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Differential q-calculus of several variables

Mohammad W. Alomari[1], Waseem G. Alshanti[2], Iqbal M. Batiha[2,3], Liliana Guran[4], Iqbal H. Jebril[2]

[1]Department of Mathematics, Faculty of Science and Information Technology, Jadara University, P.O. Box 733, Irbid, 21110, Jordan; [2]Department of Mathematics, Al Zaytoonah University of Jordan, Amman 11733, Jordan; [3]Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman, UAE; [4]Department of Hospitality Services, Faculty of Business, Babeş-Bolyai University, Horea street, no. 7, 400174, Cluj-Napoca, Romania.

This comprehensive investigation explores the application and significance of q-differential calculus in the realm of vector functions of several variables, addressing critical aspects such as q-Rolle's theorem, the q-Mean-value theorem, and q-chain rule for vector functions. Additionally, we investigate the q-gradient, q-Jacobian, and q-Hessian operators, elucidating their roles in quantifying rates of change, determining directional derivatives, and characterizing critical points of multivariate functions. Furthermore, this research provides a rigorous treatment of Multivariate and Bivariate Taylor theorems in the context of q-differential calculus, presenting analytical expansions of functions around specific points and showcasing their utility in approximating functions in higher dimensions. The q-Maximum and Minimum are demonstrated and discussed as well.

Key words and phrases: q-calculus, q-Gradient, q-Chain Rule, q-Jacobians, q-Hessian, q-Extreme Values

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1. Introduction

The realm of mathematical analysis has continuously evolved to accommodate the complexities inherent in multidimensional systems [1, 2]. Within this domain, the integration of q-calculus with the study of vector functions of several variables emerges as an innovative and promising field, offering a fresh perspective on the dynamics of multivariate functions. q-calculus, an extension of traditional calculus enriched by the parameter q, introduces intriguing variations of derivatives and integrals that have shown remarkable applications in diverse scientific disciplines. In this pursuit,

Email addresses: mwomath@gmail.com (Mohammad W. Alomari); i.jebril@zuj.edu.jo (Waseem G. Alshanti); i.batiha@zuj.edu.jo (Iqbal M. Batiha); i.jebril@zuj.edu.jo (Iqbal H. Jebril); liliana.guran@ubbcluj.ro (Liliana Guran)*

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the amalgamation of q-calculus with vector functions unveils a rich tapestry of mathematical structures, unraveling nuanced behaviors, and fostering deeper insights into the behavior of multidimensional systems. By employing q-analogues of derivatives and integrals, this research aims to probe the fundamental characteristics of vector functions across multiple variables in a q-analytic setting. The investigation seeks to shed light on the unique features and underlying properties inherent in the differentiation and integration processes, thereby providing a robust framework for understanding the intricate dynamics of vector fields in higher dimensions.

At its core, the exploration of q-calculus within the realm of vector functions of several variables is propelled by a confluence of motivations. Traditional calculus techniques, while foundational, encounter limitations when dealing with complex, high-dimensional systems prevalent in fields such as physics, engineering, finance, and computational mathematics. The introduction of q-derivatives and q-integrals offers a fresh lens through which to comprehend and analyze the behavior of multivariate functions, empowering researchers to tackle intricate phenomena that evade conventional approaches. Moreover, the utility of q-calculus extends beyond theoretical implications; it serves as a potent tool for addressing practical challenges encountered in modeling and simulating real-world systems characterized by multiple interacting variables. Embracing the capabilities of q-calculus in analyzing vector functions of several variables promises advancements in optimization, signal processing, data analysis, and the modeling of complex dynamical systems, thus paving the way for innovative methodologies with far-reaching implications across scientific disciplines.

Through analytical investigations, numerical simulations, and empirical validations, this research seeks to elucidate the distinct advantages and potential challenges associated with employing q-calculus techniques in understanding and manipulating vector fields in multiple dimensions. The ultimate goal is to establish a comprehensive understanding of q-analytic techniques applied to vector functions, thereby contributing to the development of a powerful mathematical framework for tackling complex multidimensional phenomena encountered across scientific domains.

In this work, we take $q \in (0,1)$ into consideration. The so-called q-number can be outlined as

$$\left[\alpha\right]_q = \frac{q^\alpha - 1}{q-1}, \qquad \text{for any } \alpha \in \mathbb{C}.$$

Specifically, in the case where $\alpha = n \in \mathbb{N}$, the positive q-integer is defined as

$$\left[n\right]_q = \frac{q^n - 1}{q-1} = 1 + q + q^2 + \cdots q^{n-1}.$$

As an exceptional instance, we can obtain $\left[1\right]_q = 1$ and $\left[0\right]_q = 0 = \left[\infty\right]_q$.

The q-binomial coefficient and the q-factorial of the number $[n]_q$ are defined as follows:

$$\left[0\right]_q ! = 1, \qquad \left[n\right]_q ! = \left[n\right]_q \left[n-1\right]_q \cdots \left[2\right]_q \cdot \left[1\right]_q \qquad \begin{bmatrix} n \\ j \end{bmatrix}_q = \frac{\left[n\right]_q !}{\left[j\right]_q ! \left[n-j\right]_q !}$$

in keeping with the custom that

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}_q = 1, \qquad \begin{bmatrix} 0 \\ j \end{bmatrix}_q = 0, \forall j \geq 1.$$

The so-called q-Pochammer symbol might be outlined as

$$\left(x-a\right)_q^n = \prod_{j=0}^{n-1} \left(x - q^j a\right), \text{ with } \left(x-a\right)_q^{(0)} = 1, \text{ and } \left(x-a\right)_q^{-n} = \frac{1}{\left(x - q^{-n} a\right)_q^n}.$$

A significant function of this formula is found in combinatorics. For example, this formula makes sense when $n = \infty$ and $x = 1$ and $x = a$.

$$\left(1+x\right)_q^n = \prod_{j=0}^{\infty}\left(1+q^j x\right).$$

If $q \in (0,\infty)$, then the infinite product above converges. Herein, we take up the following symbol:

$$\left(1+x\right)_q^{\alpha} = \frac{\left(1+x\right)_q^{\infty}}{\left(1+q^{\alpha}x\right)_q^{\infty}}, \forall \alpha.$$

When $\alpha = n \in \mathbb{N}$, this definition clearly corresponds with the definition of $\left(1+x\right)_q^n$.

Lemma 1. [3] *If* α *and* β *are any two numbers, we obtain*

$$\left(1+x\right)_q^{\alpha} = \frac{\left(1+x\right)_q^{\alpha+\beta}}{\left(1+q^{\alpha}x\right)_q^{\beta}}$$

and

$$D_q\left(1+x\right)_q^{\alpha} = \left[\alpha\right]_q\left(1+qx\right)_q^{\alpha-1}.$$

For every real-valued function *f*, the q-derivative is defined as

$$D_q f(x) = \frac{f(qx)-f(x)}{(q-1)x}, \qquad x \neq 0.$$

Obviously, $D_q f(x)$ goes to $f'(x)$, as $q \to 1^-$ for which the function *f* is differentiable.

Two principles Literature is familiar with q-binomial formulae. The following form represents the q-Gauss binomial:

$$\left(1+x\right)_q^n = \sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_q q^{j(j-1)/2}x^j.$$

Conversely, the form of the q-Heine's binomial formula is as follows:

$$\frac{1}{\left(1-x\right)_q^n} = \sum_{j=0}^{\infty}\begin{bmatrix} n \\ j \end{bmatrix}_q x^j.$$

Nevertheless, we can have

$$\lim_{n\to\infty}\begin{bmatrix} n \\ j \end{bmatrix}_q = \frac{1}{\left(1-q\right)\left(1-q^2\right)\ldots\left(1-q^j\right)}.$$

We now obtain two formal power series in x by using the aforementioned equality for the binomial formulas of q-Gauss and q-Heine. In other words, we obtain

$$\left(1+x\right)_q^{\infty} = \sum_{j=0}^{\infty} q^{j(j-1)/2}\frac{x^j}{\left(1-q\right)\left(1-q^2\right)\ldots\left(1-q^j\right)}, \tag{1}$$

and

$$\frac{1}{\left(1-x\right)_q^{\infty}} = \sum_{j=0}^{\infty}\frac{x^j}{\left(1-q\right)\left(1-q^2\right)\ldots\left(1-q^j\right)}. \tag{2}$$

Since these two series were used to define the q-analogue of the exponential function, they are extremely helpful in the theory of q-calculus. Based on (2), we can have

$$\frac{1}{(1-x)_q^{\infty}}=\sum_{j=0}^{n}\frac{\left(\frac{x}{1-q}\right)^j}{\left(\frac{1-q}{1-q}\right)\left(\frac{1-q^2}{1-q}\right)\dots\left(\frac{1-q^j}{1-q}\right)}=\sum_{j=0}^{n}\frac{\left(\frac{x}{1-q}\right)^j}{[1]_1[2]_q\cdots[j]_q}=\sum_{j=0}^{n}\frac{\left(\frac{x}{1-q}\right)^j}{[j]_q!}=\mathrm{e}_q^{\frac{x}{1-q}},$$

which can be rewritten as

$$\mathrm{e}_q^x=\frac{1}{(1-(1-q)x)_q^{\infty}}.$$

Likewise, using (1), the definition of the companion q-exponential function is then given as

$$(1+x)_q^{\infty}=\sum_{j=0}^{n}\frac{q^{j(j-1)/2}\left(\frac{x}{1-q}\right)^j}{\left(\frac{1-q}{1-q}\right)\left(\frac{1-q^2}{1-q}\right)\dots\left(\frac{1-q^j}{1-q}\right)}=\sum_{j=0}^{n}\frac{q^{j(j-1)/2}\left(\frac{x}{1-q}\right)^j}{[1]_1[2]_q\cdots[j]_q}=\sum_{j=0}^{n}\frac{\left(\frac{x}{1-q}\right)^j}{[j]_q!}=\mathrm{E}_q^{\frac{x}{1-q}},$$

which can be rewritten as

$$\mathrm{E}_q^x=(1+(1-q)x)_q^{\infty}.$$

The two q-exponential functions mentioned above have the following derivatives:

$$D_q\mathrm{E}_q^x=\mathrm{E}_q^{qx},\qquad D_q\mathrm{e}_q^x=\mathrm{e}_q^x$$

We see that the exponentials' additive feature is not always true, i.e.,

$$\mathrm{e}_q^x\mathrm{e}_q^y=\mathrm{e}_q^{x+y}.$$

On the other hand, the additive property is true if x and y meet the commutation relation $yx=qxy$. The relations between the two functions, E_q^x and e_q^x, are given as

$$\mathrm{E}_q^{-x}\mathrm{e}_q^x=1,\qquad \mathrm{e}_{1/q}^x=\mathrm{E}_q^x.$$

This study seeks to explore the distinctive features of q-derivatives for vector-valued functions across multiple variables, analyzing their implications for understanding the dynamics and transformations of multidimensional systems. Specifically, the research aims to establish a rigorous theoretical framework for q-calculus as applied to vector functions of several variables, defining and analyzing the properties of q-derivatives in this context. Investigating properties of q-Derivatives by exploring the unique characteristics and properties exhibited by q-derivatives of vector functions, including q-analogues of differentiation rules and their implications for higher-dimensional systems. Investigating the practical applications and implications of q-calculus techniques in modeling and analyzing complex multidimensional phenomena, highlighting the advantages and challenges of utilizing q-derivatives in various scientific and applied domains.

