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Some fixed point results with application to fractional differential equation via new type of distance spaces

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Abstract

This research has explored a novel form of distance known as the $\mathcal{T}$-distance within a b-metric space (Π, b, s) which depends on the existed b-metric on Π. Several examples illustrating this concept have been provided, along with an examination of fixed point results using this notion. Furthermore, we have presented an example as well as an application of the $\mathcal{T}$-distance in the context of fractional differential equations.

Keywords: fixed point, non-linear fractional differential equation

Mathematics Subject Classification (2010): 47H10, 47H09, 11J83.

1. Introduction

Consider a non empty set Π and a self mapping $\Omega : \Pi \to \Pi$. A point $u' \in \Pi$ is referred to as a fixed point for Ω if $\Omega u' = u'$. In the context of a metric d on Π, Ω is considered as a contraction if there is $\eta \in [0,1)$ such that the inequality $d(\Omega\mu_1, \Omega\mu_2) \leq \eta d(\mu_1, \mu_2)$, for each $\mu_1, \mu_2 \in \Pi$.

It is a common understanding within the mathematical community that the cornerstone of fixed point theory lies in the Banach contraction principle[1], which guarantees the existence of a unique fixed point for every contraction in a complete metric space. Subsequently, numerous mathematicians have undertaken various generalizations of Banach's theorem in multiple directions. These generalizations involve altering either the distance setting or the condition on the self-mapping as demonstrated in [2, 3, 4, 5, 6, 7, 8, 9, 10] and the references cited therein.

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2. $\mathcal{T}$-distance

The present study delves into a new idea concerning distance spaces within the context of a b-metric space. It is important to first revisit the definition of b-metric spaces, which is outlined as follows.

Definition 2.1. [11] *A function* $b : \Pi \times \Pi \to [0,+\infty)$ *is said to be b-metric if there is* $s \in [1,+\infty)$ *such that b satisfying:*

(1) $b(\mu_1,\mu_2) = 0$ *iff* $\mu_1 = \mu_2$,

(2) $b(\mu_1,\mu_2) = b(\mu_2,\mu_1)$, *for all* $\mu_1,\mu_2 \in \Pi$,

(3) $b(\mu_1,\mu_2) \le s[b(\mu_1,\mu_2) + b(\mu_2,\mu_3)]$, *for all* $\mu_1,\mu_2,\mu_3 \in \Pi$.

The pair (Π, b) *is called a b-metric space.*

Definition 2.2. *Let b be a b-metric on* Π. *A function* $\mathcal{T} : [0,\infty) \times \Pi \times \Pi \to [0,\infty)$ *is said to be* $\mathcal{T}$*-distance over* (Π, d) *if* $\mathcal{T}$ *satisfying:*

$(\mathcal{T}_1)$ $\mathcal{T}(t,\mu,\xi) > \frac{1}{s} b(\mu,\xi)$, *for all* $t > 0$,

$(\mathcal{T}_2)$ *for each sequences* $(\mu_n),(\xi_n)$ *in* Π *and* (t_n) *in* $(0, \infty)$, *we have*

$$\lim_{n\to\infty} b(\mu_n,\xi_n) = \lim_{n\to\infty} s\, \mathcal{T}(t_n,\mu_n,\xi_n) = L > 0 \Rightarrow \lim_{n\to\infty} t_n = 0.$$

In the following, we will present various examples of $\mathcal{T}$-distance functions. To facilitate the discussion in the remainder of this document, we will focus on the subsequent two categories of functions.

$$\Gamma = \{\eta : [0,\infty) \to [0,\infty) \text{ is continuous with } \eta^{-1}(\{0\}) = \{0\}\}.$$

$$\Theta = \{\theta : [0,\infty) \to [\frac{1}{s},\infty) \text{ is continuous with } \theta^{-1}(\{\tfrac{1}{s}\}) = \{0\}\}.$$

Example 2.3. *Let* (Π, b) *be a b-metric space with constant* $s \ge 1$, *and let* $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4 : [0, \infty) \times \Pi \times \Pi \to [0, \infty)$ *be defined as following:*

(1) $\mathcal{T}_1(t,\mu,\xi) = \frac{1}{s} b(\mu,\xi) + \eta(t)$, *where* $\eta \in \Gamma$,

(2) $\mathcal{T}_2(t,\mu,\xi) = \frac{1}{s} b(\mu,\xi) + kt$, *where* $k > 0$,

(3) $\mathcal{T}_3(t,\mu,\xi) = \theta(t)\ b(\mu,\xi)$, *where* $\theta \in \Theta$,

(4) $\mathcal{T}_4(t,\mu,\xi) = \frac{(1+t)}{s} b(\mu,\xi)$.

