, 13].

Quite number of authors estabished a various formulation of improvements and extensions of (1) in integral, discrete, and the form of half-discrete. Bathbold.T. and Laith. E. A. gave a new Hilbert's inequalities of three variables in the three forms, discrete, integral, and half discrete form [14, 15], in [15] they introduced the Discrete form of Hilbert inequality of three variables as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}\frac{a_{m,n}b_r}{(m+n+r)^{\lambda}}<C\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m+n)^{2p-\lambda-p\xi-2}a_{m,n}^{p}\right)^{\frac{1}{p}}\left(\sum_{r=1}^{\infty}r^{q+q\xi-\lambda-1}b_r^{q}\right)^{\frac{1}{q}} \tag{2}$$

where the constant $C=B\left(\frac{\lambda}{p}-\xi,\frac{\lambda}{q}+\xi\right)$ is the best constant, and $a_{m,n}$ is a double sequence with non-negative terms and b_r is a sequence with non-neqative terms. Also, in 2022, Nizar Kh. Al-Oushoush gave a new Hilbert's inequality of three variables in Discrete form [12] as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}\frac{a_{m,n}b_r}{(\frac{1}{m}+\frac{1}{n}+\frac{1}{r})^{\lambda}}\le C\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(mn)^{\lambda p-\lambda-\varsigma p}a_{m,n}^{p}}{(m+n)^{\lambda p-\lambda-\varsigma p-\frac{2p}{q}}}\right)^{\frac{1}{p}}\left(\sum_{r=1}^{\infty}r^{q\lambda+q\varsigma-\lambda+\frac{q}{p}}b_r^{q}\right)^{\frac{1}{q}} \tag{3}$$

where $C=B\left(\frac{\lambda}{p}+\varsigma,\frac{\lambda}{q}-\varsigma\right)$ is the best possible.

2. Preliminries and Lemmas

To prove the main result in this work, we will use the following famous two functions:

$$\Gamma(\vartheta)=\int_0^{\infty}t^{\vartheta-1}e^{-t}dt,\ \ \vartheta>0, \tag{4}$$

$$B\left(\varrho,\tau\right)=\int_0^{\infty}\frac{t^{\varrho-1}}{\left(t+1\right)^{\varrho+\tau}}dt,\ \ \varrho,\tau>0. \tag{5}$$

Also, there is another formula for every one of the above functions that will be used in this research

$$\frac{1}{x^{\vartheta}}=\frac{1}{\Gamma\left(\vartheta\right)}\int_0^{\infty}t^{\vartheta-1}e^{-\vartheta t}dt \tag{6}$$

$$B(\varrho,\tau)=\frac{\Gamma(\varrho)\Gamma(\tau)}{\Gamma(\varrho+\tau)} \tag{7}$$

$$B(\varrho,\tau)=B(\tau,\varrho). \tag{8}$$

Also, the following inequality is nedeed through this work:

$$\cosh x\ge 1,\qquad \forall x. \tag{9}$$

To prove the main results in this research, we give the following necessary lemmas:

Lemma 1. Let $r>1$, $\frac{1}{r}+\frac{1}{s}=1$, and $b_k>0$, then for $t>0$ and $0\le\eta<\frac{1}{r}$, we have

$$\sum_{k=1}^{\infty}e^{-[\sinh k]t}b_k\le t^{\eta-\frac{1}{r}}\Gamma^{\frac{1}{r}}(1-r\eta)\left(\sum_{k=1}^{\infty}\left(\sinh k\right)^{s\eta}e^{-[\sinh k]t}b_k^{s}\right)^{\frac{1}{s}}. \tag{10}$$

Proof: By using Hölder's inequality, and the substitution $\varpi=\sinh z$, we obtain

$$\sum_{k=1}^{\infty} e^{-[\sinh k]t} b_k = \infty \sum_{k=1} \left(\left(\sinh k\right)^{-\eta} e^{-\frac{[\sinh k]}{r}t} \right) \left(\left(\sinh k\right)^{\eta} e^{-\frac{[\sinh k]}{s}t} b_k \right)$$
$$\leq \left(\sum_{k=1}^{\infty} \left(\sinh k\right)^{-r\eta} e^{-[\sinh k]t} \right)^{\frac{1}{r}} \left(\sum_{k=1}^{\infty} \left(\sinh k\right)^{s\eta} e^{-[\sinh k]t} b_k^s) \right)^{\frac{1}{s}}$$
$$\leq \left(\int_0^{\infty} \left(\sinh z\right)^{-r\eta} e^{-[\sinh z]t} dz \right)^{\frac{1}{r}} \left(\sum_{k=1}^{\infty} \left(\sinh k\right)^{s\eta} e^{-[\sinh k]t} b_k^s) \right)^{\frac{1}{s}}$$
$$= \left(\int_0^{\infty} \varpi^{-r\eta} e^{-\varpi t} \frac{d\varpi}{\cosh \varpi} \right)^{\frac{1}{r}} \left(\sum_{k=1}^{\infty} \left(\sinh k\right)^{s\eta} e^{-[\sinh k]t} b_k^s) \right)^{\frac{1}{s}}$$
$$\leq \left(\int_0^{\infty} \varpi^{-r\eta} e^{-\varpi t} d\varpi \right)^{\frac{1}{r}} \left(\sum_{k=1}^{\infty} \left(\sinh k\right)^{s\eta} e^{-[\sinh k]t} b_k^s) \right)^{\frac{1}{s}}$$
$$= t^{\eta - \frac{1}{r}} \Gamma^{\frac{1}{r}} (1 - r\eta) \left(\sum_{k=1}^{\infty} \left(\sinh k\right)^{s\eta} e^{-[\sinh k]t} b_k^s \right)^{\frac{1}{s}}$$