2. Functions of several variables

Let f be an arbitrary function of two variables. By partial q-derivative we mean

$$D_z^q f(z,w)=\frac{\partial_q}{\partial_q z}f(z,w)=\frac{f(qz,w)-f(z,w)}{(q-1)z}\tag{3}$$

and

$$D_w^q f(z,w)=\frac{\partial_q}{\partial_q w}f(z,w)=\frac{f(z,qw)-f(z,w)}{(q-1)w}\tag{4}$$

For example, the function $f(z,w)=z^n w^m$ has q-partial derivatives such as

$$\frac{\partial_q}{\partial_q z}\left(z^n w^m\right)=\frac{q_1^n z^n w^m - z^n w^m}{(q-1)z}=[n]_q z^{n-1}w^m,\ z\neq 0,$$

and

$$\frac{\partial_q}{\partial_q w}\left(z^n w^m\right)=\frac{z^n q_2^m w^m - z^n w^m}{(q-1)w}=[m]_q z^n w^{m-1},\ w\neq 0.$$

In general, for any two functions f and g, we have

$$D_z^q(fg)(z,w)=f(qz,w)\frac{\partial_q}{\partial_q z}g(z,w)+g(z,w)\frac{\partial_q}{\partial_q z}f(z,w),$$

and

$$D_w^q(fg)(z,w)=f(z,qw)\frac{\partial_q}{\partial_q w}g(z,w)+g(z,w)\frac{\partial_q}{\partial_q w}f(z,w).$$

Let f be an arbitrary function, the k^{th}-order partial q-derivative of f is defined to be

$$\frac{\partial_q^k}{\partial_q z^k}f(z,w)=\frac{\partial_q^{k-1}}{\partial_q z^{k-1}}\left(\frac{\partial_q}{\partial_q z}f(z,w)\right)=\ldots=\frac{\partial_q}{\partial_q z}\left(\frac{\partial_q}{\partial_q z}\cdots\left(\frac{\partial_q}{\partial_q z}\left(f(z,w)\right)\right)\right), \tag{5}$$

for any $k=1,2,3,\ldots$. Consequently, the second order partial q-derivatives or simply second q-patrials are defined to be

$$\frac{\partial_q^2 f}{\partial_q z^2}=\frac{\partial_q}{\partial_q z}\left(\frac{\partial_q f}{\partial_q z}\right),\ \frac{\partial_q^2 f}{\partial_q w^2}=\frac{\partial_q}{\partial_q w}\left(\frac{\partial_q f}{\partial_q w}\right),\ \frac{\partial_q}{\partial_q w}\left(\frac{\partial_q f}{\partial_q z}\right)=\frac{\partial_q^2 f}{\partial_q w\partial_q z},$$

and

$$\frac{\partial_q}{\partial_q z}\left(\frac{\partial_q f}{\partial_q w}\right)=\frac{\partial_q^2 f}{\partial_q z\partial_q w}.$$

In special case, we have

$$\frac{\partial_q^2}{\partial_q z^2}\left(f(z,w)\right)=\frac{\partial_q}{\partial_q z}\left(\frac{\partial_q}{\partial_q z}f(z,w)\right)=\frac{qf(z,w)-(1+q)f(qz,w)+f\left(q^2 z,w\right)}{(1-q)^2 qz}.$$

and

$$\frac{\partial_q^2 f}{\partial_q w\partial_q z}=\frac{\partial_q}{\partial_q w}\left(\frac{\partial_q f}{\partial_q z}\right)=\frac{f(qz,qw)-f(z,qw)-f(qz,w)+f(z,w)}{(1-q)^2 zw}.$$

2.1. The q-Gradient and directional q-derivatives

Naturally, it's not necessarily to have the same partial q-derivatives on two different coordinates. However, this can be treated by taking the maximum of q_j over all coordinates. Despite of that, we define the local q-derivative of a function of several variables as follows:

Definition 1. *We say that $f:\mathbb{R}^n\to\mathbb{R}$ is (locally) q-differentiable provided that there exists a unique vector $G_q(z)$ such that*

$$f(qz)-f(z)=G_q(z)\cdot(q-1)z \tag{6}$$

or we write

$$f(q_1z_1,\ldots,q_nz_n)-f(z_1,\ldots,z_n)=\sum_{j=1}^{n}(q_j-1)z_jG_{q_j}(z)$$

where $q \in (0,1)$. If such vector $G_q(z)$ exits then it's unique. We call this unique vector the q-Gradient of f and denote it by $\nabla_q f(z)$.

The definition of q-Gradient is just a vector whose components are the *local* partial q_j-derivatives on each coordinates. By setting $q = \max_j \{q_j\}$, we then unify the partial derivatives in each coordinate, and thus *the total q-derivative* (which measures the q-average of change of a function f in each coordinate in direction of unit axes) or the q-Gradient of f is defined to be:

$$f(qz) - f(z) = G_q(z) \cdot (q-1)z, \qquad \text{for all fixed } q \in (0,1). \tag{7}$$

Example 1. *For $f(z) = z^2 + w^2$, we have*

$$\begin{aligned} f(qz) - f(z) &= q_1^2 z^2 + q_2^2 w^2 - z^2 - w^2 \\ &= (q_1^2 - 1)z^2 + (q_2^2 - 1)w^2 \\ &= \left[(q_1+1)ze_1 + (q_2+1)we_2\right] \cdot \left[(q_1-1)ze_1 + (q_2-1)we_2\right] \\ &= G_q(z) \cdot (q-1)z, \end{aligned}$$

so that q-Gradient is $\nabla_q f(z) = (q_1+1)ze_1 + (q_2+1)we_2$. Consequently, the q-Gradient is $\nabla_q f(z) = (q+1)ze_1 + (q+1)we_2$, where e_1 and e_2 are the unit vectors in z and w directions, respectively.

The following result holds immediately.

Proposition 1. *Let f and g be q-differentiable mappings, then the following properties are hold:*

(1) $\nabla_q\left[f(z) + g(z)\right] = \nabla_q f(z) + \nabla_q g(z)$,

(2) $\nabla_q\left[\alpha f(z)\right] = \alpha \nabla_q f(z)$,

(3) $\nabla_q\left[f(z) g(z)\right] = f(qz)[\nabla_q g(z)] + g(z)[\nabla_q f(z)]$.

Definition 2. *Let f be locally q-differentiable function. The directional q-derivative of f at the point z_0 in the direction of unit vector u is defined to be*

$$D_u^q f(z_0) = \nabla_q f(z_0) \cdot u.$$

where $q = (q_1, \cdots, q_n)_{j=1}^n$.

Consequently, for q-differentiable function f, the directional q-derivative of f in the direction of unit vector u is defined to be

$$D_u^q f(z) = \nabla_q f(z) \cdot u.$$

In particular, the q-partial derivatives of a function f in direction of a unit vector $u = (u_1, \ldots, u_j, \ldots, u_n)$ is defined as:

$$D_{z_j,u}^q f(z_1, \cdots, z_j, \cdots, z_n) = \frac{\partial_q}{\partial_q z_j} f(z) u_j.$$

For example, let us evaluate the directional q derivative of $f(z) = \sum_{j=1}^n z_j^2$ in the direction of unit vector $u = \sum_{j=1}^n a_j e_j$, i.e., $\sum_{j=1}^n a_j^2 = 1$. Therefore, $D_u^q f(z) = \nabla_q f(z) \cdot u = \left(\sum_{j=1}^n (q+1) z_j e_j\right) \cdot \left(\sum_{j=1}^n a_j e_j\right) = \sum_{j=1}^n (q+1)\, z_j a_j$. Clearly, as q tends to 1, the directional q-derivative $D_u^q f(z)$ tends to the classical directional derivative $D_u f(z) = \sum_{j=1}^n 2 z_j a_j$.

Theorem 1. *If f is q-differentiable then the first q-partial derivatives are exist and*

$$\nabla_q f(z) = \sum_{j=1}^{n} \frac{\partial_q}{\partial_q z_j} f(z) e_j$$

Proof. Since f is q-differentiable then

$$\nabla_q f(z) = \sum_{j=1}^{n} \left[\nabla_q f(z) \cdot e_j\right] e_j = \sum_{j=1}^{n} D_{e_j}^{q} f(z) e_j = \sum_{j=1}^{n} \frac{\partial_q}{\partial_q z_j} f(z) e_j$$

and this completes the proof.

Remark 1. *A more general presentation of Theorem 1 is given as follows: For* $q \in (0,1)$*, if f has continuous first* q_j*-partial derivatives* $(1 \le j \le n)$*, then f is q-differentiable and*

$$\nabla_q f(z) = \sum_{j=1}^{n} \frac{\partial_{q_j}}{\partial_{q_j} z_j} f(z) e_j.$$

Setting $q = q_j$ *for all j we refer to the Theorem 1.*

Remark 2. *Since for each unit vector u we have*

$$D_u^q f(z) = \nabla_q f(z) \cdot u = \left(\mathrm{comp}_u \nabla_q f(z)\right)\|u\| = \mathrm{comp}_u \nabla_q f(z)$$

This means that the q-derivative of f at the specified direction is equal to the scaler projection of the q-Gradient onto that direction. Moreover, if $\nabla_q f(z) \neq 0$*, then*

$$D_u^q f(z) = \nabla_q f(z) \cdot u = \left\|\nabla_q f(z)\right\| \|u\| \cos\theta = \left\|\nabla_q f(z)\right\| \cos\theta$$

where θ *is the angle between* $\nabla_q f(z)$ *and u. Since* $|\cos\theta| \le 1$*, then*

$$-\left\|\nabla_q f(z)\right\| \le D_u^q f(z) \le \left\|\nabla_q f(z)\right\|$$

for all directions u.

If u points in the direction of $\nabla_q f(z)$*, then* $D_u^q f(z) = \left\|\nabla_q f(z)\right\|$*. Also, for –u, we have* $D_{-u}^q f(z) = -\left\|\nabla_q f(z)\right\|$*.*

2.2. q-Mean value theorems

Firstly, we need to recall the q-Mean Value Theorem (q-MVT) given in [4], it states that:

Lemma 2. *For a continuous function g defined on* $[a,b]$ $(0 < a < b)$*, there exist* $\eta \in (a,b)$ *and* $\hat{q} \in (0,1)$ *such that*

$$g(b) - g(a) = D_q g(\eta)(b-a) \tag{8}$$

for all $q \in (\hat{q},1)$.

Other closely related MVTs for real valued functions of several variables was proved in [5] and [6]. In the following content, we aim to generalize Theorem 1 for the multivariable case.