Then, $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ and $\mathcal{T}_4$ *are* $\mathcal{T}$*-distance over* (Π, b).

In this context we mean by (Π, b, s) a b-metric space with constant $s \ge 1$, $\mathcal{T}$ represents a $\mathcal{T}$-distance over (Π, b, s), $P_{seq}(\Omega,\mu_0)$ refers to the Picard sequence that generated by $\mu_{n+1} = \Omega(\mu_n)$ $n = 0,1,2,\cdots$. Furthermore, $\mathcal{F}_\Omega$ denotes the set of fixed points of Ω within Π.

Definition 2.4. *Suppose there is* $\mathcal{T}$ *over* (Π, b, s). *A self mapping* $\Omega : \Pi \to \Pi$ *is said to be* $\mathcal{T}$*-contraction if for each* $\mu,\xi \in \Pi$, *we have*

$$\mathcal{T}(b(\mu,\xi),\Omega\mu,\Omega\xi) \le \frac{1}{s^2} b(\mu,\xi). \tag{1}$$

Remark 2.5. *If* $\Omega : \Pi \to \Pi$ *is* $\mathcal{T}$*-contraction, then for each distinct points* $\mu,\xi \in \Pi$, we have

$$b(\Omega\mu,\Omega\xi) < \frac{1}{s} b(\mu,\xi).$$

Lemma 2.6. *Suppose* $\Omega : \Pi \to \Pi$ *is* $\mathcal{T}$*-contraction. If* $\mu, \xi \in \mathcal{F}_\Omega$, *then* $\mu = \xi$.

Proof. Assume to the contrary; that is $\mu \ne \xi$. Then, by Remark 2.5, we have

$$b(\mu,\xi)=b(\Omega\mu,\Omega\xi)<\frac{1}{s}b(\mu,\xi),$$

a contradiction. Hence the result.

Lemma 2.7. *Suppose* Ω *is* $\mathcal{T}$*-contraction, and* $\mu_0\in\Pi$. *Then for the* $P_{seq}(\Omega,\mu_0)$, *if* $\mu_n\neq\mu_{n+1}$ *for each* $n\in\mathbb{N}$, *then*

$$\lim_{n\to\infty}b(\mu_n,\mu_{n+1})=0.$$

Proof. Since $\mu_n\neq\mu_{n+1}$ then from Remark 2.5 we have $b(\mu_n,\mu_{n+1})<b(\mu_{n-1},\mu_n)$. Hence $(b(\mu_n,\mu_{n+1}):\ n\in\mathbb{N})$ is a non-increasing sequence in $[0,\infty)$. So, there is $r\geq 0$ such that $\lim_{n\to\infty}b(\mu_n,\mu_{n+1})=r$. Suppose that $r>0$. Therefore, by $\mathcal{T}_1$ of Definition 2.2, we have

$$\frac{1}{s}b(\mu_n,\mu_{n+1})<\mathcal{T}(b(\mu_n,\mu_{n-1}),\mu_n,\mu_{n+1})\leq\frac{1}{s^2}b(\mu_{n-1},\mu_n).$$

So,

$$r\leq\lim_{n\to\infty}\mathcal{T}(b(\mu_n,\mu_{n-1}),\mu_n,\mu_{n+1})\leq\frac{1}{s}r,$$

which is a contradiction. Hence the result.