Lemma 2. Let $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, and $a_{i,j} > 0$, then for $t > 0$ and $0 < \gamma < \frac{2}{s}$, we have

$$\left(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i,j} e^{-[\sinh i + \sinh j]t} \right)^r \leq t^{\gamma r - 2\frac{r}{s}} \Gamma^{\frac{r}{s}} (2 - \gamma s) \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(\sinh i + \sinh j\right)^{\gamma r} e^{-[\sinh i + \sinh j]t} a_{i,j}^r \quad (11)$$

Proof: By applying Hölder's inequality. Use the substitutions $\sinh y = u \sinh x$ and $\sinh x = \frac{v}{1+u}$, we have

$$\left(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i,j} e^{-[\sinh i + \sinh j]t} \right)^r = \left(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left\{ \left(\sinh i + \sinh j\right)^{-\gamma} e^{-\frac{[\sinh i + \sinh j]}{s}t} \right\} \left\{ \left(\sinh i + \sinh j\right)^{\gamma} e^{-\frac{[\sinh i + \sinh j]}{r}t} a_{i,j} \right\} \right)^r$$
$$\leq \left(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(\sinh i + \sinh j\right)^{-\gamma s} e^{-[\sinh i + \sinh j]t} \right)^{\frac{r}{s}} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(\sinh i + \sinh j\right)^{\gamma r} e^{-[\sinh i + \sinh j]t} a_{i,j}^r$$
$$\leq \left(\int_0^{\infty} \int_0^{\infty} \left(\sinh x + \sinh y\right)^{-\gamma s} e^{-[\sinh x + \sinh y]t} dxdy \right)^{\frac{r}{s}} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(\sinh i + \sinh j\right)^{\gamma r} e^{-[\sinh i + \sinh j]t} a_{i,j}^r$$
$$= \left(\int_0^{\infty} \int_0^{\infty} \frac{e^{-\sinh x[1+u]t}}{(\sinh x)^{\gamma s} \left[1+u\right]^{\gamma s}} \left(\frac{dv}{(1+u)\cosh x} \right) \left(\frac{\sinh x du}{\cosh y} \right) \right)^{\frac{r}{s}}$$
$$\times \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(\sinh i + \sinh j\right)^{\gamma r} e^{-[\sinh i + \sinh j]t} a_{i,j}^r$$
$$\leq \left(\int_0^{\infty} \int_0^{\infty} \frac{e^{-\sinh x[1+u]t}}{(\sinh x)^{\gamma s - 1} \left[1+u\right]^{\gamma s + 1}} dvdu \right)^{\frac{r}{s}} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(\sinh i + \sinh j\right)^{\gamma r} e^{-[\sinh i + \sinh j]t} a_{i,j}^r$$
$$= \left(\int_0^{\infty} \int_0^{\infty} \frac{e^{-vt}}{(\frac{v}{1+u})^{\gamma s - 1} \left[1+u\right]^{\gamma s + 1}} dvdu \right)^{\frac{r}{s}} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(\sinh i + \sinh j\right)^{\gamma r} e^{-[\sinh i + \sinh j]t} a_{i,j}^r$$

$$=\left(\int_0^\infty \frac{1}{(1+u)^2}du\int_0^\infty v^{1-\gamma s}e^{-vt}dv\right)^{\frac{r}{s}}\sum_{i=1}^\infty\sum_{j=1}^\infty(\sinh i+\sinh j)^{\gamma r}e^{-[\sinh i+\sinh j]t}a_{i,j}^r$$

$$=t^{\gamma r-2\frac{r}{s}}\Gamma^{\frac{r}{s}}(2-\gamma s)\sum_{i=1}^\infty\sum_{j=1}^\infty(\sinh i+\sinh j)^{\gamma r}e^{-[\sinh i+\sinh j]t}a_{i,j}^r$$

Main Result

Theorem 1. Let $r>1$, $\frac{1}{r}+\frac{1}{s}=1$, $0<\lambda<\min\{r,s\}$, $-\frac{\lambda}{r}<\delta<\frac{\lambda}{s}$, suppose that $a_{i,j}$ is double sequence with non-negative terms, and b_k is sequence with non-neqative terms.