Theorem 2. *If f is differentiable at each point in the line segment* $\overline{ab}$*, then there exists* $\hat{q} \in (0,1)$ *and a point c on* $\overline{ab}$ *(between a and b) such that* $f(b) - f(a) = \nabla_q f(c) \cdot (b-a)$*, for all* $q \in (\hat{q},1) \cup (1,\hat{q}^{-1})$*.*

Proof. Consider the mapping $g(t) = f(a + t(b-a))$, $t \in [0,1]$. Clearly g is continuous on [0,1] and $g(0) = f(a)$, $g(1) = f(b)$. Thus, by applying the q-MVT to g we conclude that there exists $\hat{q} \in (0,1)$ and $t_0 \in (0,1)$ such that $g(1) - g(0) = D_q g(t_0)(1-0)$, for all $q \in (\hat{q},1) \cup (1,\hat{q}^{-1})$, and this means that

$$g(1)-g(0)=f(b)-f(a)=\nabla_q f\left(a+t_0(b-a)\right)\cdot(b-a)=\nabla_q f(c)\cdot(b-a)$$

for all $q\in\left(\hat{q},1\right)\cup(1,\hat{q}^{-1})$.

Remark 3. *If $f(a)=f(b)$ in Theorem 2, then $\nabla_q f(c)\cdot c=0$ which means that $\nabla_q f(c)\perp c$, where c is the vector joining the points a and b.*

Let f be differentiable on an open connected set U such that $\nabla f(z)=0$, for all $z\in U$ then f is constant. This is famous fact in the classical theory of functions of several variables. A similar result for q-Calculus of several variables is considered as follows:

Theorem 3. *Let f be q-differentiable on an open connected set U. If $\nabla_q f(z)=0$ for all $z\in U$ and all $q\in\left(\hat{q},1\right)\cup(1,\hat{q}^{-1})$ for some $\hat{q}\in(0,1)$ then f is constant.*

Proof. Let a and b be any two points in U. Since U is connected then we can join these points by a polygonal line l with vertices $a=c_0,c_1,\cdots,c_{n-1},c_n=b$. By q-MVT (Lemma 2) there exists points

$\hat{q}_1\in(0,1)$ and c_1^* between c_0 and c_1 such that $f(c_1)-f(c_0)=\nabla_q f\left(c_1^*\right)\cdot(c_1-c_0)$, $\forall q\in\left(\hat{q}_1,1\right)\cup(1,\hat{q}_1^{-1})$.

$\hat{q}_2\in(0,1)$ and c_2^* between c_1 and c_2 such that $f(c_2)-f(c_1)=\nabla_q f\left(c_2^*\right)\cdot(c_2-c_1)$, $\forall q\in\left(\hat{q}_2,1\right)\cup(1,\hat{q}_2^{-1})$.

$\vdots$

$\hat{q}_n\in(0,1)$ and c_n^* between c_{n-1} and c_n such that $f(c_n)-f(c_{n-1})=\nabla_q f\left(c_n^*\right)\cdot(c_n-c_{n-1})$, $\forall q\in\left(\hat{q}_n,1\right)\cup(1,\hat{q}_n^{-1})$.

Setting $\hat{q}=\max_{1\le j\le n}\left\{\hat{q}_j\right\}$. If $\nabla_q f(x)=0$ for all $z\in U$ and all $\forall q\in\left(\hat{q},1\right)\cup(1,\hat{q}^{-1})$, then $f(c_j)-f(c_{j-1})$ for all $j=1,\cdots,n$ and this shows that $f(c_0)=f(c_1)=\cdots=f(c_{n-1})=f(c_n)$. Since a and b are arbitrary two points in U then f must be constant.

2.3. Mixed q-partials

Theorem 4. (q-Rolle's Theorem) *Let $a,b,c,d\in\mathbb{R}$ with $0<a<b$ and $0<c<d$, and let $f:\Delta:=[a,b]\times[c,d]\to\mathbb{R}$ satisfy the following:*

(1) For each fixed $w_0\in[c,d]$, the function given by $x\mapsto f(z,w_0)$ is continuous on [a,b], q-differentiable on (a,b) and has continuous q-derivative.
(2) For each fixed $z_0\in[a,b]$, the function given by $y\mapsto \frac{\partial_q f}{\partial_q x}(z_0,y)$ is continuous on [c,d], q-differentiable on (c,d) and has continuous q-derivative.
(3) $f(a,c)+f(b,d)=f(a,d)+f(b,c)$.

Then, there exists $(z_0,w_0)\in(a,b)\times(c,d)$ such that $\frac{\partial_q^2 f}{\partial_q w\partial_q z}(z_0,w_0)=0$.

Proof. Let $\phi:[a,b]\to\mathbb{R}$ defined by $\phi(x)=f(x,d)-f(x,c)$. Then ϕ is continuous on [a,b], q-differentiable on [a,b], has continuous q-derivative, and $\phi(a)=f(a,d)-f(a,c)=f(b,d)-f(b,c)=\phi(b)$. Hence, by q-Rolle's Theorem in one variable (Lemma 2; when $g(b)=g(a)$), there is $z_0\in(a,b)$ such that

$D_q\phi(z_0)=0$, that is $\frac{\partial_q f}{\partial_q x}(z_0,d)=\frac{\partial_q f}{\partial_q x}(z_0,c)$. Now, consider the mapping $\psi:[c,d]\to\mathbb{R}$ defined by $\psi(y)=\frac{\partial_q f}{\partial_q x}(z_0,y)$. Then ψ is continuous on $[c,d]$, q-differentiable on (c,d), has continuous q-derivative and $\psi(c)=\frac{\partial_q f}{\partial_q x}(z_0,c)=\frac{\partial_q f}{\partial_q x}(z_0,d)=\psi(d)$. Hence by Lemma 2 there is $w_0\in(c,d)$ such that $D_q(w_0)=0$, that is $\frac{\partial_q^2 f}{\partial_q y\partial_q z}(z_0,w_0)=0$.

Theorem 5. (q-Mean Value Theorem) *Let $a,b,c,d\in\mathbb{R}$ with $0<a<b$ and $0<c<d$, and let $f:\Delta:=[a,b]\times[c,d]\to\mathbb{R}$ satisfy the following:*

(1) For each fixed $w_0\in[c,d]$, the function given by $x\mapsto f(z,w_0)$ is continuous on $[a,b]$, q-differentiable on (a,b) and has continuous q-derivative.
(2) For each fixed $z_0\in[a,b]$, the function given by $y\mapsto\frac{\partial_q f}{\partial_q x}(z_0,y)$ is continuous on $[c,d]$, q-differentiable on (c,d), and has continuous q-derivative.

Then, there exists $(z_0,w_0)\in(a,b)\times(c,d)$ such that

$$(b-a)(d-c)\frac{\partial_q^2 f}{\partial_q y\partial_q z}(z_0,w_0)=f(b,d)-f(a,d)-f(b,c)+f(a,c).$$

Proof. Let $t\in\mathbb{R}$ and define the mapping $F(z,w)=f(z,w)-f(a,c)-f(x,c)-f(a,y)-t(x-a)(y-c)$. Observe that $F(b,c)=F(a,d)=F(a,c)=0$. Choose

$$t=\frac{f(b,d)-f(b,c)-f(a,d)+f(a,c)}{(b-a)(d-c)}$$

so that $F(b,d)=0$, then we have $F(a,c)+F(b,d)=F(a,d)+F(b,c)$. So by Theorem 4 there is (z_0,w_0) and $\hat{q}\in(0,1)$ such that $\frac{\partial_q^2 F}{\partial_q w\partial_q z}(z_0,w_0)=0$, that is $\frac{\partial_q^2 f}{\partial_q w\partial_q z}(z_0,w_0)=t$ for all $q\in(\hat{q},1)\cup(1,\hat{q}^{-1})$ and this yields the desired result.

Corollary 1. *Let the assumptions of Theorem 5 hold. If there exist $M,m>0$ such that $m\le\frac{\partial_q^2 f}{\partial_q y\partial_q z}(z_0,w_0)\le M$ for all $(z_0,w_0)\in(a,b)\times(c,d)$, then*

$$m(b-a)(d-c)\le f(b,d)-f(a,d)-f(b,c)+f(a,c)\le M(b-a)(d-c).$$

Proof. The proof is an immediate consequence of Theorem 5.

We are now ready to prove the equality of mixed second-order partial q-derivatives provided one of them is continuous.

Theorem 6. *Let $U\subseteq\mathbb{R}^2$ be an open set and let (z_0,w_0) be any point of D. Let $f:D\to\mathbb{R}$ be such that both $\frac{\partial_q f}{\partial_q z}$ and $\frac{\partial_q f}{\partial_q w}$ exist and continuous on D. If either $\frac{\partial_q^2 f}{\partial_q w\partial_q z}$ or $\frac{\partial_q^2 f}{\partial_q z\partial_q w}$ exists on D and is continuous at (z_0,w_0), then both $\frac{\partial_q^2 f}{\partial_q w\partial_q z}(z_0,w_0)$ and $\frac{\partial_q^2 f}{\partial_q z\partial_q w}(z_0,w_0)$ exist and*

$$\frac{\partial_q^2 f}{\partial_q w\partial_q z}(z_0,w_0)=\frac{\partial_q^2 f}{\partial_q z\partial_q w}(z_0,w_0)$$

Proof. Assume that $\frac{\partial_q^2 f}{\partial_q w \partial_q x}$ exists on D and is continuous at $z_0 = (z_0, w_0)$. Let $\epsilon > 0$, by continuity there is a $\delta > 0$ such that

$$\left| \frac{\partial_q^2 f}{\partial_q w \partial_q z}(u,v) - \frac{\partial_q^2 f}{\partial_q w \partial_q z}(z_0,w_0) \right| < \epsilon \qquad \text{whenever } (u,v) \in N_\delta(z_0)$$

By the Rectangular q-Mean Value Theorem (Theorem 5), there is $(c,d) \in N_\delta(z_0)$ such that

$$f(qz_0,qw_0) - f(qz_0,w_0) - f(z_0,qw_0) + f(z_0,w_0) = (q-1)^2 z_0 w_0 \frac{\partial_q^2 f}{\partial_q y \partial_q z}(c,d).$$

The left-hand side of the above equation can be written as $F(z_0,qw_0) - F(z_0,w_0)$, where $F : (w_0 - \delta, w_0 + \delta) \to \mathbb{R}$ is defined by $F(y) := f(qz_0, y) - f(z_0, y)$. Consequently

$$\left| \frac{F(z_0,qw_0) - F(z_0,w_0)}{(q-1)^2 z_0 w_0} - \frac{\partial_q^2 f}{\partial_q w \partial_q z}(z_0,w_0) \right| = \left| \frac{\partial_q^2 f}{\partial_q w \partial_q z}(z_0,w_0) - \frac{\partial_q^2 f}{\partial_q w \partial_q x}(c,d) \right| < \epsilon.$$

Since $\frac{\partial_q f}{\partial_q w}$ exists on D, the function F is differentiable at ω_0 and

$$\frac{d_q F}{d_q y} = \frac{\partial_q f}{\partial_q y}(qz_0,w_0) - \frac{\partial_q f}{\partial_q y}(z_0,w_0)$$

Hence,

$$\left| \frac{\frac{\partial_q f}{\partial_q w}(qz_0,w_0) - \frac{\partial_q f}{\partial_q w}(z_0,w_0)}{(q-1)^2 z_0 w_0} \right| = \left| \frac{\frac{d_q F}{d_q y}(w_0) - \frac{\partial_q}{\partial_q w}\left(\frac{\partial_q f}{\partial_q z}(z_0,w_0)\right)}{(q-1)^2 z_0 w_0} \right| < \epsilon$$

Since $\epsilon > 0$ is arbitrary, we conclude that $\frac{\partial_q^2 f}{\partial_q z \partial_q w}(z_0,w_0)$ exists and is equal to $\frac{\partial_q^2 f}{\partial_q y \partial_q z}(z_0,w_0)$.