Theorem 2.8. *Suppose that* (Π,b,s) *is complete and there is* $\mathcal{T}$*-distance* $\mathcal{T}$ *on* (Π,b,s). *Assume that* $\Omega:\Pi\to\Pi$ *is* $\mathcal{T}$*-contraction. Then* $\mathcal{F}_\Omega$ *consists of only one element.*

Proof. Let $\mu_0\in\Pi$ be arbitrary and consider the $P_{seq}(\Omega,\mu_0)$. If there is $l\geq 0$ such that $\mu_l=\mu_{l+1}$, then $\mu_l\in\mathcal{F}_\Omega$. So we assume that for each $n\in\mathbb{N}$, $\mu_n\neq\mu_{n+1}$. Now, we claim that (μ_n) is a Cauchy sequence in (Π,b,s). Suppose not; that is (μ_n) is not Cauchy. Therefore, there is $\epsilon>0$ and two sub-sequences (μ_{n_k}) and (μ_{m_k}) of (μ_n) such that (m_k) is chosen as the smallest index for which

$$b(\mu_{n_k},\mu_{m_k})\geq\epsilon,\ m_k>n_k>k. \tag{2}$$

This implies that

$$b(\mu_{n_k},\mu_{m_k-1})<\epsilon. \tag{3}$$

Using the triangle inequality, Remark 2.5 and Equations (2),(3) we get

$$\epsilon\leq b(\mu_{n_k},\mu_{m_k})<\tfrac{1}{s}b(\mu_{n_k-1},\mu_{m_k-1})\leq[b(\mu_{n_k-1},\mu_{m_k})+b(\mu_{m_k},\mu_{m_k-1})].$$

So, depending on Lemma (2.7) we get

$$\epsilon\leq\liminf_{k\to+\infty}b(\mu_{n_K-1},\mu_{m_k}). \tag{4}$$

Also,

$$b(\mu_{n_k-1},\mu_{m_k})<\tfrac{1}{s}b(\mu_{n_k-2},\mu_{m_k-1})\leq[b(\mu_{n_k-2},\mu_{n_k})+b(\mu_{n_k},\mu_{m_k-1})]<s[b(\mu_{n_k-2},\mu_{n_k-1})+b(\mu_{n_k-1},\mu_{n_k})]+\epsilon].$$

Depending on Lemma (2.7) we get

$$\limsup_{k\to+\infty}b(\mu_{n_k-1},\mu_{m_k})\leq\epsilon. \tag{5}$$

By (4) and (5) we get

$$\lim_{k\to+\infty}b(\mu_{n_k-1},\mu_{m_k})=\epsilon. \tag{6}$$

Now, by the same previous argument we have

$$b(\mu_{n_k},\mu_{m_k+1})<\frac{1}{s}b(\mu_{n_k-1},\mu_{m_k}).$$

So, we get

$$\limsup_{k\to+\infty} b(\mu_{n_k},\mu_{m_k+1}) \leq \frac{\epsilon}{s}. \tag{7}$$

On the other hand

$$\epsilon \leq b(\mu_{n_k},\mu_{m_k}) \leq s[b(\mu_{n_k},\mu_{m_k+1}) + b(\mu_{m_k+1},\mu_{m_k})].$$

So, we get

$$\frac{\epsilon}{s} \leq \liminf_{k\to+\infty} b(\mu_{n_k},\mu_{m_k+1}). \tag{8}$$

By (7) and (8) we get

$$\lim_{k\to+\infty} b(\mu_{n_k},\mu_{m_k+1}) = \frac{\epsilon}{s}. \tag{9}$$

So, using condition 2.1, we get

$$\frac{1}{s} b(\mu_{n_k},\mu_{m_k+1}) < \mathcal{T}(b(\mu_{n_k-1},\mu_{m_k}),\mu_{n_k},\mu_{m_k+1}) \leq \frac{1}{s^2} b(\mu_{n_k-1},\mu_{m_k}).$$

Therefore, $\lim_{k\to\infty} \mathcal{T}(b(\mu_{n_k-1},\mu_{m_k-1}),\mu_{n_k},\mu_{m_k}) = \frac{\epsilon}{s^2}$, and so, $\lim_{k\to\infty} b(\mu_{n_k-1},\mu_{m_k-1}) = 0$, a contradiction. Hence, (μ_n) is Cauchy, so there is $\mu \in \Pi$ such that (μ_n) converges to some $\mu \in \Pi$. Now, by Remark 2.5, we have

$$b(\Omega\mu,\mu_{n+1}) = b(\Omega\mu,\Omega\mu_n) < \frac{1}{s} b(\mu_n,\mu).$$

Therefore, As the value of n approaches to ∞, we obtain $b(\Omega\mu,\mu) \leq 0$, and so $\mu \in \mathcal{F}_\Omega$. The uniqueness derived from the insights presented in Lemma 2.6.