If $\sum_{i=1}^\infty\sum_{j=1}^\infty(\sinh i+\sinh j)^{-\lambda-r\delta+2\frac{r}{s}}a_{i,j}<\infty$ and $\sum_{k=1}^\infty(\sinh k)^{-\lambda+s\delta+\frac{s}{r}}b_k<\infty$, then

$$\sum_{i=1}^\infty\sum_{j=1}^\infty\sum_{k=1}^\infty\frac{a_{i,j}b_k}{(\sinh i+\sinh j+\sinh k)^\lambda}\le C\left(\sum_{i=1}^\infty\sum_{j=1}^\infty(\sinh i+\sinh j)^{-\lambda-r\delta+2\frac{r}{s}}a_{i,j}^r\right)^{\frac{1}{r}}\times\left(\sum_{k=1}^\infty(\sinh k)^{-\lambda+s\delta+\frac{s}{r}}b_k^s\right)^{\frac{1}{s}}\quad(12)$$

where $C=B\left(\frac{\lambda}{r}+\delta,\frac{\lambda}{s}-\delta\right)$ is the best possible.

Proof: Let

$$I=\sum_{i=1}^\infty\sum_{j=1}^\infty\sum_{k=1}^\infty\frac{a_{i,j}b_k}{(\sinh i+\sinh j+\sinh k)^\lambda}=\left(\frac{1}{\Gamma(\lambda)}\sum_{i=1}^\infty\sum_{j=1}^\infty\sum_{k=1}^\infty a_{i,j}b_k\int_0^\infty t^{\lambda-1}e^{-[\sinh i+\sinh j+\sinh k]t}dt\right)$$

$$=\frac{1}{\Gamma(\lambda)}\int_0^\infty\left(t^{\frac{\lambda-1}{r}+\delta}\sum_{i=1}^\infty\sum_{j=1}^\infty a_{i,j}e^{-[\sinh i+\sinh j]t}\right)\left(t^{\frac{\lambda-1}{s}-\delta}\sum_{k=1}^\infty e^{-[\sinh k]t}b_k\right)dt$$

$$\le\frac{1}{\Gamma(\lambda)}\left(\int_0^\infty t^{\lambda-1+r\delta}\left(\sum_{i=1}^\infty\sum_{j=1}^\infty a_{i,j}e^{-[\sinh i+\sinh j]t}\right)^r dt\right)^{\frac{1}{r}}\times\left(\int_0^\infty t^{\lambda-1-s\delta}\left(\sum_{k=1}^\infty e^{-[\sinh k]t}b_k\right)^s dt\right)^{\frac{1}{s}}\quad(13)$$

Putting the inequalities (10), (11) in (13) we get:

$$I\le\frac{1}{\Gamma(\lambda)}\left(\int_0^\infty t^{\lambda-1+r\delta}t^{\gamma r-2\frac{r}{s}}\Gamma^{\frac{r}{s}}(2-\gamma s)\sum_{i=1}^\infty\sum_{j=1}^\infty(\sinh i+\sinh j)^{\gamma r}e^{-[\sinh i+\sinh j]t}a_{i,j}^r dt\right)^{\frac{1}{r}}$$

$$\times\left(\int_0^\infty t^{\lambda-1-s\delta}\left(t^{\eta-\frac{1}{r}}\Gamma^{\frac{1}{r}}(1-r\eta)\left(\sum_{k=1}^\infty(\sinh k)^{s\eta}e^{-[\sinh k]t}b_k^s\right)^{\frac{1}{s}}\right)^s dt\right)^{\frac{1}{s}}$$

$$=\frac{\Gamma^{\frac{1}{s}}(2-\gamma s)\Gamma^{\frac{1}{r}}(1-r\eta)}{\Gamma(\lambda)}\left(\sum_{i=1}^\infty\sum_{j=1}^\infty(\sinh i+\sinh j)^{\gamma r}a_{i,j}^r\int_0^\infty t^{\lambda-1+r\delta+\gamma r-2\frac{r}{s}}e^{-(\sinh i+\sinh j)t}dt\right)^{\frac{1}{r}}$$

$$\times\left(\sum_{k=1}^\infty(\sinh k)^{s\eta}b_k^s\int_0^\infty t^{\lambda-1-s\delta+s\eta-\frac{s}{r}}e^{-[\sinh k]t}dt\right)^{\frac{1}{s}}$$