Example 2. *For the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by*

$$f(z,w) = \begin{cases} zw\,\frac{z^2 - w^2}{z^2 + w^2}, & (z,w) \neq (0,0) \\ 0, & (z,w) = (0,0) \end{cases}$$

it is clear that $\frac{\partial_q f}{\partial_q z}$, $\frac{\partial_q f}{\partial_q w}$, $\frac{\partial_q^2 f}{\partial_q w \partial_q z}$ and $\frac{\partial_q^2 f}{\partial_q z \partial_q w}$ exist on $\mathbb{R}^2 \setminus \{(0,0)\}$. Moreover, $\frac{\partial_q f}{\partial_q z}(0,w_0) = -w_0$ and $\frac{\partial_q f}{\partial_q w}(z_0,0) = z_0$ for all $z,w \in \mathbb{R}$. Hence, $\frac{\partial_q^2 f}{\partial_q w \partial_q z}(0,0) = -1 \neq 1 = \frac{\partial_q^2 f}{\partial_q z \partial_q w}(0,0)$. Thus by Theorem 6, it follows that neither $\frac{\partial_q^2 f}{\partial_q w \partial_q z}$ nor $\frac{\partial_q^2 f}{\partial_q z \partial_q w}$ can be continuous at (0,0).

Theorem 7. (Multivariate q-Mean Value Theorem (q-*MMVT*)) *Let $D \subset \mathbb{R}^n$ and D° denote the interior of D. Suppose z_0, z_1 are distinct points of D such that $H = \{z_0 + t(z_1 - z_0) \in \mathbb{R}^n : t \in (0,1)\} \subseteq D^\circ$. Let $u = (u_1, \cdots, u_n)$ be a unit vector given as $u = \frac{z_1 - z_0}{\|z_1 - z_0\|}$ and let $f : D \to \mathbb{R}$ be a continuous function such that $D_u^q f$ exists at each point of H for all $q \in (0,1)$. Then there is $c_0 \in H$ such that*

$$f(z_1) - f(z_0) = \|z_1 - z_0\| D_u^q f(c_0)$$

where $q=(q_1,\cdots,q_n)$ *such that* $q_j=\frac{z_0^{(j)}+qt_0\left(z_1^{(j)}-z_0^{(j)}\right)}{z_0^{(j)}+t_0\left(z_1^{(j)}-z_0^{(j)}\right)}$; $j=1,\cdots,n$, $t_0\in(0,1)$ *and* $q\neq q_j$ *for all j.*

Proof. Let $q\in(0,1)$ be any point and $z_0=\left(z_0^{(1)},\dots,z_0^{(n)}\right)$ and $z_1=\left(z_1^{(1)},\dots,z_1^{(n)}\right)$ be distinct points of *D*. Consider the function $F:[0,1]\to\mathbb{R}$ defined by

$$F(t)=f(z(t))=f(z_0+t(z_1-z_0))=f\left(z_0^{(1)}+t\left(z_1^{(1)}-z_0^{(1)}\right),\dots,z_0^{(n)}+t\left(z_1^{(n)}-z_0^{(n)}\right)\right).$$

Clearly, *F* is continuous on [0,1]. So that, for any $t_0\in(0,1)$ with $t\neq t_0$, we have

$$z_j(t)=z_0^{(j)}+t\left(z_1^{(j)}-z_0^{(j)}\right)+(t-t_0)\left(z_1^{(j)}-z_0^{(j)}\right)=z_0^{(j)}+(t-t_0)\|z_1-z_0\|u_j$$

for all $j=1,\cdots,n$.

Hence,

$$\begin{aligned}\frac{F(qt+(1-q)t_0)-F(t_0)}{(q-1)(t-t_0)}&=\frac{f\left(z_0^{(1)}+t\left(z_1^{(1)}-z_0^{(1)}\right),\dots,z_0^{(n)}+t\left(z_1^{(n)}-z_0^{(n)}\right)\right)-f(z(t_0))}{(q-1)(t-t_0)}\\&=\frac{f\left(z_0^{(1)}+(t-t_0)\|z_1-z_0\|^{(q)}u_1,\dots,z_0^{(n)}+(t-t_0)\|z_1-z_0\|^{(q)}u_{nj}\right)-f(z_1(t_0),\dots,z_n(t_0))}{(q-1)(t-t_0)}.\end{aligned}$$

Multiplying both denominator and numerator by $\|z_1-z_0\|$ we get that

$$\begin{aligned}\left.\frac{d_qF(t)}{d_qt}\right|_{t=t_0}&=\|z_1-z_0\|\sum_{j=1}^{n}\frac{\partial_{q_j}f(z_j(t_0))}{\partial_{q_j}z_j}\frac{d_qz_j(t_0)}{d_qt}u_j\\&=\|z_1-z_0\|\left(\sum_{j=1}^{n}\frac{\partial_{q_j}f(z_j(t_0))}{\partial_{q_j}z_j}\frac{d_qz_j(t_0)}{d_qt}e_j\right)\cdot\left(\sum_{j=1}^{n}u_je_j\right)\\&=\|z_1-z_0\|\nabla_qf(z(t))\cdot u\Big|_{t=t_0}\\&=\|z_1-z_0\|D_u^qf(z(t))\Big|_{t=t_0}\end{aligned}$$

where $q=(q_1,\cdots,q_n)$ such that $q_j=\frac{z_0^{(j)}+qt_0\left(z_1^{(j)}-z_0^{(j)}\right)}{z_0^{(j)}+t_0\left(z_1^{(j)}-z_0^{(j)}\right)}$, $j=1,\cdots,n$.

Thus, by the classical MVT applied to *F*, there is $s\in(0,1)$ such that $F(1)-F(0)=(1-0)\frac{d_qF(s)}{d_qt}$. Therefore, there is a $c_0=(c_1,\cdots,c_n)=(z_1(s),\cdots,z_n(s))$ which is a point in *H* satisfies that

$$f(z_1)-f(z_0)=\|z_1-z_0\|D_u^qf(c_0)$$

and this completes the proof.

Corollary 2. *Assume the assumptions of Theorem 7 are hold. Then there is* $c_0\in H$ *such that*

$$f(z_1)-f(z_0)=\|z_1-z_0\|\sum_{j=0}^{n}\frac{\partial_{q_j}f}{\partial_{q_j}z_j}(c_0)$$

where $q=\left(q_j\right)_{j=1}^{n}$ *such that* $q_j=\frac{z_0^{(j)}+qt_0\left(z_1^{(j)}-z_0^{(j)}\right)}{z_0^{(j)}+t_0\left(z_1^{(j)}-z_0^{(j)}\right)}$; $j=1,\cdots,n$, $t_0\in(0,1)$ *and* $q\neq q_j$ *for all j.*

Proof. The proof is straightforward.

3. Chain Rule.

We recall that, if f is continuously differentiable in an open set U and $\ell=\ell(t)$ is differentiable curve lies in U then the composition $f\circ\ell$ is differentiable and $\frac{d}{dt}f\left(\ell(t)\right)=\nabla f\left(\ell(t)\right)\cdot\ell'(t)$. Does the same result hold for multivariable q-calculus? Roughly, the answer is yes, however we have different formula.

Theorem 8. *Let f is continuously q-differentiable in an open set U and* $\ell=\ell(t)$ $(0<a\leq t\leq b)$ *is strictly q-increasing and q-differentiable curve lies in U then the composition* $f\circ\ell$ *is q-differentiable and*

$$\frac{d_q}{d_qt}f\left(\ell(t)\right)=\nabla_q f\left(\ell(t)\right)\cdot\frac{d_q}{d_qt}\ell(t)=\sum_{j=1}^{n}\frac{\partial_{q_j}}{\partial_{q_j}t}f\left(\ell(t)\right)\frac{d_q}{d_qt}r_j(t) \tag{9}$$

where $q_j=\frac{r_j(qt)}{r_j(t)}$, $1\leq j\leq n$ *and* $q=\left(q_j\right)_{1\leq j\leq n}$.

Proof. Let $\ell(t)=\sum_{j=1}^{n}r_j(t)e_j$ be strictly q-increasing and q-differentiable curve lies in an open set U.

Now, let $f(z)=f\left(z_1,\ldots,z_j,\ldots z_n\right)$ such that $z_j=r_j(t)$. But f is continuously q-differentiable, therefore

$$\begin{aligned}\frac{\partial_{q_j}}{\partial_{q_j}t}f\left(r_1(t),\ldots,r_j(t),\ldots r_n(t)\right)&=\frac{f\left(r_1(t),\ldots,r_j(t),\ldots r_n(t)\right)-f\left(r_1(t),\ldots,r_j(t)\frac{r_j(qt)}{r_j(t)},\ldots r_n(t)\right)}{\left(1-\frac{r_j(qt)}{r_j(t)}\right)r_j(t)}\frac{\left(1-\frac{r_j(qt)}{r_j(t)}\right)r_j(t)}{(1-q)t}\\&=\frac{f\left(r_1(t),\ldots,r_j(t),\ldots r_n(t)\right)-f\left(r_1(t),\ldots,r_j(t)\frac{r_j(qt)}{r_j(t)},\ldots r_n(t)\right)}{\left(1-\frac{r_j(qt)}{r_j(t)}\right)r_j(t)}\left(\frac{r_j(t)-r_j(qt)}{(1-q)t}\right)\\&=\frac{\partial_{q_j}}{\partial_{q_j}t}f\left(\ell(t)\right)\frac{d_q}{d_qt}r_j(t),\end{aligned}$$

where $q_j=\frac{r_j(qt)}{r_j(t)}$, $1\leq j\leq n$. Summing up over j from 1 to n, we get

$$\begin{aligned}\frac{d_q}{d_qt}f\left(\ell(t)\right)=\sum_{j=1}^{n}\frac{\partial_{q_j}}{\partial_{q_j}t}f\left(\ell(t)\right)\frac{d_q}{d_qt}r_j(t)&=\left(\sum_{j=1}^{n}\frac{\partial_{q_j}}{\partial_{q_j}t}f\left(\ell(t)\right)e_j\right)\cdot\left(\sum_{j=1}^{n}\frac{d_q}{d_qt}r_j(t)e_j\right)\\&=\nabla_q f\left(\ell(t)\right)\cdot\frac{d_q}{d_qt}\ell(t)\end{aligned}$$