Example 2.9. *Let* $\Pi = [0, 1]$ *and let* $\Omega : \Pi \to \Pi$ *be defined by* $\Omega\mu = \frac{1-\mu^2}{10+\mu^2}$. *Then* Ω *has a unique fixed point.*

Proof. Let $\Pi = [0, 1]$ provided with the b-metric $b(\mu,\upsilon) = |\mu-\upsilon|^2$. Then

$\Omega : \Pi \to \Pi$ which defined by $\Omega\mu = \frac{1-\mu^2}{10+\mu^2}$ is a self map. Let $\mathcal{T}$: $[0, \infty) \times \Pi \times \Pi \to [0,\infty)$ be defined by $\mathcal{T}(t,\mu,\upsilon) = \frac{t+1}{s} b(\mu,\upsilon)$.

Now, for each $\mu,\upsilon \in [0,1]$ we have

$$\begin{aligned} b(\Omega\mu,\Omega\upsilon) &= \left|\Omega\mu - \Omega\upsilon\right|^2 \\ &= \left|\frac{1-\mu^2}{10+\mu^2} - \frac{1-\upsilon^2}{10+\upsilon^2}\right|^2 \\ &= \left(\frac{11}{(10+\mu^2)(10+\upsilon^2)}\right)^2 |\mu^2-\upsilon^2|^2 \leq 4\left(\frac{11}{100}\right)^2 |\mu-\upsilon|^2 \\ &= \left(\frac{11}{50}\right)^2 b(\mu,\upsilon). \end{aligned}$$

So,

$$\begin{aligned} \mathcal{T}(b(\mu,\upsilon),\Omega\mu,\Omega\upsilon) &= \frac{1+b(\mu,\upsilon)}{2} b(\Omega\mu,\Omega\upsilon) \\ &= \frac{1+|\mu^2-\upsilon^2|^2}{2} |\Omega\mu-\Omega\upsilon|^2 \leq \frac{5}{2}\left(\frac{11}{50}\right)^2 b(\mu,\upsilon) \leq \frac{1}{4} b(\mu,\upsilon). \end{aligned}$$

As a result, Ω satisfies all the conditions specified in Theorem 2.8, and Theorem 2.8 guarantees that $\mathcal{F}_\Omega$ consists of a single element.

Based on Theorem 2.8, we are able to deduce the subsequent corollaries.

Corollary 2.10. *Suppose that* (Π, b, s) *is complete and* $\theta \in \Theta$. *Suppose* $\Omega : \Pi \to \Pi$ *is a self map satisfies*

$$s^2 b(\Omega\mu, \Omega\xi) \leq \frac{b(\mu,\xi)}{\theta(b(\mu,\xi))}. \tag{10}$$

Then $\mathcal{F}_\Omega$ *consists of one element.*

Corollary 2.11. *Suppose that* (Π, b, s) *is complete and* $\Omega : \Pi \to \Pi$ *is a self map satisfies*

$$s b(\Omega\mu, \Omega\xi) \leq \frac{b(\mu,\xi)}{1 + b(\mu,\xi)}. \tag{11}$$

Then $\mathcal{F}_\Omega$ *consists of one element.*

Corollary 2.12. *Suppose that* (Π, b, s) *is complete and* $\eta \in \Gamma$. *Suppose* $\Omega : \Pi \to \Pi$ *is a self map satisfies*

$$b(\Omega\mu, \Omega\xi) \leq \frac{1}{s} b(\mu,\xi) - s\ \eta(b(\mu,\xi)). \tag{12}$$

Then $\mathcal{F}_\Omega$ *consists of one element.*

The subsequent theorem can be proven using the identical technique employed in Theorem 2.2.

Theorem 2.13. *Suppose* (Π, b, s) *is complete and* $\Omega : \Pi \to \Pi$ *is a self map satisfies the following condition for all* $\mu, \xi \in \Pi$

$$\mathcal{T}(b(\Omega\mu, \Omega\xi), \Omega\mu, \Omega\xi) \leq \frac{1}{s^2} b(\mu,\xi). \tag{13}$$

Then $\mathcal{F}_f$ consists of one element.