$$=\frac{\Gamma^{\frac{1}{s}}(2-\gamma s)\Gamma^{\frac{1}{r}}(1-r\eta)}{\Gamma(\lambda)}\left(\Gamma\left(\lambda+r\delta+\gamma r-2\frac{r}{s}\right)\sum_{i=1}^\infty\sum_{j=1}^\infty(\sinh i+\sinh j)^{-\lambda-r\delta+2\frac{r}{s}}a_{i,j}^r\right)^{\frac{1}{r}}$$

$$\times\left(\Gamma\left(\lambda-s\delta+s\eta-\frac{s}{r}\right)\sum_{k=1}^\infty(\sinh k)^{-\lambda+s\delta+\frac{s}{r}}b_k^s\right)^{\frac{1}{s}}=C_{\psi,\phi}\left(\sum_{i=1}^\infty\sum_{j=1}^\infty(\sinh i+\sinh j)^{-\lambda-r\delta+2\frac{r}{s}}a_{i,j}^r\right)^{\frac{1}{r}}\left(\sum_{k=1}^\infty(\sinh k)^{-\lambda+s\delta+\frac{s}{r}}b_k^s\right)^{\frac{1}{s}}$$

where $C_{\psi,\varphi} = \frac{\Gamma^{\frac{1}{s}}(2-\gamma s)\Gamma^{\frac{1}{r}}(1-r\eta)\Gamma^{\frac{1}{r}}\left(\lambda+r\delta+\gamma r-2\frac{r}{s}\right)\Gamma^{\frac{1}{s}}\left(\lambda-s\delta+s\eta-\frac{s}{r}\right)}{\Gamma(\lambda)}$.

Set $\psi = \frac{2-\lambda-\delta r+\frac{2r}{s}}{rs}$ and $\varphi = \frac{1-\lambda+\delta s+\frac{s}{r}}{rs}$, we get inequality (10) with constant $C_{\frac{2-\lambda-\delta r+\frac{2r}{s}}{rs},\frac{1-\lambda+\delta s+\frac{s}{r}}{rs}} = C_{\psi,\varphi} = C.$

Next, we want to prove that the constant C in (12) is the best possible. Define two sequences as follows: $\tilde{a}_{i,j} = 0,$ for $i = j = 1,$ $\tilde{a}_{i,j} = (\sinh i + \sinh j)^{\frac{\lambda+r\delta-2\frac{r}{s}-\epsilon-2}{r}} (\cosh i \cosh j)^{\frac{1}{r}}$ for $i, j \geq 2,$ $\tilde{b}_k = 0,$ for $k = 1$ and $\tilde{b}_k = (\sinh k)^{\frac{\lambda-s\delta-\frac{s}{r}-\epsilon-1}{s}}$ for $k \geq 2$, use the same substitutions as in Lemma 1 and Lemma 2. Suppose that C is not the best possible, then $\exists Q$, where $0 < Q < C$, noting that $\frac{\cosh i \cosh j}{(\sinh i+\sinh j)^{\epsilon+2}} < 1,$ for $i, j \geq 1$ such that

$$I \leq Q\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(\sinh i + \sinh j)^{-\lambda-r\delta+2\frac{r}{s}} \tilde{a}_{l,j}^{r}\right)^{\frac{1}{r}} \left(\sum_{k=1}^{\infty}(\sinh k)^{-\lambda+s\delta+\frac{s}{r}} \tilde{b}_{i}^{s}\right)^{\frac{1}{s}}$$

$$= Q\left(\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}(\sinh i + \sinh j)^{-\lambda-r\delta+2\frac{r}{s}} \left((\sinh i + \sinh j)^{\frac{\lambda+r\delta-2\frac{r}{s}-\epsilon-2}{r}} (\cosh i \cosh j)^{\frac{1}{r}}\right)^{r}\right)^{\frac{1}{r}}$$

$$\times\left(\sum_{k=2}^{\infty}(\sinh k)^{-\lambda+s\delta+\frac{s}{r}} \left((\sinh k)^{\frac{\lambda-s\delta-\frac{s}{r}-\epsilon-1}{s}}\right)^{s}\right)^{\frac{1}{s}} = Q\left(\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}\frac{\cosh i\cosh j}{(\sinh i + \sinh j)^{\epsilon+2}}\right)^{\frac{1}{r}}\left(\sum_{k=2}^{\infty}(\sinh k)^{-\epsilon-1}\right)^{\frac{1}{s}}$$

$$= Q\left(\int_1^{\infty}\int_1^{\infty}(\sinh x + \sinh y)^{-\epsilon-2}\cosh x\cosh y\,dxdy\right)^{\frac{1}{r}}\left(\int_1^{\infty}(\sinh z)^{-\epsilon-1}dz\right)^{\frac{1}{s}}$$

$$< Q\left(\int_1^{\infty}\int_1^{\infty}(\sinh x)^{-\epsilon-2}\left(1+\frac{\sinh y}{\sinh x}\right)^{-\epsilon-2}\cosh x\cosh y\,dxdy\right)^{\frac{1}{r}}\left(\int_1^{\infty}(\sinh z)^{-\epsilon-1}dz\right)^{\frac{1}{s}}$$