Remark 4. *As* $q\to1$ *we have* $q_j=\frac{r_j(qt)}{r_j(t)}\to1$ *for all j, and so*

$$\nabla_q f\left(\ell(t)\right)\cdot\frac{d_q}{d_qt}\ell(t)=\frac{d_q}{d_qt}f\left(\ell(t)\right)\to\frac{d}{dt}f\left(\ell(t)\right)=\nabla f\left(\ell(t)\right)\cdot\frac{d}{dt}\ell(t).$$

Example 3. *Let us find* $\frac{d_q}{d_q t} f(\ell(t))$ *given that* $\ell(t) = ti + t^2 j + t^3 k$ *and* $f(z) = z^3 + w^3 + z^3$. *The first step is to find* q_j. *Simply,* $q_j = \frac{r_j(qt)}{r_j(t)}$ *so that* $q_1 = q$, $q_2 = q^2$ *and* $q_3 = q^3$. *The second step is determine* $\nabla_q f(z)$, *which is*

$$\begin{aligned}\nabla_q f(z) &= \frac{\partial_{q_1}}{\partial_{q_1} x} f(z) i + \frac{\partial_{q_2}}{\partial_{q_2} y} f(z) j + \frac{\partial_{q_3}}{\partial_{q_3} z} f(z) k \\ &= [3]_{q_1} z^2 i + [3]_{q_2} w^2 j + [3]_{q_3} z^2 k \\ &= [3]_q z^2 i + [3]_{q^2} w^2 j + [3]_{q^3} z^2 k.\end{aligned}$$

The third step is to find $\frac{d_q}{d_q t} \ell(t)$, *which is*

$$\frac{d_q}{d_q t} \ell(t) = i + [2]_q tj + [3]_q t^2 k.$$

Using (9), *we have*

$$\begin{aligned}\frac{d_q}{d_q t} f(\ell(t)) &= \nabla_q f(\ell(t)) \cdot \frac{d_q}{d_q t} \ell(t) \\ &= \left([3]_q t^2 i + [3]_{q^2} t^4 j + [3]_{q^3} t^6 k\right) \cdot \left(i + [2]_q tj + [3]_q t^2 k\right) \\ &= [3]_q t^2 + [2]_q [3]_{q^2} t^4 + [3]_q [3]_{q^3} t^6.\end{aligned}$$

Consequently, the q-derivative of $f(\ell(t))$ *is the directional q-derivative of f in the direction of q-tangent vector* $D_q \ell(t)$.

Proposition 2. *Let f be continuously q-differentiable on an open set U. If* $\ell = \ell(t)$ $(0 < a \le t \le b)$ *is continuous and q-differentiable curve that lies in U, then* $f \circ \ell$ *is q-differentiable. Moreover, there exists c(t) between l(qt) and l(t) such that*

$$\frac{d_q}{d_q t} f(\ell(t)) = \nabla_q f(c(t)) \cdot D_q \ell(t), \tag{10}$$

for all $q \in (\hat{q}, 1)$ and some $\hat{q} \in (0,1)$.

Proof. It is sufficient to show that there exists *c*(*t*) between *l*(*qt*) and *l*(*t*) such that (10) holds. But this follows by Theorem 8, i.e., there exists $c(t) = \ell(qt) + s_0 [\ell(t) - \ell(qt)]$ for some $s_0 \in (0,1)$ such that

$$f(\ell(qt)) - f(\ell(t)) = \nabla_q f(c(t)) \cdot [\ell(qt) - \ell(t)]$$

for all $q \in (\hat{q}, 1)$ and some $\hat{q} \in (0,1)$. Dividing both sides by $(q-1)t$ we get

$$\frac{f(\ell(qt)) - f(\ell(t))}{(q-1)t} = \nabla_q f(c(t)) \cdot \left[\frac{\ell(qt) - \ell(t)}{(q-1)t}\right].$$

Thus, we can have

$$D_q f(\ell(t)) = \nabla_q f(c(t)) \cdot D_q \ell(t),$$

which ends the proof.

Remark 5. *We note that, as* q *tends to* 1, $c(t)=\ell(qt)+s_0\left[\ell(t)-\ell(qt)\right]$ *tends to l(t) and since* $\nabla_q f$ *is continuous then* $\nabla_1 f(c)\to\nabla_1 f(\ell(t))$ *and* (10) *reduces to the classical version, that is*

$$\frac{d}{dt}f(\ell(t))=\nabla f(\ell(t))\cdot\ell'(t).$$

Despite of there is no closed formula to evaluate the $\frac{d_q}{d_q t}f(\ell(t))$, it could be handled by utilizing the directional q-derivative as follows:

$$D_u^q f(z)=\nabla_q f(z)\cdot u.$$

Substituting $z=\ell(t)$ and $u=\frac{d_q}{d_q t}\ell(t)$ therefore

$$D_u^q f(\ell(t))=\nabla_q f(\ell(t))\cdot\frac{d_q}{d_q t}\ell(t). \tag{11}$$

Remark 6. *In general, it is not true that* $\dfrac{d_q}{d_q t}f(\ell(t))=D_u^q f(\ell(t))$. *But, we will take this as convention or allegory.*

4. Higher order Partial q-derivatives

Let $f:\mathbb{R}^3\to\mathbb{R}$ and $\ell:\mathbb{R}\to\mathbb{R}^3$ such that $f=f(z,w,z)$ and $\ell(t,s)=\ell_1(t)i+\ell_2(t)j+\ell_3(t)k$. Then the composition $f\circ\ell:\mathbb{R}\to\mathbb{R}$ is well defined and its derivative is given by

$$\frac{d_q}{d_q t}f(\ell(t))=\nabla_{\hat{q}}f(\ell(t))\cdot\frac{d_q\ell}{d_q t}=\frac{\partial_{\hat{q}_1}f}{\partial_{\hat{q}_1}x}\frac{d_q x}{d_q t}+\frac{\partial_{\hat{q}_2}f}{\partial_{\hat{q}_2}y}\frac{d_q y}{d_q t}+\frac{\partial_{\hat{q}_3}f}{\partial_{\hat{q}_3}z}\frac{d_q z}{d_q t}, \tag{12}$$

where $\hat{q}_1=\frac{x(qt)}{x(t)}$, $\hat{q}_2=\frac{y(qt)}{y(t)}$ and $\hat{q}_3=\frac{z(qt)}{z(t)}$.

In general, for $f:\mathbb{R}^3\to\mathbb{R}$ and $\ell:\mathbb{R}^2\to\mathbb{R}^3$ such that $f=f(z,w,z)$ and $\ell(t,s)=\ell_1(t,s)i+\ell_2(t,s)j+\ell_3(t,s)k$. Consider

$$\hat{q}_{(1,1)}=\frac{\ell_1(qt,s)}{\ell_1(t,s)},\ \hat{q}_{(1,2)}=\frac{\ell_2(qt,s)}{\ell_2(t,s)},\ \hat{q}_{(1,3)}=\frac{\ell_3(qt,s)}{\ell_3(t,s)},$$

and

$$\hat{q}_{(2,1)}=\frac{\ell_1(t,qs)}{\ell_1(t,s)},\ \hat{q}_{(2,2)}=\frac{\ell_2(t,qs)}{\ell_2(t,s)},\ \hat{q}_{(2,3)}=\frac{\ell_3(t,qs)}{\ell_3(t,s)}.$$

The q-chain rule in this case is given by

$$\frac{\partial_q f}{\partial_q t}:=\nabla_{\hat{q}_1}f(\ell(t,s))\cdot\frac{\partial_q\ell}{\partial_q t}=\frac{\partial_{\hat{q}(1,1)}f}{\partial_{\hat{q}(1,1)}x}\frac{\partial_q x}{\partial_q t}+\frac{\partial_{\hat{q}(1,2)}f}{\partial_{\hat{q}(1,2)}y}\frac{\partial_q y}{\partial_q t}+\frac{\partial_{\hat{q}(1,3)}f}{\partial_{\hat{q}(1,3)}z}\frac{\partial_q z}{\partial_q t}, \tag{13}$$

and

$$\frac{\partial_q f}{\partial_q s}:=\nabla_{\hat{q}_2}f(\ell(t,s))\cdot\frac{\partial_q\ell}{\partial_q s}=\frac{\partial_{\hat{q}(2,1)}f}{\partial_{\hat{q}(2,1)}x}\frac{\partial_q x}{\partial_q s}+\frac{\partial_{\hat{q}(2,2)}f}{\partial_{\hat{q}(2,2)}y}\frac{\partial_q y}{\partial_q s}+\frac{\partial_{\hat{q}(2,3)}f}{\partial_{\hat{q}(2,3)}z}\frac{\partial_q z}{\partial_q s}, \tag{14}$$

where $\hat{q}_1=\left(\hat{q}_{(1,1)},\hat{q}_{(1,2)},\hat{q}_{(1,3)}\right)$ and $\hat{q}_2=\left(\hat{q}_{(2,1)},\hat{q}_{(2,2)},\hat{q}_{(2,3)}\right)$.

Example 4. *Let* $\ell(t,s)=ts^2 i+t^2s^3 j+t^3sk$ *and* $f(z)=z^3+w^3+z^3$. *Then*

$$\hat{q}_{(1,1)}=\frac{\ell_1(qt,s)}{\ell_1(t,s)}=q,\ \ \hat{q}_{(1,2)}=\frac{\ell_2(qt,s)}{\ell_2(t,s)}=q^2,\ \ \hat{q}_{(1,3)}=\frac{\ell_3(qt,s)}{\ell_3(t,s)}=q^3,$$
$$\hat{q}_{(2,1)}=\frac{\ell_1(t,qs)}{\ell_1(t,s)}=q^2,\ \ \hat{q}_{(2,2)}=\frac{\ell_2(t,qs)}{\ell_2(t,s)}=q^3,\ \ \hat{q}_{(2,3)}=\frac{\ell_3(t,qs)}{\ell_3(t,s)}=q.$$

Moreover, we obtain

$$\begin{aligned}\nabla_q f(z)&=\frac{\partial_{q_1}}{\partial_{q_1}x}f(z)i+\frac{\partial_{q_2}}{\partial_{q_2}y}f(z)j+\frac{\partial_{q_3}}{\partial_{q_3}z}f(z)k\\&=[3]_{q_1}z^2i+[3]_{q_2}w^2j+[3]_{q_3}z^2k,\end{aligned}\tag{15}$$

where

$$\begin{aligned}\frac{\partial_{\hat{q}_{(1,1)}}f}{\partial_{\hat{q}_{(1,1)}}x}&=\frac{\partial_q f}{\partial_q x}=[3]_q z^2, &&(q_1=q),\\ \frac{\partial_{\hat{q}_{(1,2)}}f}{\partial_{\hat{q}_{(1,2)}}y}&=\frac{\partial_{q^2}f}{\partial_{q^2}y}=[3]_{q^2}w^2, &&(q_2=q^2),\\ \frac{\partial_{\hat{q}_{(1,3)}}f}{\partial_{\hat{q}_{(1,3)}}z}&=\frac{\partial_{q^3}f}{\partial_{q^3}z}=[3]_{q^3}w^2, &&(q_3=q^3).\end{aligned}$$