3. Application

The exploration of resolving fractional differential equations and integral equations through the application of fixed point theory has emerged as a central focus in contemporary academic investigations. We suggest that those with a keen interest delve into reputable references [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22] for deeper understanding and valuable perspectives on this subject matter.

Let $E = C[0,1]$ be the set of continuous functions and consider the b-metric b on E as follows:

$$b(g,h) = \|g - h\|_\infty^2 = \max_{\mu \in [0,1]} |g(\mu) - h(\mu)|^2.$$

Then $(E, b, 2)$ is a complete b-metric space with $s = 2$.

Consider the following boundary fractional differential equation

$$\begin{cases} \mathcal{D}^\alpha g(\mu) + \Lambda(\mu, g(\mu)) = 0, \ 0 \leq \mu \leq 1, 1 < \alpha < 2 \\ g(0) = g(1) = 0, \end{cases} \tag{14}$$

where $\mathcal{D}^\alpha$ is the Caputo fractional derivative of order α and $\Lambda : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

The Green's function associated to (14) is defined as:

$$G(\mu,\xi) = \begin{cases} \mu(1-\xi)^{\alpha-1} - (\mu - \xi)^{\alpha-1}, & 0 \leq \xi \leq \mu \leq 1, \\ \dfrac{\mu(1-\xi)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq \mu \leq \xi \leq 1. \end{cases}$$

Theorem 3.1. *Suppose that*

$$|\Lambda(\mu, z_1) - \Lambda(\mu, z_2)| \leq \sqrt{\frac{1-4k}{2}}\, | z_1 - z_2 |,\ k < \frac{1}{4},$$

for all $\mu \in [0, 1]$ *and* $z_1, z_2 \in \mathbb{R}$. *Then,*

$$g(\mu) = \int_0^1 G(\mu,\xi)\Lambda(\xi, g(\xi))d\xi, \tag{15}$$

for all $\mu \in [0, 1]$ *and* $g \in E$ *has a unique solution.*

Proof. The solution of 15 is equivalent to the fixed point the operator $S : E \to E$ which defined as:

$$S(g(\mu)) = \int_0^1 G(\mu,\xi)\Lambda(\xi, g(\xi))d\xi,$$

For any $g, h \in E$ we have

$$\begin{aligned}
\left|S(g(\mu)) - S(h(\mu))\right|^2 &= \left|\int_0^1 G(\mu,\xi)(\Lambda(\xi, g(\xi)) - \Lambda(\xi, h(\xi)))d\xi\right|^2 \\
&\leq \left(\int_0^1 G(\mu,\xi)\,|\,\Lambda(\xi, g(\xi)) - \Lambda(\xi, h(\xi))\,|\,d\xi\right)^2 \\
&\leq \left(\int_0^1 G(\mu,\xi)\sqrt{\frac{1-4k}{2}}\,|\,g(\xi) - h(\xi)\,|\,d\xi\right)^2 \\
&\leq \frac{1-4k}{2}\,\|\,g - h\,\|_\infty^2 \left(\sup_{\mu\in[0,1]}\int_0^1 G(\mu,\xi)d\xi\right)^2 \\
&\leq \frac{1-4k}{2}\,\|\,g - h\,\|_\infty^2 .
\end{aligned}$$

Hence,

$$\left\|S(g(\mu)) - S(h(\mu))\right\|_\infty^2 \leq \frac{1-4k}{2}\left\|g - h\right\|_\infty^2 .$$

Define $\mathcal{T} : [0,\infty)\times E\times E \to [0,\infty)$ by $\mathcal{T}(t, g, h) = \frac{1}{2}b(g,h) + kt, 0 < k < \frac{1}{4}$. Therefore,

Theorem 2.8 guarantees the existence of one and only one fixed point for S.

Conclusion

This research has introduced a novel form of distance spaces that are equipped with a b-metric. This enables us to establish a fresh set of contractions and present fixed point results. Furthermore, we reinforce our findings by providing an example and demonstrating an application.

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