$$< Q\left(\int_{\sinh 1(1+u)}^{\infty}\int_{\frac{\sinh 1}{\sinh x}}^{\infty}\left(\frac{v}{1+u}\right)^{-\epsilon-2}(1+u)^{-\epsilon-2}\cosh x\cosh y\left(\frac{\sinh x\,du}{\cosh y}\right)\left(\frac{dv}{(1+u)\cosh x}\right)\right)^{\frac{1}{r}}\times\left(\int_{\sinh 1}^{\infty}(\varpi)^{-\epsilon-1}\frac{d\varpi}{\cosh z}\right)^{\frac{1}{s}}$$

$$< Q\left(\int_1^{\infty}\int_0^{\infty}(v)^{-\epsilon-1}(1+u)^{-2}dudv\right)^{\frac{1}{r}}\left(\int_1^{\infty}\varpi^{-\epsilon-1}d\varpi\right)^{\frac{1}{s}} = Q\left(\frac{1}{\epsilon}\right)^{\frac{1}{r}}\left(\frac{1}{\epsilon}\right)^{\frac{1}{s}} = \frac{Q}{\epsilon} \tag{14}$$

Next, we estimat the left-hand side of (13), let $\chi = \frac{\sinh\omega}{(\sinh i+\sinh j)(2\cosh i\cosh j)^{\frac{1}{\frac{\lambda-\epsilon}{s}-\delta-1}}}$, where ω is less than *x,y,* and z, and suppose that $\frac{(2)^{\frac{\lambda-\epsilon}{s}-\delta-1}(\cosh i\cosh j)^{\frac{1}{r}+\frac{\lambda-\epsilon}{s}-\delta-1}}{\cosh\varpi} > 1,$ and $b = \frac{\sinh 2}{(\sinh i+\sinh j)(2\cosh i\cosh j)^{\frac{1}{\frac{\lambda-\epsilon}{s}-\delta-1}}}$. (Noting that from the conditions for λ and δ in the above theorem, we can obtain that $\left(\frac{\lambda-\epsilon}{s}-\delta-1<0\right),$ and using the substitutions as in lemma 2 we get

$$I=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{\tilde{a}_{i,j}\tilde{b}_{k}}{(\sinh i+\sinh j+\sinh k)^{\lambda}}$$

$$=\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}\sum_{k=2}^{\infty}\frac{(\sinh i+\sinh j)^{\frac{\lambda+r\delta-2\frac{r}{s}-\epsilon-2}{r}}\left(\cosh i\cosh j\right)^{\frac{1}{r}}(\sinh k)^{\frac{\lambda-s\delta-\frac{s}{r}-\epsilon-1}{s}}}{(\sinh i+\sinh j+\sinh k)^{\lambda}}$$

$$=\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}\sum_{k=2}^{\infty}\frac{(\sinh i+\sinh j)^{\frac{\lambda+r\delta-2\frac{r}{s}-\epsilon-2}{r}}\left(\cosh i\cosh j\right)^{\frac{1}{r}}(\sinh k)^{\frac{\lambda-s\delta-\frac{s}{r}-\epsilon-1}{s}}}{(\sinh i+\sinh j)^{\lambda}(1+\frac{\sinh k}{\sinh i+\sinh j})^{\lambda}}$$

$$=\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}\sum_{k=2}^{\infty}\frac{(\sinh i+\sinh j)^{\frac{\lambda+r\delta-2\frac{r}{s}-\epsilon-2}{r}-\lambda}\left(\cosh i\cosh j\right)^{\frac{1}{r}}(\sinh k)^{\frac{\lambda-\epsilon}{s}-\delta-1}}{(1+\frac{\sinh k}{\sinh i+\sinh j})^{\lambda}}$$

$$=\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}(\sinh i+\sinh j)^{\frac{\lambda+r\delta-2\frac{r}{s}-\epsilon-2}{r}-\lambda}\left(\cosh i\cosh j\right)^{\frac{1}{r}}\sum_{k=2}^{\infty}\frac{(\sinh k)^{\frac{\lambda-\epsilon}{s}-\delta-1}}{(1+\frac{\sinh k}{\sinh i+\sinh j})^{\lambda}}$$

$$\geq\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}(\sinh i+\sinh j)^{\frac{\lambda+r\delta-2\frac{r}{s}-\epsilon-2}{r}-\lambda}\left(\cosh i\cosh j\right)^{\frac{1}{r}}\left(\int_{2}^{\infty}\frac{(\sinh\omega)^{\frac{\lambda-\epsilon}{s}-\delta-1}\left(\cosh r\right)^{\frac{1}{s}}}{(1+\frac{\sinh\omega}{\sinh i+\sinh j})^{\lambda}}d\varpi\right)$$