Also, we can find

$$\frac{\partial_q x}{\partial_q t}=s^2,\qquad \frac{\partial_q y}{\partial_q t}=[2]_{q^2}ts^3,\qquad \frac{\partial_q z}{\partial_q t}=[3]_{q^3}t^2s.$$

On applying (13), *we have*

$$\begin{aligned}\frac{\partial_q f}{\partial_q t}&=\frac{\partial_{\hat{q}_{(1,1)}}f}{\partial_{\hat{q}_{(1,1)}}x}\frac{\partial_q x}{\partial_q t}+\frac{\partial_{\hat{q}_{(1,2)}}f}{\partial_{\hat{q}_{(1,2)}}y}\frac{\partial_q y}{\partial_q t}+\frac{\partial_{\hat{q}_{(1,3)}}f}{\partial_{\hat{q}_{(1,3)}}z}\frac{\partial_q z}{\partial_q t}\\&=[3]_q z^2s^2+[3]_{q^2}w^2[2]_{q^2}ts^3+[3]_{q^3}z^2[3]_{q^3}t^2s\\&=[3]_q t^2s^6+[3]_{q^2}[2]_{q^2}t^5s^9+[3]_{q^3}[3]_{q^3}t^{11}s^4,\end{aligned}$$

i.e.,

$$\begin{aligned}\frac{\partial_q f}{\partial_q s}&=\frac{\partial_{\hat{q}_{(2,1)}}f}{\partial_{\hat{q}_{(2,1)}}x}\frac{\partial_q x}{\partial_q s}+\frac{\partial_{\hat{q}_{(2,2)}}f}{\partial_{\hat{q}_{(2,2)}}y}\frac{\partial_q y}{\partial_q s}+\frac{\partial_{\hat{q}_{(2,3)}}f}{\partial_{\hat{q}_{(2,3)}}z}\frac{\partial_q z}{\partial_q s}\\&=[3]_{q^2}z^2[2]_{q^2}ts+[3]_{q^3}w^2[3]_{q^3}t^2s^2+[3]_q z^2t^3\\&=[3]_{q^2}[2]_{q^2}t^3s^5+[3]_{q^3}[3]_{q^3}t^6s^8+[3]_q t^9s^2.\end{aligned}$$

Noting that on setting $q_1=q^2$, $q_2=q^3$ and $q_3=q$, one can find $\frac{\partial_q x}{\partial_q t}$, $\frac{\partial_q y}{\partial_q t}$ and $\frac{\partial_q z}{\partial_q t}$. Let $f:\mathbb{R}^m\to\mathbb{R}$ and $\ell:\mathbb{R}^n\to\mathbb{R}^m$ such that $f=f(z)$ and $\ell(t)=\sum_{k=1}^m r_k(t)e_k$, where $z\in\mathbb{R}^m$ and $t\in\mathbb{R}^n$. If

$$\hat{q}_{(k,j)} = \frac{r_k\left(t_1,\ldots,qt_j,\ldots,t_n\right)}{r_k\left(t_1,\ldots,t_j,\ldots,t_n\right)}, 1 \le j \le n, 1 \le k \le m,$$

then the q-chain rule is given by

$$\frac{\partial_q f}{\partial_q t_j} := \nabla_{\hat{q}} f\left(\ell(t)\right) \cdot \frac{\partial_q \ell}{\partial_q t_j} = \sum_{k=1}^{m} \frac{\partial_{\hat{q}_{(k,j)}} f}{\partial_{\hat{q}_{(k,j)}} r_j} \frac{\partial_q r_j}{\partial_q t_j} \qquad \text{forall } j = 1,\cdots,n, \tag{16}$$

where $\hat{q}$ is the $m \times n$-matrix $\left[\hat{q}_{(k,j)}\right]_{m\times n}$.

Definition 3. *A map $S : \mathbb{R}^n \to \mathbb{R}^m$ is said to be q-differentiable if there is a linear map $\mathcal{J}_q\left(S\right) : \mathbb{R}^n \to \mathbb{R}^m$ such that*

$$\frac{\left\|S(qz) - S(z) - (q-1)z \cdot \mathcal{J}_q(S)\right\|}{(q-1)\|z\|} = 0.$$

Similar to the classical calculus the q-Jacobian matrix is given by

$$\mathcal{J}_q(\mathbf{S}) := \begin{pmatrix} \frac{\partial_q S_1}{\partial_q z_1} & \frac{\partial_q S_1}{\partial_q z_2} & \cdots & \frac{\partial_q S_1}{\partial_q z_n} \\ \frac{\partial_q S_2}{\partial_q z_1} & \frac{\partial_q S_2}{\partial_q z_2} & \cdots & \frac{\partial_q S_2}{\partial_q z_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial_q S_m}{\partial_q z_1} & \frac{\partial_q S_m}{\partial_q z_2} & \cdots & \frac{\partial_q S_m}{\partial_q z_n} \end{pmatrix},$$

or equivalently

$$\mathcal{J}_q(\mathbf{S}) = \begin{pmatrix} \nabla_q S_1 \\ \nabla_q S_2 \\ \vdots \\ \nabla_q S_m \end{pmatrix}.$$

To analyze the q-derivative of the composition of two mappings, namely $S(z)$ and $z(t)$, in higher dimensional spaces such that $S : \mathbb{R}^m \to \mathbb{R}^p$ given by $S(z) = \left(S_1(z),\ldots,S_m(z)\right)$ and $z : \mathbb{R}^n \to \mathbb{R}^m$ defined by $z(t) = \left(z_1(t),\ldots,z_m(t)\right)$, where $t = \left(t_1,\ldots,t_n\right)$, we should note that if

$$\hat{q}_{(j,k)} = \hat{q}_{jk} = \frac{z_k\left(t_1,\ldots,qt_j,\ldots,t_n\right)}{z_k\left(t_1,\ldots,t_j,\ldots,t_n\right)}, 1 \le j \le n, 1 \le k \le m,$$

we have

$$\frac{\partial_{q_1} S_1}{\partial_{q_1} t_1} = \sum_{k=1}^{m} \frac{\partial_{\hat{q}_{1k}} S_i}{\partial_{\hat{q}_{1k}} z_k} \frac{\partial_q z_k}{\partial_q t_1} = \frac{\partial_{\hat{q}_{11}} S_i}{\partial_{\hat{q}_{11}} z_1} \frac{\partial_q z_1}{\partial_q t_1} + \frac{\partial_{\hat{q}_{12}} S_i}{\partial_{\hat{q}_{12}} z_2} \frac{\partial_q z_2}{\partial_q t_1} + \ldots + \frac{\partial_{\hat{q}_{1m}} S_i}{\partial_{\hat{q}_{1m}} z_m} \frac{\partial_q z_m}{\partial_q t_1}$$

$$\frac{\partial_{q_2} S_i}{\partial_{q_2} t_2} = \sum_{k=1}^{m} \frac{\partial_{\hat{q}_{2k}} S_i}{\partial_{\hat{q}_{2k}} z_k} \frac{\partial_q z_k}{\partial_q t_2} = \frac{\partial_{\hat{q}_{21}} S_i}{\partial_{\hat{q}_{21}} z_1} \frac{\partial_q z_1}{\partial_q t_2} + \frac{\partial_{\hat{q}_{22}} S_i}{\partial_{\hat{q}_{22}} z_2} \frac{\partial_q z_2}{\partial_q t_2} + \ldots + \frac{\partial_{\hat{q}_{2m}} S_i}{\partial_{\hat{q}_{2m}} z_m} \frac{\partial_q z_m}{\partial_q t_2}$$

$$\vdots$$

$$\frac{\partial_{q_n} S_i}{\partial_{q_n} t_n} = \sum_{k=1}^{m} \frac{\partial_{\hat{q}_{kn}} S_i}{\partial_{\hat{q}_{nk}} z_k} \frac{\partial_q z_k}{\partial_q t_n} = \frac{\partial_{\hat{q}_{n1}} S_i}{\partial_{\hat{q}_{n1}} z_1} \frac{\partial_q z_1}{\partial_q t_n} + \frac{\partial_{\hat{q}_{n2}} S_i}{\partial_{\hat{q}_{n2}} z_2} \frac{\partial_q z_2}{\partial_q t_n} + \ldots + \frac{\partial_{\hat{q}_{nm}} S_i}{\partial_{\hat{q}_{nm}} z_m} \frac{\partial_q z_m}{\partial_q t_n},$$

for each $i=1,2,\cdots,p$. Each term represents the component of j^{th}-unit base vector e_j, and so it gives the q-gradient for each S_i, i.e.,

$$\nabla_q S_i\left(z(t)\right)=\sum_{j=1}^{n}\frac{\partial_{q_j}S_i}{\partial_{q_j}t_j}e_j=\sum_{j=1}^{n}\sum_{k=1}^{m}\frac{\partial_{\hat{q}_{jk}}S_i}{\partial_{\hat{q}_{jk}}z_k}\frac{\partial_q z_k}{\partial_q t_j}e_j,\ 1\le i\le p.$$

for all $1\le i\le p$, where $q_j=\left(\hat{q}_{jk}\right)_{1\times m}$ for all $1\le j\le n$ and $1\le k\le m$ and $q=\left(q_1,\cdots,q_n\right)$.