$$=\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}(\sinh i+\sinh j)^{\frac{\lambda+r\delta-2\frac{r}{s}-\epsilon-2}{r}-\lambda}\left(\cosh i\cosh j\right)^{\frac{1}{r}}$$

$$\times\int_{b}^{\infty}\frac{\left[\left(2\cosh i\cosh j\right)^{\frac{1}{\frac{\lambda-\epsilon}{s}-\delta-1}}(\sinh i+\sinh j)\chi\right]^{\frac{\lambda-\epsilon}{s}-\delta-1}}{(1+\left(2\cosh i\cosh j\right)^{\frac{1}{\frac{\lambda-\epsilon}{s}-\delta-1}}\chi)^{\lambda}}\left(\frac{\left(2\cosh i\cosh j\right)}{\cosh\varpi(\sinh i+\sinh j)^{-1}}d\chi\right)$$

$$=2\sum_{l=2}^{\infty}\sum_{j=2}^{\infty}\frac{\cosh i\cosh j}{(\sinh i+\sinh j)^{\epsilon+2}}\int_{b}^{\infty}\frac{(\chi)^{\frac{\lambda-\epsilon}{s}-\delta-1}}{(1+\chi)^{\lambda}}\left(\frac{(2)^{\frac{\lambda-\epsilon}{s}-\delta-1}\left(\cosh i\cosh j\right)^{\frac{1}{r}+\frac{\lambda-\epsilon}{s}-\delta-1}}{\cosh\varpi}d\chi\right)$$

$$\geq 2\int_{2}^{\infty}\int_{2}^{\infty}\frac{\cosh x\cosh y}{(\sinh x+\sinh y)^{\epsilon+2}}\left(\int_{b}^{\infty}\frac{(\chi)^{\frac{\lambda-\epsilon}{s}-\delta-1}}{(1+\chi)^{\lambda}}d\chi\right)dxdy$$

$$\geq 2\int_{2}^{\infty}\int_{2}^{\infty}\left(\sinh x\right)^{-\epsilon-2}(1+\frac{\sinh y}{\sinh x})^{-\epsilon-2}\left(\cosh x\cosh y\right)dxdy\int_{b}^{\infty}\frac{(\chi)^{\frac{\lambda-\epsilon}{s}-\delta-1}}{(1+\chi)^{\lambda}}d\chi$$

$$=2\int_{\frac{\sinh 2}{\sinh x}}^{\infty}\int_{\sinh 2(1+u)}^{\infty}\left(\frac{v}{1+u}\right)^{-\epsilon-2}(1+u)^{\epsilon-2}\left(\frac{\cosh x\cosh y\,dv}{(1+u)\cosh x}\right)\left(\frac{\sinh x\,du}{\cosh y}\right)\int_{b}^{\infty}\frac{(\chi)^{\frac{\lambda-s\delta-\frac{s}{r}-\epsilon-1}{s}}}{(1+\chi)^{\lambda}}d\chi$$

$$=2\int_{\frac{\sinh 2}{\sinh x}}^{\infty}\int_{\sinh 2(1+u)}^{\infty}\left(\frac{v}{1+u}\right)^{-\epsilon-1}(1+u)^{\epsilon-2}\left(\frac{dvdu}{(1+u)}\right)\int_{b}^{\infty}\frac{(\chi)^{\frac{\lambda-s\delta-\frac{s}{r}-\epsilon-1}{s}}}{(1+\chi)^{\lambda}}d\chi$$

Since $\frac{\sinh 2}{\sinh x} \leq 1, \forall x \geq 2,$ we obtain:

$$I \geq 2\left(\left(\int_1^\infty (1+u)^{-2} \int_{\sinh 2(1+u)}^\infty v^{-\epsilon-1} dv\right) du\right)\left(\int_0^\infty \frac{(\chi)^{\frac{\lambda-\epsilon}{s}-\delta-1}}{(1+\chi)^\lambda} d\chi - \int_0^b \frac{(\chi)^{\frac{\lambda-\epsilon}{s}-\delta-1}}{(1+\chi)^\lambda} d\chi\right)$$

$$= \frac{2}{\epsilon(\epsilon+1)2^{\epsilon+1}\left(\sinh 2\right)^\epsilon}\left(B\left(\frac{\lambda}{s}+\epsilon+\delta, \frac{\lambda}{r}-\epsilon-\delta\right) - \int_0^b (\chi)^{\frac{\lambda-\epsilon}{s}-\delta-1} d\chi\right)$$

$$= \frac{B\left(\frac{\lambda}{s}+\epsilon+\delta, \frac{\lambda}{r}-\epsilon-\delta\right)}{\epsilon(\epsilon+1)\left(2\sinh 2\right)^\epsilon} - O(1) \tag{15}$$

when $\epsilon \to 0^+$ in (14) and (15), we obtain a contradiction. By this the proof of theorem is completed.

In the following theorem, the reverse form of the main inequality 1 will be introduced, a constant C in this theorem is also, the best.

Theorem 2. Let $0 < r < 1$, $\frac{1}{r}+\frac{1}{s}=1$, $0 < \lambda < \min\{r,s\}$, $-\frac{\lambda}{r} < \varsigma < \frac{\lambda}{s}$, suppose that $a_{i,j}$ is a non-negative double sequence and b_k is a non-negative sequence. If $\sum_{i=1}^\infty \sum_{j=1}^\infty (\sinh i + \sinh j)^{-\lambda - r\delta + \frac{2r}{s}} a_{i,j} < \infty$ and $\sum_{k=1}^\infty (\sinh k)^{s\delta - \lambda + \frac{r}{s}} b_k^s < \infty$, then

$$\sum_{i=1}^\infty \sum_{j=1}^\infty \sum_{k=1}^\infty \frac{a_{i,j} b_k}{(\sinh i + \sinh j + \sinh k)^\lambda} \geq C\left(\sum_{i=1}^\infty \sum_{j=1}^\infty (\sinh i + \sinh j)^{-\lambda - r\delta + \frac{2r}{s}} a_{i,j}\right)^{\frac{1}{r}} \times \left(\sum_{k=1}^\infty (\sinh k)^{s\delta - \lambda + \frac{r}{s}} b_k^s\right)^{\frac{1}{s}} \tag{16}$$

where $C = B\left(\frac{\lambda}{r}+\delta, \frac{\lambda}{s}-\delta\right)$ is the best possible.

Proof: To prove this result, we will use the same procedures that are used in the above Theorem, here, we will use the reverse Hölder inequality, so, we will leave it.

3. Equivalent Forms

Theorem 3: Using the same conditions of Theorem 1, we have an equivalent form of (12) as follows:

$$\sum_{k=1}^\infty (\sinh k)^{(r-1)-r\delta-1}\left(\sum_{j=1}^\infty \sum_{k=1}^\infty \frac{a_{i,j}}{(\sinh i + \sinh j + \sinh k)^\lambda}\right)^r \leq C^r \sum_{i=1}^\infty \sum_{j=1}^\infty \left(\sinh i + \sinh j\right)^{-\lambda - r\delta + \frac{2r}{s}} a_{i,j} \tag{17}$$

and

$$\sum_{j=1}^\infty \sum_{j=1}^\infty \left(\sinh i + \sinh j\right)^{s\delta + \lambda(s-1) - 2}\left(\sum_{k=1}^\infty \frac{b_k}{(\sinh i + \sinh j + \sinh k)^\lambda}\right)^s \leq C^s \sum_{k=1}^\infty \left(\sinh k\right)^{s\delta - \lambda + \frac{s}{r}} b_k^s \tag{18}$$

Both of (17) and (18) are equivalent to (12) and C^r and C^s are the best possible.

Proof: To prove (17), set

$$b_k = (\sinh k)^{(r-1)-r\delta-1}\left(\sum_{i=1}^\infty \sum_{j=1}^\infty \frac{a_{i,j}}{(\sinh i + \sinh j + \sinh k)^\lambda}\right)^{r-1}$$

By using (12), we obtain

$$\sum_{k=1}^{\infty}(\sinh k)^{(r-1)-r\delta-1}\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{a_{i,j}}{(\sinh i+\sinh j+\sinh k)^{\lambda}}\right)^{r}$$

$$=\sum_{k=1}^{\infty}r^{-\lambda-r\varsigma-1}\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{a_{i,j}}{(\sinh i+\sinh j+\sinh k)^{\lambda}}\right)^{r-1}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{a_{i,j}}{(\sinh i+\sinh j+\sinh k)^{\lambda}}$$

$$\leq\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{a_{i,j}b_k}{(\sinh i+\sinh j+\sinh k)^{\lambda}}\leq C\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(\sinh i+\sinh j)^{-\lambda-r\delta+\frac{2r}{s}}\ a_{i,j}\right)^{\frac{1}{r}}\left(\sum_{k=1}^{\infty}(\sinh k)^{s\delta-\lambda+\frac{s}{r}}b_k^{s}\right)^{\frac{1}{s}}$$

$$=C\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(\sinh i+\sinh j)^{-\lambda-r\delta+\frac{2r}{s}}\ a_{i,j}\right)^{\frac{1}{r}}\times\left(\sum_{k=1}^{\infty}(\sinh k)^{(r-1)-r\delta-1}\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{a_{i,j}}{(\sinh i+\sinh j+\sinh k)^{\lambda}}\right)^{r}\right)^{\frac{1}{s}} \quad (19)$$

Multiply both sides of (19) by $\left(\sum_{k=1}^{\infty}(\sinh k)^{(r-1)-r\delta-1}\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{a_{i,j}}{(\sinh i+\sinh j+\sinh k)^{\lambda}}\right)^{r}\right)^{\frac{-1}{s}}$, we get (17).