Putting this in n-rows for each $i=1,\cdots,p$, then the general q-Jacobian matrix is given by

$$\mathcal{J}_q\left(\mathbf{S}\left(\mathbf{z}(\mathbf{t})\right)\right)=\begin{pmatrix}\frac{\partial_{\mathbf{q}_1}S_1}{\partial_{\mathbf{q}_1}t_1} & \frac{\partial_{\mathbf{q}_1}S_1}{\partial_{\mathbf{q}_1}t_2} & \cdots & \frac{\partial_{\mathbf{q}_1}S_1}{\partial_{\mathbf{q}_1}t_n}\\ \frac{\partial_{\mathbf{q}_2}S_2}{\partial_{\mathbf{q}_2}t_1} & \frac{\partial_{\mathbf{q}_2}S_2}{\partial_{\mathbf{q}_2}t_2} & \cdots & \frac{\partial_{\mathbf{q}_2}S_2}{\partial_{\mathbf{q}_2}t_n}\\ \vdots & \vdots & \vdots & \vdots\\ \frac{\partial_{\mathbf{q}_n}S_p}{\partial_{\mathbf{q}_n}t_1} & \frac{\partial_{\mathbf{q}_n}S_p}{\partial_{\mathbf{q}_n}t_2} & \cdots & \frac{\partial_{\mathbf{q}_n}S_p}{\partial_{\mathbf{q}_n}t_n}\end{pmatrix}=\begin{pmatrix}\sum_{k=1}^{m}\frac{\partial_{\hat{q}_{1k}}S_1}{\partial_{\hat{q}_{1k}}z_k}\frac{\partial_q z_k}{\partial_q t_1} & \sum_{k=1}^{m}\frac{\partial_{\hat{q}_{2k}}S_1}{\partial_{\hat{q}_{2k}}z_k}\frac{\partial_q z_k}{\partial_q t_2} & \cdots & \sum_{k=1}^{m}\frac{\partial_{\hat{q}_{kn}}S_1}{\partial_{\hat{q}_{nk}}z_k}\frac{\partial_q z_k}{\partial_q t_n}\\ \sum_{k=1}^{m}\frac{\partial_{\hat{q}_{1k}}S_2}{\partial_{\hat{q}_{1k}}z_k}\frac{\partial_q z_k}{\partial_q t_1} & \sum_{k=1}^{m}\frac{\partial_{\hat{q}_{2k}}S_2}{\partial_{\hat{q}_{2k}}z_k}\frac{\partial_q z_k}{\partial_q t_2} & \cdots & \sum_{k=1}^{m}\frac{\partial_{\hat{q}_{kn}}S_2}{\partial_{\hat{q}_{nk}}z_k}\frac{\partial_q z_k}{\partial_q t_n}\\ \vdots & \vdots & \vdots & \vdots\\ \sum_{k=1}^{m}\frac{\partial_{\hat{q}_{1k}}S_p}{\partial_{\hat{q}_{1k}}z_k}\frac{\partial_q z_k}{\partial_q t_1} & \sum_{k=1}^{m}\frac{\partial_{\hat{q}_{2k}}S_p}{\partial_{\hat{q}_{2k}}z_k}\frac{\partial_q z_k}{\partial_q t_2} & \cdots & \sum_{k=1}^{m}\frac{\partial_{\hat{q}_{kn}}S_p}{\partial_{\hat{q}_{nk}}z_k}\frac{\partial_q z_k}{\partial_q t_n}\end{pmatrix}_{p\times n}.$$

On the other hand, the Hessian matrix can be defined as:

$$\mathcal{H}_q\left(S_i\right):=\begin{pmatrix}\frac{\partial_q^2 S_i}{\partial_q z_1^2} & \frac{\partial_q^2 S_i}{\partial_q z_1\partial_q z_2} & \cdots & \frac{\partial_q^2 S_i}{\partial_q z_1\partial_q z_n}\\ \frac{\partial_q^2 S_i}{\partial_q z_2\partial_q z_1} & \frac{\partial_q^2 S_i}{\partial_q z_2^2} & \cdots & \frac{\partial_q^2 S_i}{\partial_q z_2\partial_q z_n}\\ \vdots & \vdots & \vdots & \vdots\\ \frac{\partial_q^2 S_i}{\partial_q z_n\partial_q z_1} & \frac{\partial_q^2 S_i}{\partial_q z_n\partial_q z_2} & \cdots & \frac{\partial_q^2 S_i}{\partial_q z_n^2}\end{pmatrix},$$

for all $1\le i\le p$ such that

$$\mathcal{H}_q\left(S\right):=\left(\mathcal{H}_q\left(S_1\right)\quad \mathcal{H}_q\left(S_2\right)\quad\cdots\quad \mathcal{H}_q\left(S_p\right)\right)^{\mathrm{T}}.$$

Moreover, q-Tensors The Hessian matrix is defined as

$$\mathcal{T}_q\left(S_i\right):=D_q\left(\mathcal{H}_q\left(S_i\right)\right)=\begin{pmatrix}\frac{\partial_q^3 S_i}{\partial_q z_1\partial_q z_1\partial_q z_1} & \frac{\partial_q^3 S_i}{\partial_q z_1\partial_q z_1\partial_q z_2} & \cdots & \frac{\partial_q^3 S_i}{\partial_q z_1\partial_q z_1\partial_q z_n}\\ \frac{\partial_q^3 S_i}{\partial_q z_2\partial_q z_2\partial_q z_1} & \frac{\partial_q^3 S_i}{\partial_q z_2\partial_q z_2\partial_q z_2} & \cdots & \frac{\partial_q^3 S_i}{\partial_q z_2\partial_q z_2\partial_q z_n}\\ \vdots & \vdots & \vdots & \vdots\\ \frac{\partial_q^3 S_i}{\partial_q z_n\partial_q z_n\partial_q z_1} & \frac{\partial_q^3 S_i}{\partial_q z_n\partial_q z_n\partial_q z_2} & \cdots & \frac{\partial_q^3 S_i}{\partial_q z_n\partial_q z_n\partial_q z_n}\end{pmatrix}=\begin{pmatrix}\frac{\partial_q^3 S_i}{\partial_q z_1^3} & \frac{\partial_q^3 S_i}{\partial_q z_1^2\partial_q z_2} & \cdots & \frac{\partial_q^3 S_i}{\partial_q z_1^2\partial_q z_n}\\ \frac{\partial_q^3 S_i}{\partial_q z_2^2\partial_q z_1} & \frac{\partial_q^3 S_i}{\partial_q z_2^3} & \cdots & \frac{\partial_q^3 S_i}{\partial_q z_2^2\partial_q z_n}\\ \vdots & \vdots & \vdots & \vdots\\ \frac{\partial_q^3 S_i}{\partial_q z_n^2\partial_q z_1} & \frac{\partial_q^3 S_i}{\partial_q z_n^2\partial_q z_2} & \cdots & \frac{\partial_q^3 S_i}{\partial_q z_n^3}\end{pmatrix}$$

for all $1\le i\le p$ such that

$$\mathcal{T}_q\left(S\right):=\left(\mathcal{T}_q\left(S_1\right)\quad \mathcal{T}_q\left(S_2\right)\quad\cdots\quad \mathcal{T}_q\left(S_p\right)\right)^{\mathrm{T}}$$

Theorem 9. *Let f be nonconstant and continuously q-differentiable on an open set U. If $\nabla_q f\ne 0$ at each point of U then $\nabla_q f$ is perpendicular to the level curve of f that passes through that point.*

Proof. Pick a point $z_0=\left(z_0^1,\cdots,z_0^n\right)$ in U and assume $\nabla_q f\left(z_0\right)\ne 0$. The level curve of f that passes through that point is $f(z)=f\left(z_0\right)=c$. Obviously, by implicit function theorem this curve could be parameterized in neighborhood of z_0 by a curve $\ell(t)=\sum_{j=1}^{n} r_j(t)e_j$ which continuously q-differentiable for all t in some interval I such that $D_q\ell(t)\neq 0$.

For some $t_0\in I$, $\ell(t_0)=\sum_{j=1}^{n} r_j(t_0)e_j=(z_0^1,\cdots,z_0^n)$. Since f is constant at z_0 on the curve, then $f(\ell(t))=c$ for all $t\in I$. Therefore,

$$\frac{d_q}{d_q t}f(\ell(t))=D_u^q f(\ell(t))=\nabla_q f(\ell(t))\cdot D_q\ell(t)=0, \qquad \text{where } u=D_q\ell(t)$$

which implies that $\nabla_q f(\ell(t))\cdot D_q\ell(t)=0$ and hence $\nabla_q f(\ell(t))\perp D_q\ell(t)$.

Let $f:U\subseteq\mathbb{R}^2\to\mathbb{R}$ be q-differentiable function and assume its directional q-derivative in the direction of a unit vector u

$$D_u^q f(z)=\nabla_q f(z)\cdot u=\sum_{j=0}^{n}\frac{\partial_q f}{\partial_q z_j}(z)u_j$$

exists.

Since $D_u^q f$ is a real-valued function of two variables, we can further consider its directional derivatives along u at points of U, as follows: $f_0=D_u^q f:=f$ and suppose $f_k:=D_u^{k-1,q}\left(D_u^q f\right)=D_u^{k,q}f$ has been defined for $k=0,1,\cdots,n-1$ for all $n\in\mathbb{N}$. If the directional q-derivative of f_{n-1} on U along u exists, then we denote it by $D_u^{n,q}f$ and call it the n-th order directional q-derivative of f on U along u. In symbols, we write

$$D_u^{2,q}f(z)=\sum_{j_2=1}^{n}\frac{\partial_q f}{\partial_q x_{j_2}}\left(\sum_{j_1=0}^{n}\frac{\partial_q f}{\partial_q z_{j_1}}(z)u_{j_1}\right)u_{j_2}:=\sum_{j_2=1}^{n}\sum_{j_1=0}^{n}\frac{\partial_q f}{\partial_q z_{j_1}\partial_q x_{j_2}}(z)u_{j_1}u_{j_2}$$

and in general,

$$D_u^{n,q}f(z)=\sum_{j_n=1}^{n}\frac{\partial_q f}{\partial_q x_{j_n}}\left(\cdots\left(\sum_{j_3=1}^{n}\frac{\partial_q f}{\partial_q z_{j_3}}\left(\sum_{j_2=1}^{n}\frac{\partial_q f}{\partial_q x_{j_2}}\left(\sum_{j_1=0}^{n}\frac{\partial_q f}{\partial_q z_{j_1}}(z)u_{j_1}\right)u_{j_2}\right)u_{j_3}\right)\right)u_{j_n}$$

Theorem 10. (Multivariate q-Taylor Theorem) *Let $q\in(0,1)$. Let $U\subset\mathbb{R}^n$ and U° denotes the interior of U. Suppose $z_0=\left(z_0^{(1)},\ldots,z_0^{(n)}\right)$ and $z_1=\left(z_1^{(1)},\ldots,z_1^{(n)}\right)$ are distinct points of U such that*

$$H=\left\{z_0+t(z_1-z_0)\in\mathbb{R}^n : t\in(0,1)\right\}\subseteq U^\circ.$$

Let $u_q=(u_1,\cdots,u_n)$ be a unit vector given as $u_q=\frac{z_1-z_0}{\|z_1-z_0\|^{(q)}}$. Let n be a nonnegative integer and $f:U\to\mathbb{R}$ be such that $D_u^{k,q}f$ exist and continuous for all $k=0,1,\cdots,n$ and furthermore, $D_u^{n+1,q}f$ exists at every point of H and for all $q\in(0,1)$. Then there is $c_0\in H\setminus\{z_0,z_1\}$ such that

$$f(z_1)=\sum_{j=0}^{n}\frac{r^j}{[j]_q!}\left(D_u^{j,q}f\right)(z_0)+\frac{r^{n+1}}{[n+1]_q!}\left(D_u^{n+1,q}f\right)(c_0) \tag{17}$$

where $r=\|z_1-z_0\|^{(q)}$ and $q=(q_j)_{j=1}^{n}$ such that $q_j=\dfrac{z_0^{(j)}+qt_0\left(z_1^{(j)}-z_0^{(j)}\right)}{z_0^{(j)}+t_0\left(z_1^{(j)}-z_0^{(j)}\right)}$; $j=1,\cdots,n$, and for all $t_0\in(0,1)$.

Proof. For $j = 0,1,\cdots,n$, define $S_i : U \to \mathbb{R}$ by $S_j = D_u^{j,q} f$ and $F_j : [0,1] \to \mathbb{R}$ by $F_j(t) = S_j\left(z(t)\right)$ for $t \in [0,1]$. Define $f_{n+1} : H \setminus \{z_0, z_1\} \to \mathbb{R}$ by $f_{n+1} = D_u^{n+1,q} f$ and $F_{n+1} : (0,1) \to \mathbb{R}$ by $F_{n+1}(t) = f_{n+1}\left(z(t)\right)$ for $t \in (0,1)$.

By definition of directional q-derivative and as in the proof of q-MMVT, if $n \geq 1$ then $F := F_0$ is q-differentiable on (0,1) and

$$D_q F\left(t\right) = rD_u^{0,q} f\left(z\left(t\right)\right) = rS_1\left(z\left(t\right)\right) = rF_1\left(t\right) \qquad \forall t \in (0,1).$$

Similarly, if $n \geq 2$, then F_1 is q-differentiable on (0,1) and $F_1(t) = rF_2(t)$ for all $t \in (0,1])$. Hence $F := F_0$ is twice q-differentiable on (0,1) and

$$D_q^2 F\left(t\right) = rD_q F_1\left(t\right) = r^2 F_2\left(t\right) \qquad \forall t \in (0,1).$$

Continuing in this way, we see that for $j = 0,1,\cdots,n$, the j-th order q-derivative of F exists on (0,1) and $D_q^{(j)} F(t) = r^j F_j(t)$ for all $t \in (0,1)$. Moreover, the (n+1)-th order q-derivative of F exists on (0,1) and $D_q^{(n+1)} f(t) = r^{n+1} F_{n+1}(t)$ for all $t \in (0,1)$. Applying the q-Taylor's theorem of one-variable on F, there exists $\eta \in (0,1)$ such that

$$F\left(1\right) = \sum_{j=0}^{n} \frac{D_q^{(j)} F\left(0\right)}{\left[j\right]_q !} + \frac{D_q^{(n+1)} F\left(\eta\right)}{\left[n+1\right]_q !}.$$

In other words, there is $c_0 \in H \setminus \{z_0, z_1\}$ such that (18) holds, and this proves the required result.

Now, given any $h, k \in \mathbb{R}$, we define *the partial q-differential operator* $\mathcal{D}_{h,k}^q$ as follows:

$$\mathcal{D}_{h,k}^q := h \frac{\partial_q}{\partial_q z} + k \frac{\partial_q}{\partial_q w}.$$

Clearly, $\mathcal{D}_{h,k}^{n,q}$ transforms a real-valued function of two variables to another real-valued function of two variables.

The operator notation $\mathcal{D}_{h,k}^q$ has the property that we can consider successive q-composites of $\mathcal{D}_{h,k}^q$ and these allow us to consider a combination of the n-th order partial q-derivatives at once. Therefore, for any $n \in \mathbb{N}$ we define

$$\mathcal{D}_{h,k}^{n,q} := \left(h \frac{\partial_q}{\partial_q x} + k \frac{\partial_q}{\partial_q w}\right)_q^n := \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q h^{n-j} k^j \frac{\partial_q^n}{\partial_q z^{n-j} \partial_q w^j}$$

Namely, for open subset U of $\mathbb{R}^2$ and function $f : U \to \mathbb{R}$ has continuous partial derivatives of order $\leq n$ at every point of U, then $\mathcal{D}_{h,k}^{n,q} : U \to \mathbb{R}$ is the function defined by

$$\left(\mathcal{D}_{h,k}^{n,q} f\right)\left(z_0, w_0\right) := \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q h^{n-j} k^j \frac{\partial_q^n}{\partial_q z^{n-j} \partial_q w^j} \left(z_0, w_0\right)$$

where (z_0, w_0) varies over U. For instance, the second successive q-composites is

$$\left(\mathcal{D}_{h,k}^{2,q} f\right)\left(z_0, w_0\right) := h^2 \frac{\partial_q^2 f}{\partial_q z^2}\left(z_0, w_0\right) + \left[2\right]_q hk \frac{\partial_q^2 f}{\partial_q z^{2-j} \partial_q w^j}\left(z_0, w_0\right) + k^2 \frac{\partial_q^2 f}{\partial_q w^2}\left(z_0, w_0\right).$$

Bivariate q-Taylor Theorem via *the partial q-differential operator* $\mathcal{D}_{h,k}^{n,q}$

Corollary 3. (Bivariate q-Taylor Theorem) *Let* $q \in (0,1)$. *Let* $U \subset \mathbb{R}^n$ *and* U° *denotes the interior of U. Suppose* $z_0 = \left(z_0^{(1)}, \ldots, z_0^{(n)}\right)$ *and* $z_1 = \left(z_1^{(1)}, \ldots, z_1^{(n)}\right)$ *are distinct points of U such that*

$$H = \left\{z_0 + t\left(z_1 - z_0\right) \in \mathbb{R}^n : t \in (0,1)\right\} \subseteq U^\circ.$$

Let $u_q = \left(u_1, \cdots, u_n\right)$ be a unit vector given as $u_q = \frac{z_1 - z_0}{\|z_1 - z_0\|^{(q)}}$. Let n be a nonnegative integer and $f : U \to \mathbb{R}$ be such that $D_u^{k,q} f$ exist and continuous for all $k = 0, 1, \cdots, n$ and furthermore, $D_u^{n+1,q} f$ exists at every point of H and for all $q \in (0,1)$. Then there is $c_0 \in H \setminus \{z_0, z_1\}$ such that

$$f\left(z_1, w_1\right) = \sum_{j=0}^{n} \frac{1}{[j]_q!}\left(\mathcal{D}_{h,k}^{n,q} f\right)\left(z_0, w_0\right) + \frac{1}{[n+1]_q!}\left(\mathcal{D}_{h,k}^{n+1,q} f\right)(c,d) \tag{18}$$

where $h = x - z_0$, $k = y - w_0$ and $r = \|z_1 - z_0\|^{(q)}$.

Proof. The proof is similar to that one given in proving Theorem 10.

The partial q-differential operator $\mathcal{D}_{h,k}^q$ can be generalized to a formal linear N-th order partial q-differential operator in n variables as follows:

$$P\left(z, \partial_q\right) = \sum_{|\alpha| \le N} I_\alpha(z) \partial_q^\alpha$$

where $\alpha = \left(\alpha_1, \alpha_2, \cdots, \alpha_n\right) \in \mathbb{N}_0^n$, $\partial_q^\alpha = \partial_q^{\alpha_1} \partial_q^{\alpha_2} \cdots \partial_q^{\alpha_n}$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and $I_\alpha(z) := I_{\alpha_1}(z) \cdots I_{\alpha_n}(z)$ are functions on some open domain in n-dimensional space.

For instance, the the k-th order partial q-derivatives

$$\left(\sum_{j=1}^{n} h_j \frac{\partial_q}{\partial_q z_j}\right)^k = \sum_{|\alpha| = k} \begin{bmatrix} k \\ \alpha \end{bmatrix}_q h^\alpha \frac{\partial_q^\alpha}{\partial_q z^\alpha}$$

where $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$ and $\begin{bmatrix} k \\ \alpha \end{bmatrix}_q = \frac{[k]_q!}{[\alpha_1]_q! [\alpha_2]_q! \cdots [\alpha_n]_q!} = \frac{[k]_q!}{[\alpha]_q!}$, such that $k = |\alpha| \in \mathbb{N}_0$.

5. *q*-Maximum-Minimum problems with applications

Definition 4. *Suppose that f is function of several variables and* z_0 *is a point in the domain.*

(1) The function f is said to be has a local q-maximum at a point z_0 *provided that* $f(z_0) \ge f(qz_0)$. *If* $f(z) \ge f(qz_0)$ *for z in the domain of f then* z_0 *is called absolute q-maximum point.*

(2) The function f is said to be has a local q-minimum at a point z_0 *provided that* $f(z_0) \le f(qz_0)$. *If* $f(z) \le f(qz_0)$ *for z in the domain of f then* z_0 *is called absolute q-minimum point.*

Theorem 11. *If* f *has local* q-*extreme value at* z_0, *then* $\nabla_q f(z_0) = 0$ *or* $\nabla_q f(z_0)$ *does not exists.*

Proof. The proof is similar to classical case of multivariable Calculus.

In general, the interior points of the domain at which the q-gradient is zeros of does not exists are called q-critical points. In particular, q-critical points at which the q-gradient is zeros are called q-stationary points and those do not give rise to local q-extreme values are called q-saddle points.

For example, the function $f(z,w) = 2z^2 + w^2 - xy - 7y$ has a q-gradient $\nabla_q f(z,w) = \left[2(q+1)x - y\right]i + \left[(q+1)y - x - 7\right]j$. To find the q-stationary points we set $\nabla_q f(z,w) = 0$.

This gives $x_q = \frac{7}{2(q+1)^2-1}$ and $y_q = \frac{14(q+1)}{2(q+1)^2-1}$. So, the point (x_q, y_q) is the only q-stationary point. Furthermore, it can be shown that (x_q, y_q) is a local q-minimum value of f such that $f(x_q, y_q) = -\frac{98(2q^3+4q^2+2q-1)}{(2q^2+4q+1)^2}$.

On the other hand, as $q \to 1$ we then refer to the original problem in classical calculus, i.e., $\nabla f(z,w) = [4x - y]i + [2y - x - 7]j$ with stationary point $(z_1, w_1) = (1,4)$, however this point gives rise to local minimum value $f(z_1, w_1) = -14$.

Theorem 12. *Suppose f has a continuous second partial q-derivatives in a neighborhood of a point* $z_0 = (z_0, w_0)$ *and* $\nabla_q f(z_0) = 0$. *Set*

$$A_q = \frac{\partial_q^2 f}{\partial_q z^2}(z_0), \qquad B_q = \frac{\partial_q^2 f}{\partial_q w \partial_q z}(z_0), \qquad C_q = \frac{\partial_q^2 f}{\partial_q w^2}(z_0)$$

and for the discriminant $D_q = A_q C_q - B_q^2$, *if*

(1) $D_q < 0$, *then* z_0 *is a q-saddle point.*

(2) $D_q > 0$, *then f has*

- *(a) a local q-minimum at* z_0 *whenever* $A_q > 0$.
- *(b) a local q-maximum at* z_0 *whenever* $A_q < 0$.

Proof. Consider the matrix $M = \begin{pmatrix} A_q & B_q \\ B_q & C_q \end{pmatrix}$. If $D_q < 0$, then M is negative definite, i.e.,

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} A_q & B_q \\ B_q & C_q \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} < 0 \qquad \text{if } (z,w) \neq (0,0)$$

that is, $A_q z^2 + 2B_q xy + C_q w^2 < 0$.

6. Further Studies and Recommendations

In this work, we introduce and study q-Calculus for functions of several variables by presenting the main concepts and analyses. This foundational study paves the way for further considerations and elaborations on other related concepts in q-Calculus. An important extension of this work involves the study of double, triple, and line integrals. In light of this study, we plan to investigate the Green and Stokes theorems within the framework of q-Calculus. It should be noted that this is the first systematic study that facilitates and advances the understanding of differential q-Calculus, bringing it closer to practical applications and deeper theoretical insights.

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