To prove (18), set $a_{i,j}=(\sinh i+\sinh j)^{s\delta+\lambda(s-1)-2}\left(\sum_{k=1}^{\infty}\frac{b_k}{(\sinh i+\sinh j+\sinh k)^{\lambda}}\right)^{s-1}$, by using (12), we get

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(\sinh i+\sinh j)^{s\delta+\lambda(s-1)-2}\left(\sum_{k=1}^{\infty}\frac{b_k}{(\sinh i+\sinh j+\sinh k)^{\lambda}}\right)^{s}$$

$$=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(\sinh i+\sinh j)^{s\delta+\lambda(s-1)-2}\left(\sum_{k=1}^{\infty}\frac{b_k}{(\sinh i+\sinh j+\sinh k)^{\lambda}}\right)^{s-1}\times\sum_{k=1}^{\infty}\frac{b_k}{(\sinh i+\sinh j+\sinh k)^{\lambda}}$$

$$=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{a_{i,j}b_k}{(\sinh i+\sinh j+\sinh k)^{\lambda}}\leq C\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(\sinh i+\sinh j)^{-\lambda-r\delta+\frac{2r}{s}}\ a_{i,j}\right)^{\frac{1}{r}}\left(\sum_{k=1}^{\infty}(\sinh k)^{s\delta-\lambda+\frac{r}{s}}b_k^{s}\right)^{\frac{1}{s}}$$

$$=C\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(\sinh i+\sinh j)^{s\delta+\lambda(s-1)-2}\left(\sum_{k=1}^{\infty}\frac{b_k}{(\sinh i+\sinh j+\sinh k)^{\lambda}}\right)^{s}\right)^{\frac{1}{r}}\times\left(\sum_{k=1}^{\infty}(\sinh k)^{s\delta-\lambda+\frac{r}{s}}b_k^{s}\right)^{\frac{1}{s}}. \quad (20)$$

By multiply both sides of (20) by $\left((\sinh i+\sinh j)^{s\delta+\lambda(s-1)-2}\left(\sum_{k=1}^{\infty}\frac{b_k}{(\sinh i+\sinh j+\sinh k)^{\lambda}}\right)^{s}\right)^{-\frac{1}{r}}$ we get (18).

Also, we can get (12) from (17) or (12) from (18) as follows (using Hölder inequality):

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{a_{i,j}b_k}{(\sinh i+\sinh j+\sinh k)^{\lambda}}=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left((\sinh i+\sinh j)^{-\frac{s\delta+\lambda(s-1)-2}{s}}\ a_{i,j}\right)$$

$$\times\left((\sinh i+\sinh j)^{\frac{s\delta+\lambda(s-1)-2}{s}}\sum_{k=1}^{\infty}\frac{b_k}{(\sinh i+\sinh j+\sinh k)^{\lambda}}\right)$$

$$\leq\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left((\sinh i+\sinh j)^{-\lambda-r\delta+\frac{2r}{s}}\right)a_{i,j}\right)^{\frac{1}{r}}$$

$$\times\left((\sinh i+\sinh j)^{s\delta+\lambda(s-1)-2}\left(\sum_{k=1}^{\infty}\frac{b_k}{(\sinh i+\sinh j+\sinh k)^{\lambda}}\right)^{s}\right)^{\frac{1}{s}}$$

$$\leq C\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left((\sinh i+\sinh j)^{-\lambda-r\delta+\frac{2r}{s}}\right)a_{i,j}\right)^{\frac{1}{r}}\left(\sum_{k=1}^{\infty}(\sinh k)^{s\delta-\lambda+\frac{r}{s}}b_k^{s}\right)^{\frac{1}{s}}$$

thus, we get the equivalence relation between (18) and (12).

Theorem 4: Applying the same conditions of Theorem 2, we have:

$$\sum_{k=1}^{\infty}(\sinh k)^{(r-1)-r\delta-1}\left(\sum_{k=1}^{\infty}\frac{b_k}{(\sinh i+\sinh j+\sinh k)^{\lambda}}\right)^{s}\geq C^{r}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\left(\sinh i+\sinh j\right)^{-\lambda-r\delta+\frac{2r}{s}}\right)a_{i,j}, \tag{21}$$

and

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\sinh i+\sinh j\right)^{s\delta+\lambda(s-1)-2}\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{a_{i,j}}{(\sinh i+\sinh j+\sinh k)^{\lambda}}\right)^{r}\geq C^{s}\sum_{k=1}^{\infty}(\sinh k)^{s\delta-\lambda+\frac{r}{s}}b_k^{s}. \tag{22}$$

Both (21) and (22) are equivalent to (16), and the constants are the best possible.

Proof: Since the proofs of (21) and (22) are the same as the proofs of (17) and (18), so we leave it.

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